On the radially symmetric solutions of a BVP for a class of nonlinear elliptic partial differential equations

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Abstract. Uniqueness and comparison theorems are proved for the BVP of the form

$$\Delta u(x) + g(x, u(x),\ |\nabla u(x)|) = 0, \quad x \in B, u|_\Gamma = a \in \mathbb{R}\ (\Gamma := \partial B),$$

where B is the unit ball in $\mathbb{R}^n$ centered at the origin ($n \geq 2$). We investigate radially symmetric solutions, their dependence on the parameter $a \in \mathbb{R}$, and their concavity.

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Introduction

Radially symmetric solutions to Dirichlet problem for the nonlinearly perturbed Laplace operator are investigated by many authors, see e.g. [1]-[4].

In [1] it is proven for a wide class of perturbations that the smooth positive solutions of the homogeneous Dirichlet problem in a ball are necessarily radially symmetric. The perturbation of the Laplacian in the paper [2], is $f(u)$ with a locally Lipschitz function f; a BVP with a condition at infinity is considered, reduced to an ODE problem and sufficient conditions are given that guarantee the solvability of the original problem. In the papers [3], [4] nonlinear ODE-BVP-s (partly related to perturbed Laplacian) are considered on the intervals (a, b) and $(0, 1)$ respectively; the term y'' is perturbed with the sum

$$g(x, y') + f(x, y), \qquad \frac{n-1}{x}y' + f(x, y)$$

respectively (where g is locally Lipschitz), and sufficient conditions (certain additional restrictions on f and g) of the existence and uniqueness of the positive solution y are presented. These cases do not cover the general case of perturbations y'' of the form

$$\frac{n-1}{x}y' + f(x, y, y') \quad x \in (0, R), \quad 0 < R < \infty,$$

i.e. the case of the perturbations Δu with $f(|x|, u, \pm|\nabla u|)$.

We remark that fundamental results for the investigations in [1] were already given in [5]. The contribution of the author of [5] to the theory of radially symmetric solutions of nonlinear elliptic PDE-s (mainly on the whole space $\mathbb{R}^n$ and more generally for the m-Laplacian) can be found in [6].

This paper is in final form and no version of it will be submitted for publication elsewhere.

Recently G. Bognár [7] considered the following BVP in the unit ball $B := \{x \in \mathbb{R}^n | \rho \equiv |x| < 1\}$ $(\Gamma := \partial B)$:

(A_1) $\Delta u(x) + \exp(\lambda u(x) + \kappa|\nabla u(x)|) = 0$ $x \in B$; $\kappa, \lambda \leq 0$ are constants,

(A_2) $u \in C^2(B) \cap C(\overline{B}), u(x) = v(|x|) \equiv v(\rho)$,

(A_3) $u|_\Gamma = a = \text{const.} \quad a \geq 0$.

Existence and uniqueness results were established by the author and it was shown that the solution u depends monotonically on the parameter a.

The purpose of the present paper is to prove uniqueness, monotonicity, and concavity results for the solutions of the more general BVP: **Problem 1:**

$$\Delta u(x) + f(|x|,\ u(x),\ |\nabla u(x)|) = 0 \quad x \in B, \tag{1.1}$$

$$u \in C^2(B) \cap C(\overline{B}),\ \exists v : [0,1] \to \mathbb{R} : v(\rho) \equiv v(|x|) = u(x) \quad \forall x \in \overline{B}, \tag{1.2}$$

$$u|_\Gamma = a \in \mathbb{R}. \tag{1.3}$$

Here $f \in C(G_a;\ (0,\infty))$, $a \in \mathbb{R}$ is arbitrarily fixed; $G_a := [0,1] \times [a,\infty) \times [0,\infty)$, B is the unit ball centered at the origin, and $\rho := |x| \quad x \in \overline{B}$.

The method used here is, partly, a modification of that of [7]. Some results are proved without using radial symmetricity. These proofs are based upon the techniques communicated in [9].

To prepare our general results we formulate some of them in simplified versions:

Theorem A. If $f \in C(G_a; (0,\infty))$ and $f(\rho, t, \beta)$ is strongly decreasing in $t \in [a,\infty)$, then **Problem 1** may have no more than one solution.

Theorem B. If $f \in C(G_a; (0,\infty))$ and $f(\rho, t_1, t_2)$ is nonincreasing both in $t_1 \in [a,\infty)$, and $t_2 \in [0,\infty)$, then for the (radially symmetric) solutions u_1 and u_2 of **Problem 1** with the property:

$$u_1|_\Gamma \equiv a_1 > u_2|_\Gamma \equiv a_2 \geq a$$

inequalities

$$v_1(\rho) \equiv u_1(|x|) \geq v_2(\rho) \equiv u_2(|x|) \quad x \in \overline{B},\ v_1'(\rho) \geq v_2'(\rho) \quad \rho \in [0,1)$$

hold.

