# TRANSVERSAL HOMOCLINICS IN NONLINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Bifurcation of transversal homoclinics is studied for a pair of ordinary differential equations with periodic perturbations when the first unperturbed equation has a manifold of homoclinic solutions and the second unperturbed equation is vanishing. Such ordinary differential equations often arise in perturbed autonomous Hamiltonian systems.

## 1. Introduction

Let us consider the system of ordinary differential equations given by

$$\begin{aligned} \dot{x} &= f(x,y) + \epsilon h(x,y,t,\epsilon), \\ \dot{y} &= \epsilon\Big(Ay + g(y) + p(x,y,t,\epsilon) + \epsilon q(y,t,\epsilon)\Big), \end{aligned} \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, $\epsilon \neq 0$ is sufficiently small, $A$ is an $m \times m$ matrix, and all mappings are smooth, 1-periodic in the time variable $t \in \mathbb{R}$ and such that

(i) $f(0,\cdot) = 0$, $g(0) = 0$, $g_y(0) = 0$, $p(0,\cdot,\cdot,\cdot) = 0$. Here $g_y$ means the derivative of $g$ with respect to $y$. Similar notations are used below.

(ii) The eigenvalues of $A$ and $f_x(0,\cdot)$ lie off the imaginary axis.

(iii) There exists a smooth mapping $\gamma(\theta,y,t) \neq 0$, where $\theta \in \mathbb{R}^{d-1}$, $d \geq 1$ and $y$ is small, such that

$$\begin{aligned} &\dot{\gamma}(\theta,y,t) = f(\gamma(\theta,y,t),y), \quad \gamma(\theta,y,t) = O\big(\mathrm{e}^{-c_1|t|}\big) \\ &\gamma_y(\theta,y,t) = O\big(\mathrm{e}^{-c_1|t|}\big), \quad \gamma_{yy}(\theta,y,t) = O\big(\mathrm{e}^{-c_1|t|}\big) \end{aligned}$$

for a constant $c_1 > 0$, and uniformly for $\theta, y$. Moreover, we suppose

$$d = \dim W^s(y) \cap W^u(y) = \dim T_{\gamma(\theta,y,t)} W^s(y) \cap T_{\gamma(\theta,y,t)} W^u(y)\,.$$

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Here $W^{s(u)}(y)$ is the stable (unstable) manifold to $x=0$ of $\dot{x}=f(x,y)$, respectively, and $T_zW^{s(u)}(y)$ is the tangent bundle of $W^{s(u)}(y)$ at $z\in W^{s(u)}(y)$, respectively.

Consequently, assumption (iii) means that equation $\dot{x}=f(x,y)$ has a nondegenerate homoclinic manifold [5,7,10]

$$W_h(y)=W^s(y)\cap W^u(y)=\Big\{\gamma(\theta,y,t)\mid\theta\in\mathbb{R}^{d-1},\,t\in\mathbb{R}\Big\}.$$

We suppose that $\overline{W}_h(y)$ are compact. We are interested in homoclinic solutions of (1.1) near the family $W_h(y)$. Moreover, we search for transversal such solutions to show chaos for (1.1) [2,5,10].

Systems like (1.1) are investigated in [3], where the existence of chaos is proved, but the situation of this note is not included in [3]. Usually such systems occur in perturbed Hamiltonian systems [7,10], but in this note, equation $\dot{x}=f(x,y)$ has not to be necessary Hamiltonian in $x$ uniformly for $y$ small. For proving our results, we follow [3]. Related results are studied also in the papers [1,8,11,12].

## 2. Transversal Homoclinics

We take in (1.1) the following change of variables

$$\begin{aligned}&x(t)=\gamma(\theta,\epsilon y(t),t)+\epsilon z(t)\,,\\&y\leftrightarrow\epsilon y,\quad t\leftrightarrow t+\alpha\,,\end{aligned}$$

then by (iii) we get

$$\begin{aligned}\dot{z}=\;&f_x(\gamma(\theta,0,t),0)z+h(\gamma(\theta,0,t),0,t+\alpha,0)\\&-\gamma_y(\theta,0,t)p(\gamma(\theta,0,t),0,t+\alpha,0)+O(\epsilon),\\ \dot{y}=\;&\epsilon\Big(\big(A+p_y(\gamma(\theta,0,t),0,t+\alpha,0)\big)y+p_\epsilon(\gamma(\theta,0,t),0,t+\alpha,0)\\&\quad+q(0,t+\alpha,0)+p_x(\gamma(\theta,0,t),0,t+\alpha,0)z+O\big(\mathrm{e}^{-c_1|t|}\big)+O(\epsilon)\Big)\\&+p(\gamma(\theta,0,t),0,t+\alpha,0)\,.\end{aligned}\tag{2.1}$$