Finally a concavity result:

Theorem C. Let $f \in C(G_a; (0,\infty))$, and let $f(\rho, t_1, t_2)$ be nonincreasing both in $t_1, \in [a,\infty)$, and $t_2 \in [0,\infty)$. Then there exists a constant $K(a)$ such, that $0 < f \leq K(a) < \infty$, and any of the assumptions $(C_1), (C_2)$

$$f(t, a + \frac{K(a)}{n}\frac{1-t^2}{2},\ \frac{K(a)}{n}t) \geq K(a)(1 - \frac{1}{n}) \quad t \in [0,1], \tag{C_1}$$

$$(C_2) \qquad f(t, a+\frac{K(a)}{2n},\ \frac{K(a)}{n}) \geq K(a)(1-\frac{1}{n}) \quad t \in [0,1]$$

guarantees the concavity of the solution of **Problem 1.** For the case

$$f(\rho, u, |\nabla u|) \equiv \exp\ (\lambda u + \kappa|\nabla u|) \quad \lambda, \kappa \leq 0$$

considered in [7], the assumption (C_1) turnes into (C_3):

$$(C_3) \qquad \exp\left\{\lambda[a+\frac{e^{\lambda a}}{n}\frac{1-t^2}{2}]+\kappa\frac{e^{\lambda a}}{n}t\right\} \geq e^{\lambda a}(1-\frac{1}{n}) \quad t \in [0,1].$$

One of the simplest sufficient conditions for the concavity of u for this special case is

$$(C_4) \qquad \kappa \leq \lambda(\leq 0),\ -1 \leq \kappa e^{\lambda a}.$$

1. Uniqueness results.

We shall prove (under the corresponding conditions) two theorems on the uniqueness of solution of **Problem 1.** The first one will be a consequence of a classical, simple uniqueness theorem related to the problem more general than **Problem 1.**

Theorem 1. Let $w(t) := f(\alpha, t, \beta) \quad t \in [a, \infty)$ for every $\alpha \in [0,1], \quad \beta \in [0, \infty)$ fixed be strongly decreasing in t on the interval $[a, \infty)$; then **Problem 1** has no more than one solution.

Instead of a direct proof consider **Problem 2** (mentioned above) in an arbitrary bounded domain Ω of $\mathbb{R}^n$ with the boundary $\Gamma := \partial\Omega$:

Problem 2.

$$(4) \qquad u \in C^2(\Omega) \cap C(\overline{\Omega})$$

$$(5) \qquad (\Delta u)(x) + g(x, u(x), u_{x_1}(x), \ldots, u_{x_n}(x)) = 0 \quad x \in \Omega,$$

$$(6) \qquad u|_\Gamma = \varphi \in C(\Gamma),$$

where

$$g \in C(\overline{\Omega} \times \mathbb{R}^{n+1}).$$

Theorem 2. If $w(t) := g(\underline{\alpha}, t, \underline{\beta}) \quad t \in \mathbb{R}$ is strongly decreasing in t for any

$$\underline{\alpha} \in \Omega, \quad \underline{\beta} \in \mathbb{R}^n$$

fixed, then **Problem 2** has no more than one solution.

This theorem is very close to the Theorem 9.3 (p.208) of the book [8].

Proof. Suppose that there exist two different solutions of **Problem 2**: u_1 and u_2. Define $u := u_1 - u_2$ and suppose that there exists a point $y \in \Omega$ such, that $u(y) \neq 0$. Without loss of the generality it may be supposed that $u(y) < 0$. Letting

$$m := \min_{x \in \overline{B}} \ u(x)$$

we see, that $m < 0$, and there exists a point $x_0 \in \Omega$ of global minimum of the function $u(x) \quad x \in \overline{\Omega}$ i.e. $\exists \ x_0 \in \Omega$ such that

$$0 > m = u(x_0) \leq u(x) \quad \forall x \in \overline{\Omega}.$$

Consequently we have

$$(\Delta u)(x_0) \geq 0, \quad u_{x_i}(x_0) = 0 \quad i = \overline{1, n}. \tag{7}$$

On the other hand we know, that

$$(\Delta u_1)(x_0) + g(x_0, \ u_1(x_0), \ (grad \ u_1)(x_0)) = 0, \tag{8}$$

$$(\Delta u_2)(x_0) + g(x_0, u_2(x_0), \ (grad \ u_2)(x_0)) = 0, \tag{9}$$

therefore subtracting (9) from (8) we have

$$(\Delta u)(x_0) = g(x_0, \ u_2(x_0), \ (grad \ u_2)(x_0)) - g(x_0, u_1(x_0), \ (grad \ u_1)(x_0)). \tag{10}$$

Here arguments $(grad \ u_2)(x_0)$, $(grad u_1)(x_0)$ are common in virtue of equalities $u_{x_i}(x_0) = 0 \quad i = \overline{1, n}$ (see (7)), therefore using the relations

$$0 > u(x_0) = m \equiv u_1(x_0) - u_2(x_0)$$

and their consequence $u_2(x_0) > u_1(x_0)$; from the monotonicity condition on $w(t)$ we get

$$f(x_0, u_2(x_0), \ (grad \ u_2)(x_0)) - f(x_0, u_1(x_0), \ (grad \ u_1)(x_0)) < 0.$$

So, in (10) we have

$$(\Delta u) \ (x_0) < 0, \tag{11}$$

that contradicts inequality of (7). Theorem 2 is proved.

Remark 1. For the proof of Theorem 1 it is enough to apply Theorem 2 for the case $\Omega := B$ with the nonlinear part g (appearing in **Problem 2**) defined by the formula

$$g(x, u, u_{x_1}, \ldots, u_{x_n}) := f\left(|x|, u, \left(\sum_{i=1}^{n} u_{x_i}^2\right)^{1/2}\right) \quad x \in \overline{B},$$

where f is the nonlinearity appearing in **Problem 1.**

Next we explain another result on the uniqueness of the solution to the **Problem 1** without assumption on strong decrease of $f(\alpha, t, \beta)$ in t. However we need that $f(\alpha, t_1, t_2)$ is nonincreasing both in t_1 and t_2. Here in the proof we will use the radial symmetricity of the solutions.

Theorem 3. Let function f appearing in differential equation of **Problem 1** satisfy conditions:

(i) $w(t) := f(\alpha, t, \beta)$ is nonincreasing in $t \in [a, \infty)$ for every fixed $\alpha, \beta (\alpha \in [0,1],\ \beta \in [0, \infty))$, and

(ii) $\tilde{w}(t) := f(\alpha, \beta, t)$ is nonincreasing in $t \in [0, \infty)$ for every fixed α, β $(\alpha \in [0,1],\ \beta \in [a, \infty))$.

Then **Problem 1** has no more, than one solution.

Proof. Suppose, that there exist two different solutions: $u_1, u_2 (u_1(x) = v_1(|x|),\quad u_2(x) = v_2(|x|) x \in \overline{B})$ of **Problem 1** with the same boundary value $a \in \mathbb{R}$. We introduce the notation

$$v(\rho) := v_1(\rho) - v_2(\rho) \quad \rho \in [0,1].$$

From the assumption, that$f > 0$ and u_1, u_2 are solutions of Problem 1 (especially they are superharmonic and radially symmetric in B) easily follows that

$$\begin{aligned} &v \in C^2([0,1)) \cap C([0,1]), \quad v(1) = 0, \quad v'(0) = 0, \\ &\Delta u_i(x) + f(|x|,\ u_i(x),\ |\nabla u_i(x)|) = \\ &= v_i''(\rho) + \frac{n-1}{\rho} v_i'(\rho) + f(\rho, v_i(\rho), -v_i'(\rho)) = 0 \quad x \in B,\ \rho \in (0,1) \quad i = 1,2, \end{aligned} \tag{12}$$

and the multiplied by ρ^{n-1} version of the last equality of(12) holds:

$$(\rho^{n-1} v_i'(\rho))' + \rho^{n-1} f(\rho, v_i(\rho), -v_i'(\rho)) = 0 \quad \rho \in [0,1), \quad i = 1,2. \tag{13}$$

It can be supposed – without loss of the generality – that there exists a point $a_1 \in [0,1]$ such, that $v(a_1) > 0$. Using the continuity of v on $[0,1]$ it is trivial, that the interval $(0,1)$ also contains a point a_1 such, that $v(a_1) > 0$. Let us fix such a point a_1 for the sequel. Our aim is to construct an interval $[\alpha, \beta] \subseteq [0,1]$ such that

$$v(\rho) > 0\ \rho \in [\alpha, \beta],\ v'(\rho) < 0\ \rho \in (\alpha, \beta],\ v'(\alpha) = 0.$$

Let be

$$b := \sup\{\rho \in [0,1) |\ v(\rho) > 0\}$$

It is clear, that

$$b \in (0,1],\ v(b) = 0,$$

and that

$$a_1 \in (0,b).$$

Further let

$$d := \inf\{\rho \in (a_1, b] | v(\rho) = 0\}.$$

It is clear, that

$$v(d) = 0.$$

Then let be

$$c := 0 \quad \text{if} \quad v(\rho) > 0 \quad \rho \in [0, a_1], \tag{14}$$

and

$$c := \sup\{\rho \in [0, a_1) | v(\rho) = 0\} \quad \text{otherwise.} \tag{15}$$

In the case of (2.43)

$$v(c) = 0$$

holds automatically. Further, denoting by M the maximum of the function v on $[c,d]$ ($M > 0$) let us introduce

$$e := \sup\{\rho \in [c,d] | v(\rho) = M\}.$$

We remark that for the case of (14) $e \in [c,d] \equiv [0,d]$ and

$$v'(e) \equiv v'(0) = 0 \tag{16}$$

in virtue of (12) if $e = c = 0$; and $v'(e) = 0$ if $e \in (c,d) \equiv (0,d)$ using the fact that $v(e) = M$ i.e. e is a point of interior global maximum of the function v on the interval $[c,d]$. In the case of (15)

$$e \in (c,d),\ v(e) = M,\ v'(e) = 0 \tag{17}$$

hold automatically because e is an interior point of global maximum of v on $[c,d]$.