Now we consider the variational equation given by

$$\dot{u}=f_x(\gamma(\theta,0,t),0)u\,.\tag{2.2}$$

According to (iii), we note that the system

$$\Big\{\frac{\partial}{\partial\theta_i}\gamma(\theta,0,t)\Big\}_{i=1}^{d-1}\cup\dot{\gamma}(\theta,0,t)$$

is a family of bounded solutions of (2.2), where $\theta=\big(\theta_1,\theta_2,\cdots,\theta_{d-1}\big)$. We can assume that these vectors are linearly independent. Then this family represents a basis of bounded solutions of (2.2). Let $U_\theta(t)$ denote a fundamental solution of (2.2) with $u_{\theta j}(t)$ the $j$th column of $U_\theta(t)$ and define $U_\theta^\perp(t)=\big(U_\theta(t)^{-1}\big)^*$, where $*$ is a transposition with respect to a scalar product $\langle\cdot,\cdot\rangle$ on $\mathbb{R}^n$. We can suppose that $u_{\theta j}(t)$ and $u_{\theta j+d}^\perp(t)$, $j=1,2,\cdots,d$ form bases of the bounded solutions of (2.2) and of the adjoint equation

$$\dot{u}=-f_x(\gamma(\theta,0,t),0)^*u\,,\tag{2.3}$$

respectively, where $u_{\theta j}^\perp(t)$ is the $j$th column of $U_\theta^\perp(t)$. Moreover, we can assume the smoothness of $U_\theta(t)$ on both $\theta$ and $t$. We note that $U_\theta^\perp(t)$ is a fundamental solution of (2.3).

Now by following [3], we get the following result.

**Theorem 2.1.** *Let us define a mapping*

$$M\colon \mathbb{R}^{d-1}\times\mathbb{R}\to\mathbb{R}^d,\quad M=(M_1,M_2,\cdots,M_d)\,,$$

*by*

$$M_l(\theta,\alpha)=\int\limits_{-\infty}^{\infty}\big\langle u^{\perp}_{\theta l+d}(t),h(\gamma(\theta,0,t),0,t+\alpha,0)\big\rangle\,dt \tag{2.4}$$
$$-\int\limits_{-\infty}^{\infty}\big\langle u^{\perp}_{\theta l+d}(t),\gamma_y(\theta,0,t)p(\gamma(\theta,0,t),0,t+\alpha,0)\big\rangle\,dt\,.$$

*If there is a simple root $(\theta_0,\alpha_0)$ of $M(\theta,\alpha)=0$, i.e. $M(\theta_0,\alpha_0)=0$ and the matrix $M_{(\theta,\alpha)}(\theta_0,\alpha_0)$ is nonsingular, then (1.1) has for any $\epsilon\neq 0$ sufficiently small a transversal homoclinic solution near $\gamma(\theta_0,0,\cdot+\alpha_0)\times 0$.*

*Proof.* Since the proof is very similar as of Theorem 2.10 of [3], so we only sketch it here. Let us define the following Banach spaces

$$Z=\Big\{z\in C(\mathbb{R},\mathbb{R}^n)\big|\,|z|=\sup_t|z(t)|<\infty\Big\}\,,$$
$$Y_\theta=\Big\{h\in Z\big|\int\limits_{-\infty}^{\infty}\langle h(t),u^{\perp}_{\theta i+d}(t)\rangle\,dt=0\quad\text{for any}\quad i=1,2,\dots,d\Big\}\,,$$
$$X=\Big\{v\in C(\mathbb{R},\mathbb{R}^m)\big|\,|v|=\sup_t|v(t)|<\infty\Big\}\,.$$

We need the following two results.