The assumption $v'(\rho) \geq 0 \quad \rho \in [e,d)$ leads to contradiction in both cases (14) and (15), because if $d_1 < d$, and $d_1 \to d$ then

$$v(e) + \int_e^{d_1} v'(\rho)\ d\rho \to v(d) = 0$$

and

$$v(e) + \int_e^{d_1} v'(\rho)\ d\rho \equiv M + \int_e^{d_1} v'(\rho)\ d\rho \geq M > 0 \quad (\forall d_1 \in (e,d)).$$

Consequently there exists a point $\beta \in (e, d)$ such, that $v'(\beta) < 0$. Fixing such a point, it is easy to show – using (16) and continuity of v' on $[0, 1)$ – that there exists an interval $[\alpha, \beta] \subseteq [c, d]$ such, that

$$v'(\rho) < 0 \quad \rho \in (\alpha, \beta],\ v'(\alpha) = 0,\ v(\rho) > 0 \quad \rho \in [\alpha, \beta]. \tag{18}$$

Namely, for the both of the cases (14) and (15) α may be choosen as

$$\alpha := \sup\{\rho \in [e, \beta) | v'(\rho) = 0\} \equiv \sup \mathcal{M} \tag{19}$$

because the set $\mathcal{M}$ is non empty ($e \in \mathcal{M}$), and using the property $v' \in C[0, 1)$ (see (12))

$$\alpha \in [e, \beta), \quad v'(\alpha) = 0. \tag{20}$$

The next step of the proof is the using of the validity of differential equation of **Problem 1** for v_1, v_2 on the interval $I \equiv (\alpha, \beta)$ choiced above:

$$(\rho^{n-1} v_1'(\rho))' + \rho^{n-1} f(\rho, v_1(\rho), -v_1'(\rho)) = 0,$$

$$(\rho^{n-1} v_2'(\rho))' + \rho^{n-1} f(\rho, v_2(\rho), -v_2'(\rho)) = 0,$$

from which after subtracting we get

$$(\rho^{n-1} v'(\rho))' + \rho^{n-1}[f(\rho, v_1, -v_1') - f(\rho, v_2, -v_2')] = 0$$

i.e.

$$(\rho^{n-1} v'(\rho))' = \rho^{n-1}[f(\rho, v_2, -v_2') - f(\rho, v_1, -v_1')].$$

Subtracting and adding in the brackets of the right hand side the term

$$f(\rho,\ v_1(\rho), -v_2'(\rho))$$

we get

$$(\rho^{n-1} v'(\rho))' = \delta_1(\rho) + \delta_2(\rho) \equiv \delta(\rho) \quad \rho \in [\alpha, \beta], \tag{21}$$

where

$$\delta_1(\rho) := \rho^{n-1}[f(\rho, v_2(\rho), -v_2'(\rho)) - f(\rho, v_1(\rho), -v_2'(\rho)] \quad \rho \in [\alpha, \beta],$$

$$\delta_2(\rho) := \rho^{n-1}[f(\rho, v_1(\rho), -v_2'(\rho)) - f(\rho, v_1(\rho), -v_1'(\rho)] \quad \rho \in [\alpha, \beta].$$

Of course $\delta_i \in C[\alpha, \beta] \quad i = 1, 2$. Moreover, taking into account the choice of the interval $[\alpha, \beta]$ we have the relations

$$v(\rho) \equiv v_1(\rho) - v_2(\rho) > 0 \quad \rho \in [\alpha, \beta], \quad v'(\rho) \equiv v_1'(\rho) - v_2'(\rho) < 0 \quad \rho \in (\alpha, \beta]. \tag{22}$$

They imply the inequalities

$$\delta_i(\rho) \geq 0 \quad \rho \in [\alpha, \beta] \tag{23}$$

in virtue of the monotonicity - assumptions (i), (ii) of the theorem. Summarising the precedings results, we get

$$\delta \in C[\alpha, \beta], \quad \delta(\rho) \geq 0 \quad \rho \in [\alpha, \beta]. \tag{24}$$

Integrating equality (21) over the interval (α, β) we get after rearranging:

$$\beta^{n-1} v'(\beta) = \alpha^{n-1} v'(\alpha) + \int_\alpha^\beta \delta(\rho)\, d\rho,$$

from which using equality $v'(\alpha) = 0$ we get

$$\beta^{n-1} v'(\beta) = \int_\alpha^\beta \delta(\rho)\, d\rho \geq 0,$$

consequently $v'(\beta) \geq 0$ that contradicts the choice of β as a point, such, that $v'(\beta) < 0$. Theorem is proved.

Now, let us formulate a weakly generalized Problem 1, namely **Problem 3:**

$$u \in C^2(B_0^R) \cap C(\overline{B_0^R}), \tag{25}$$

$$\Delta u(x) + f(|x|, u(x),\ |\nabla u(x)|) = 0 \quad x \in B_0^R, \tag{26}$$

$$\exists v : [0, R] \to \mathbb{R},\ v(|x|) = u(x) \quad x \in \overline{B}_0^R \ (|x| \in [0, R]), \tag{27}$$

$$u|_\Gamma = a \in \mathbb{R}, \tag{28}$$

where $a \in \mathbb{R}$ is arbitrarily fixed; $R \in (0, \infty)$,

$$B_0^R := \{x \in \mathbb{R}^n |\ |x| < R\},\ \Gamma = \partial B_0^R \equiv \{x \in \mathbb{R}^n |\ |x| = R\},$$

and

$$f \in C(G_{a,R}; (0, \infty), \tag{29}$$

where

$$G_{a,R} := [0, R] \times [a, \infty) \times [0, \infty).$$

Theorem 4. Let function f satisfy the monotonicity conditions: (i) $w(t) := f(\alpha, t, \beta)$ is nonincreasing in $t \in [a, \infty)$ for every fixed $\alpha, \beta \quad (\alpha \in [0, R],\ \beta \in [0, \infty))$,

(ii) $\tilde{w}(t) := f(\alpha, \beta, t)$ is nonincreasing in $t \in [0, \infty)$ for every fixed $\alpha, \beta \quad (\alpha \in [0, R],\ \beta \in [a, \infty))$.

Then **Problem 3** has no more than one solution.

Proof. The arguments used in the proof of Theorem 3 applied to $[0,R]$ instead of $[0,1]$ show the validity of Theorem 4.

2. Comparison results

Theorem 5. Suppose that all of the assumptions included in the formulation of **Problem 3** are fulfilled, moreover assumptions (i), (ii) of Theorem 4 hold. Consider the problems

$$u_i \in C^2(B_0^R) \cap C(\overline{B_0^R}) \quad i=1,2, \tag{30}$$

$$\Delta u_i(x) + f(|x|,\ u_i(x),\ |\nabla u_i(x)|) = 0 \quad x \in B_0^R, \quad i=1,2, \tag{31}$$

$$\exists v_i : [0,R] \to \mathbb{R},\ v_i(|x|) = u_i(x) \quad x \in \overline{B}_0^R \quad (|x| \in [0,R]), \quad i=1,2, \tag{32}$$

$$u_i|_\Gamma = a_i \in \mathbb{R} \quad i=1,2, \tag{33}$$

where

$$a_1,, a_2 \in \mathbb{R}, \quad a_1 > a_2 \geq a.$$

If $u_i \sim v_i \quad i=1,2$ are solutions of problems (30) - (33), then

$$v_1(\rho) \geq v_2(\rho) \quad \rho \in [0,R], \tag{34}$$

$$(0 \geq) v_1'(\rho) \geq v_2'(\rho) \quad \rho \in [0,R],\ v_1'(0) = v_2'(0) = 0. \tag{35}$$

Proof. Let us begin with the proof of inequality (34). We introduce the notation

$$v(\rho) := v_1(\rho) - v_2(\rho) \quad \rho \in [0,R].$$

The arguments used for the derivation of the relations (12), (13) applied to B_0^R instead of B_0^1 give

$$v \in C^2([0,R)) \cap C([0,R]),\ v(R) = a_1 - a_2 > 0, \quad v'(0) = 0, \tag{36}$$

and

$$(\rho^{n-1} v_i'(\rho))' + \rho^{n-1} f(\rho, v_i(\rho), -v_i'(\rho)) = 0 \quad \rho \in [0,R); \quad i=1,2. \tag{37}$$

If $v(\rho) > 0 \quad \rho \in [0,R]$ is also fulfilled, then $v_1(\rho) > v_2(\rho) \quad \rho \in [0,R]$ and (34) is proved. In the case, when there exists a point $b_1 \in [0,R)$ such that $v(b_1) = 0$ let

$$b := \sup\{\rho \in [0,R) | v(\rho) = 0\}. \tag{38}$$

It is clear that

$$b \in [0, R), \quad v(b) = 0.$$

If $b = 0$, then

$$v_1(\rho) > v_2(\rho) \quad \rho \in (0, R], \quad v_1(0) = v_2(0), \tag{39}$$

so (34) is fulfilled. If $b > 0$, then $b \in (0, R)$ and Theorem 4 applied to the ball B_0^b gives

$$v_1(\rho) = v_2(\rho) \quad \rho \in [0, b]. \tag{40}$$

On the other hand $v(R) > 0$, and the definition of b implies the inequality

$$v_1(\rho) > v_2(\rho) \quad \rho \in (b, R]. \tag{41}$$

Relations (40) combined with (41) give

$$v_1(\rho) \geq v_2(\rho) \quad \rho \in [0, R].$$

Next we prove the inequality (35). Suppose the contrary. Then using also the first one of the relations in (36) there exists a point $c_1 \in (0, R)$ such that $v'(c_1) < 0$. Introduce the notation