**Claim 1.** ([3]) *The nonhomogeneous equation*

$$\dot z=f_x(\gamma(\theta,0,t),0)z+h(t),\quad h\in Z$$

*has a solution $z\in Z$ if and only if $h\in Y_\theta$. The solution is unique if it satisfies $\int\limits_{-\infty}^{\infty}\langle z(t),u_{\theta i}(t)\rangle\,dt=0$ for any $i=1,2,\dots,d$. This solution is smooth in $\theta$ and $h$.*

**Claim 2.** ([3]) *For $\epsilon\neq 0$ sufficiently small, the nonhomogeneous equation*

$$\dot y=\epsilon\Big(\big(A+p_y(\gamma(\theta,0,t),0,t+\alpha,0)\big)y+w\Big),\quad w\in X$$

*has a unique solution in $X$ which we denote $t\to y(t,\alpha,\theta,\epsilon)$. This solution satisfies $|y|\le c_2|w|$ for a constant $c_2>0$, and $\left|\frac{\partial y}{\partial\alpha}\right|=O(\epsilon|w|)$. If in addition $\int\limits_{-\infty}^{\infty}|w(s)|\,ds<\infty$ then $|y|\le c_3|\epsilon|\int\limits_{-\infty}^{\infty}|w(s)|\,ds$ for a constant $c_3>0$.*

Now by using the standard way of Lyapunov-Schmidt like in [3], we can solve (2.1) to get the statement of the theorem. ☐

We note that usually we start with a system of the form

$$\begin{aligned} \dot{x} &= f_1(x,y) + \epsilon h_1(x,y,t,\epsilon)\,, \\ \dot{y} &= \epsilon g_1(x,y,t,\epsilon)\,. \end{aligned} \tag{2.5}$$

Then we suppose that $f_1(x,y)=0$ has a smooth solution $x=\psi(y)$ and by changing the variables, we can suppose that $f_1(0,y)=0$. Then we consider the equation $\dot{y}=\epsilon g_1(0,y,t,\epsilon)$ and we take its averaged equation $\dot{y}=\epsilon\int\limits_0^1 g_1(0,y,t,0)\,dt$ (see [9]). Let $y=0$ be a hyperbolic root of $\int\limits_0^1 g_1(0,y,t,0)\,dt=0$, i.e. $\int\limits_0^1 g_1(0,0,t,0)\,dt=0$ and the matrix $\int\limits_0^1 g_{1y}(0,0,t,0)\,dt$ has no eigenvalues on the imaginary axis. By taking in (2.5) the usual averaging change of variables of the form $y\leftrightarrow y+\epsilon H(y,t)$, where $H$ is smooth and 1-periodic in $t$, we arrive at the system like (1.1). So let us take $y(t)=v(t)+\epsilon H(v(t),t)$ in (1.1). Then we get

$$\begin{aligned} \dot{x} &= f(x,v)+\epsilon\big(f_y(x,v)H(v,t)+h(x,v,t,0)\big)+O(\epsilon^2) \\ &= f_1(x,v)+\epsilon h_1(x,v,t,\epsilon)\,, \\ \dot{v} &= \epsilon\big(I+\epsilon H_v(v,t)\big)^{-1}\Big(Av+g(v+\epsilon H(v,t))-H_t(v,t) \\ &+\epsilon AH(v,t)+p(x,v+\epsilon H(v,t),t,\epsilon)+\epsilon q(v+\epsilon H(v,t),t,\epsilon)\Big) \\ &= \epsilon g_1(x,v,t,\epsilon)\,. \end{aligned} \tag{2.6}$$

The unperturbed equation of (2.6) has the same form as for (1.1). For the mapping $M=(M_1,M_2,\cdots,M_d)$ of (2.4) in terms of (2.6), we have

$$\begin{aligned} M_l(\theta,\alpha) = &-\int\limits_{-\infty}^{\infty}\big\langle u_{\theta l+d}^{\perp}(t), f_y(\gamma(\theta,0,t),0)H(0,t+\alpha)+\gamma_y(\theta,0,t)H_t(0,t+\alpha)\big\rangle\,dt \\ &+\int\limits_{-\infty}^{\infty}\big\langle u_{\theta l+d}^{\perp}(t), h_1(\gamma(\theta,0,t),0,t+\alpha,0)-\gamma_y(\theta,0,t)g_1(\gamma(\theta,0,t),0,t+\alpha,0)\big\rangle\,dt\,. \end{aligned}$$