$$c := \sup\{c_1 \in (0, R] | v'(c_1) < 0\}. \tag{42}$$

It is clear that $c \in (0, R]$ and $v'(c) \leq 0$. Then we consider the three possible cases

$$v(\rho) > 0 \quad \rho \in [0, R], \tag{A}$$

$$v(0) = 0, \quad v(\rho) > 0 \quad \rho \in (0, R] \quad (b = 0), \tag{B}$$

$$v(\rho) \equiv 0 \quad \rho \in [0, b], \quad v(\rho) > 0 \quad \rho \in (b, R] \quad (b \in (0, R)). \tag{C}$$

In the cases (A),(B) let us choose a point $d \in (0, c)$ such, that $v'(d) < 0$. Then we define the set $\mathcal{M}$:

$$\mathcal{M} := \{\rho \in [0, d) |\ v'(\rho) = 0\}.$$

It is obvious, that $\mathcal{M} \neq \emptyset$ because $v'(0) = 0$ (see the last of the relations in (36)). Then let

$$e := \sup \mathcal{M}.$$

It is trivial that

$$e \in [0, d),\ v'(e) = 0,\ v'(\rho) < 0 \quad \rho \in (e, d].$$

Summarising, in the cases (A), (B) we have

$$v(\rho) > 0 \quad \rho \in (e, d],\ v(e) \geq 0;\ v'(\rho) < 0 \quad \rho \in (e, d],\ v'(e) = 0,$$

consequently, the same arguments as in the proof of Theorems 3, 4, applied to the interval $(\alpha, \beta) := (e, d)$ lead to the inequality $v'(d) \geq 0$ that contradicts the choice of d for which $v'(d) < 0$.

For the case (C), first, remark that in virtue of the inequality (41)

$$v(\rho) > 0 \quad \rho \in (b, R], \tag{43}$$

moreover

$$v(b) = 0, \quad v(\rho) = 0 \quad \rho \in [0, b) \quad (v_1'(\rho) = v_2'(\rho) \quad \rho \in [0, b)) \tag{44}$$

according to the definition of b and to Theorem 4 on the uniqueness in the ball B_0^b. Now (44) - using the property $v \in C^2[0, 1)$- implies $v'(b) = 0$, consequently we have the same situation as in the case (B), but on the interval $[b, R]$ instead of interval $[0, R]$. The theorem is proven.

Remark 2. In fact, we proved a stronger result, than inequality (34) : namely, may occour three and only the following three cases:

$$v_1(\rho) > v_2(\rho) \quad \rho \in [0, R], \tag{A}$$

or

$$v_1(\rho) > v_2(\rho) \quad \rho \in (0, R], \quad v_1(0) = v_2(0), \tag{B}$$

or there exists a number $b \in (0, R)$ such that

$$v_1(\rho) = v_2(\rho) \quad \rho \in [0, b], \quad v_1(\rho) > v_2(\rho) \quad \rho \in (b, R]. \tag{C}$$

On the other hand inequality (35)

$$(0 \geq)\ v_1'(\rho) \geq v_2'(\rho) \quad \rho \in [0, R] \quad (v_1'(0) = v_2'(0) = 0)$$

– in general – cannot be replaced by another, stronger one under assumptions of Theorem 5 (see e.g. the case, when f does not depend on argument u).

Theorem 6. All of the statements of Theorem 5 remain - except for inequality (35) - if in conditions of Theorem 5 assumptions (i), (ii) of Theorem 4 are replaced by condition:

$$w(t) := f(|x|,\ t,\ |\nabla u|) \sim f(\alpha, t, \beta)$$

is strongly decreasing in $t \in [a, \infty)$ for every fixed $\alpha, \beta \quad (\alpha \in [0, R], \quad \beta \in [0, \infty))$.

This theorem is a corollary of a general comparison result, namely:

Theorem 7. Let u_1, u_2 be solutions of **Problem 2** satisfying conditions

$$u_i|_\Gamma = \varphi_i \in C(\Gamma) \quad i = 1, 2; \quad \varphi_1 \geq \varphi_2,$$

and suppose that function

$$w(t) := f(\underline{\alpha}, t, \underline{\beta}) \quad t \in \mathbb{R}$$

is strongly decreasing in $t \in \mathbb{R}$ for any $\underline{\alpha} \in \Omega$, $\underline{\beta} \in \mathbb{R}^n$ fixed. Then

$$u_1(x) \geq u_2(x) \quad x \in \Omega.$$

Moreover, if there exists a point $y \in \Gamma$ such that $\varphi_1(y) > \varphi_2(y)$, then may occour two, and only the following two cases:

$$(A) \qquad u_1(x) > u_2(x) \quad \forall x \in \Omega,$$

or there exists a subset $\Omega_1 \neq \emptyset$ of Ω such that

$$0 < \mu(\Omega_1) \leq \mu(\Omega)$$

(μ is the n -dimensional Lebesgue measure) and

$$(B) \qquad u_1(x) > u_2(x) \quad \forall x \in \Omega_1; \quad u_1(x) = u_2(x) \quad \forall x \in \Omega \backslash \Omega_1.$$

Proof. Let $u := u_1 - u_2$, and suppose that there exists a point $y \in \Omega$ such that $u(y) < 0$. Then there is a point $x_0 \in \Omega$ with the property:

$$u(x_0) = \min_{x \in \overline{\Omega}} u(x) \equiv m < 0,$$

and all that remains is to repeat the proof of Theorem 2 for to get a contradiction. Theorem is proven.

3. Concavity results.

Here we will present certain results on the concavity of the function $v : [0,1] \to \mathbb{R}$, defined in the Introduction ((1.2)) by the relation $v(|x|) = u(x) \quad x \in \overline{B}$, where the function u is supposed to be a solution of **Problem 1.**

Theorem 8. Let $a \in \mathbb{R}$ in **Problem 1** be fixed, and suppose that

$$\text{(i)} \qquad w(t) := f(\alpha, t, \beta)$$

is nonincreasing in $t \in [a, \infty)$ for every α, β fixed $(\alpha \in [0,1],\ \beta \in [0, \infty))$,

$$\text{(ii)} \qquad \tilde{w}(t) := f(\alpha, \beta, t)$$

is nonincreasing in $t \in [0, \infty)$ for every α, β fixed $(\alpha \in [0,1],\ \beta \in [a, \infty))$.

If, in addition,

$$\text{(iii)} \qquad f(t, a + \frac{K_a}{n}\frac{1-t^2}{2}, \frac{K_a}{n}t) \geq K_a(1 - \frac{1}{n}) \quad t \in [0,1),$$

where

$$K_a := \sup_{Ga} f (= \max_{\rho \in [0,1]} f(\rho, a, 0))$$

then function v is concave (in non strong sense) on the interval $[0,1)$.

In other words - if $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is a curve in $\mathbb{R}^3$:

$$\gamma : \gamma_1 = t,\ \gamma_2 = a + \frac{K_a(1-t^2)}{2n},\ \gamma_3 = \frac{K_a t}{n} \quad t \in [0,1),$$

then condition (iii) means that

$$\text{(iv)} \qquad f|_\gamma \geq K_a(1 - \frac{1}{n}).$$

Proof. Assumptions of the Theorem guarantee the uniqueness (see Theorem 3 in above) of the solution $u \sim v$ to the **Problem 1.** We know (see (12), (13)) that v has the following properties:

$$\begin{aligned} & v \in C^2[0,1) \cap C[0,1], \quad v(1) = a, \quad v'(0) = 0, \\ (45) \qquad & \Delta u(x) + f(|x|, u(x), |\nabla u(x)|) = v''(\rho) + \frac{n-1}{\rho} v'(\rho) + f(\rho, v(\rho), -v'(\rho)) = 0 \\ & x \in B, \quad \rho \in (0,1), \end{aligned}$$

and

$$(46) \qquad (\rho^{n-1} v'(\rho))' + \rho^{n-1} f(\rho, v(\rho), -v'(\rho)) = 0 \quad \rho \in [0,1).$$

Integrating equality (46) over the inteval $[\delta, t] (0 < \delta < t < 1)$ we get

$$(47) \qquad t^{n-1} v'(t) = \delta^{n-1} v'(\delta) - \int_\delta^t \rho^{n-1} f(\rho, v(\rho), -v'(\rho))\, d\rho$$

from which passing to the limit as $\delta \to 0 + 0$ we obtain

$$(48) \qquad t^{n-1} v'(t) = -\int_0^t \rho^{n-1} f(\rho, v(\rho), -v'(\rho))\, d\rho \quad \forall t \in (0,1).$$

Using the notation

$$\nu := -v' \quad (\nu(t) := -v'(t) \quad \forall t \in [0,1])$$

the first and second of the relations of (45) give

$$v(t)-v(t_1)=\int_t^{t_1}\nu(s)ds \quad 0\le t<t_1<1,\ v(t)-v(t_1)\to v(t)-a \text{ as } \mathrm{t}_1\to 1-0,$$

consequently there exists the improper integral

$$\int_t^1 \nu(s)ds := \lim_{t_1\to 1-0}\int_t^{t_1}\nu(s)ds \quad t\in[0,1),$$

and

$$v(t)=a+\int_t^1 \nu(s)\ ds \quad \forall t\in[0,1], \tag{49}$$

From (48),(49) we obtain that function ν satisfies equality

$$\nu(t)=\int_0^t\left(\frac{\rho}{t}\right)^{n-1} f(\rho,a+\int_\rho^1\nu(s)\ ds,\ \nu(\rho))\ d\rho \quad t\in[0+0,1) \tag{50}$$

which is understood at $t=0+0$ in the limit sense. From the definition of K_a and equality (50) we get the inequality

$$(0\le)\nu(t)\le\frac{K_a}{n}t \quad \forall t\in[0,1). \tag{51}$$

To prove the theorem we have to show that

$$\nu'(t)\ge 0 \qquad t\in[0,1), \tag{52}$$

i.e.-using the last of the equalities in (45) for $\rho\in[0+0,1)$ combined with (50) - the inequality

$$\begin{aligned}&\nu'(t)\equiv f(t,a+\int_t^1\nu(s)ds,\ \nu(t))-\\ &-\frac{n-1}{t}\int_0^t\left(\frac{\rho}{t}\right)^{n-1} f(\rho,a+\int_\rho^1\nu(s)ds,\ \nu(\rho))\ d\rho\ge 0 \quad \rho\in[0+0,1).\end{aligned} \tag{53}$$

From (51) we obtain that

$$\begin{aligned}&\nu'(t)\ge f(t,a+\frac{K_a}{n}\frac{1-t^2}{2},\frac{K_a}{n}t)-\frac{n-1}{t}\nu(t)\ge\\ &\ge f(t,a+\frac{K_a}{n}\frac{1-t^2}{2},\frac{K_a}{n}t)-\frac{n-1}{t}\frac{K_a}{n}t\ge 0 \quad t\in[0,1)\end{aligned} \tag{54}$$

in virtue of conditon (iii). Theorem is proven.

Some concrete sufficient conditions for the special case of **Problem 1**, when

$$f(\rho, u, |\nabla u|) = e^{\lambda u + \mathcal{K}|\nabla u|} \quad \lambda, \mathcal{K} \in \mathbb{R};\ \lambda, \mathcal{K} \le 0 \tag{55}$$

are presented in the following

Theorem 9. Let $a \in \mathbb{R}$ be arbitrarily fixed in **Problem 1** with nonlinearity f of the form in (55). Then solution $u \sim v$ of **Problem 1** exists ([7]), is unique, and any of the following conditions (i) - (vi) guarantees the nonstrong concavity of solution v on $[0, 1)$; where we use the notation

$$d_n := \ln\left[\left(1 - \frac{1}{n}\right)^n\right] \quad n \in \mathbb{N},\ n \quad \text{is fixed} \quad n \ge 2 \quad (d_n < 0),$$

$$\lambda = \mathcal{K} = 0, \tag{i}$$

$$\lambda = 0,\ 0 > \mathcal{K} \ge d_n, \tag{ii}$$

$$\mathcal{K} = 0, \quad 0 > \frac{\lambda e^{\lambda a}}{2} \ge d_n, \tag{iii}$$

$$\mathcal{K} < \lambda < 0, \quad \mathcal{K} e^{\lambda a} \ge d_n, \tag{iv}$$

$$\mathcal{K} = \lambda < 0,\ \lambda e^{\lambda a} \ge d_n, \tag{v}$$

$$\lambda < \mathcal{K} < 0, \quad \frac{e^{\lambda a} \cdot \lambda}{2}\left(1 + \frac{\mathcal{K}^2}{\lambda^2}\right) \ge\ d_n. \tag{vi}$$

Proof. It is enough to prove that inequality (iii) of Theorem 8 is fulfilled in every of the cases (i) - (vi) of the present Theorem. Using that

$$f(t_1, t_2, t_3) \sim f(t_2, t_3) = e^{\lambda t_2 + \mathcal{K} t_3} \quad t_2 \in [a, \infty),\ t_3 \in [0, \infty)$$

and relations

$$f(a, 0) = e^{\lambda a} \ge f(t_2, t_3) \quad t_2 \in [a, \infty), \quad t_3 \in [0, \infty)$$

we get that $K_a = e^{\lambda a}$. Substituting this value into inequality (iii) of Theorem 8, the desirable inequality gains the form

$$e^{\lambda[a + \frac{e^{\lambda a}}{n}\frac{1-t^2}{2}] + \mathcal{K}\ \frac{e^{\lambda a}}{n} t} \ge e^{\lambda a}(1 - \frac{1}{n}) \quad t \in [0, 1)$$

i.e.

$$e^{e^{\lambda a}[\lambda \frac{1-t^2}{2} + \mathcal{K}t]\frac{1}{n}} \geq (1 - \frac{1}{n}) \quad t \in [0, 1)$$

i.e.

$$e^{\lambda a}[\lambda \frac{1-t^2}{2} + \mathcal{K}t] \equiv g(t) \geq \ln[(1 - \frac{1}{n})^n] \equiv \ d_n \quad t \in [0, 1).$$

It is easy to prove in every of the cases (i) - (vi) that

$$\min_{t \in [0,1]} g(t) \geq d_n,$$

which completes the proof.

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