Assumption (iii) for $\omega(t)=\gamma_y(\theta,0,t)H(0,t+\alpha)$ gives

$$\begin{aligned} \dot{\omega}(t) &= f_x(\gamma(\theta,0,t),0)\omega(t) \\ &\quad + f_y(\gamma(\theta,0,t),0)H(0,t+\alpha)+\gamma_y(\theta,0,t)H_t(0,t+\alpha)\,. \end{aligned} \tag{2.7}$$

Since $\omega\in Z$, equation (2.7) and Claim 1 imply

$$f_y(\gamma(\theta,0,t),0)H(0,t+\alpha)+\gamma_y(\theta,0,t)H_t(0,t+\alpha)\in Y_\theta\,.$$

Hence we get

$$\begin{aligned} M_l(\theta,\alpha) = &\int\limits_{-\infty}^{\infty}\big\langle u_{\theta l+d}^{\perp}(t), h_1(\gamma(\theta,0,t),0,t+\alpha,0)\big\rangle\,dt \\ &-\int\limits_{-\infty}^{\infty}\big\langle u_{\theta l+d}^{\perp}(t), \gamma_y(\theta,0,t)g_1(\gamma(\theta,0,t),0,t+\alpha,0)\big\rangle\,dt\,. \end{aligned} \tag{2.8}$$

When $f_1(0,\cdot)=0$ in (2.5), then (2.8) expresses the mapping $M$ in terms of (2.5) without using its averaged form (1.1).

Generally, when $f_1(\psi(y),y)=0$ and $\gamma(\theta,y,t)$ are homoclinics to the hyperbolic fixed points $x=\psi(y)$ of $\dot{x}=f_1(x,y)$, and $y=y_0$ is a hyperbolic root of the equation $\int_0^1 g_1(\psi(y),y,t)\,dt=0$, then the mapping $M=(M_1,M_2,\cdots,M_d)$ has the form

$$
\begin{aligned}
M_l(\theta,\alpha)=&\int_{-\infty}^{\infty}\left\langle u^{\perp}_{\theta l+d}(t),h_1(\gamma(\theta,y_0,t),y_0,t+\alpha,0)\right\rangle dt\\
&-\int_{-\infty}^{\infty}\left\langle u^{\perp}_{\theta l+d}(t),\gamma_y(\theta,y_0,t)g_1(\gamma(\theta,y_0,t),y_0,t+\alpha,0)\right\rangle dt\,,
\end{aligned}
\tag{2.9}
$$

where (2.2) has to be replaced by

$$\dot{u}=f_x(\gamma(\theta,y_0,t),y_0)u\,.$$

## 3. An Example

Let us consider the system

$$
\begin{aligned}
\ddot{z}&=z-(v^2+\dot{v}^2)z(z^2+w^2+u)+\epsilon\delta\dot{v}\,,\\
\ddot{w}&=w-(v^2+\dot{v}^2)w(z^2+w^2+u)\,,\\
\dot{u}&=(1+v^2+\dot{v}^2)u+\epsilon w^2\,,\\
\ddot{v}+v&=\epsilon\big((1-v^2)\dot{v}+w\big)\,,
\end{aligned}
\tag{3.1}
$$

where $\delta$ is a constant and $\epsilon$ is a small parameter. By taking the polar coordinates

$$v=y\sin\phi,\quad \dot{v}=y\cos\phi\,,$$

(3.1) possesses the form

$$
\begin{aligned}
x_1'&=x_2/g_2(y,\phi,x,\epsilon)\,,\\
x_2'&=\big(x_1-y^2x_1(x_1^2+x_3^2+x_5)+\epsilon\delta y\cos\phi\big)/g_2(y,\phi,x,\epsilon)\,,\\
x_3'&=x_4/g_2(y,\phi,x,\epsilon)\,,\\
x_4'&=\big(x_3-y^2x_3(x_1^2+x_3^2+x_5)\big)/g_2(y,\phi,x,\epsilon)\,,\\
x_5'&=\big((1+y^2)x_5+\epsilon x_3^2\big)/g_2(y,\phi,x,\epsilon)\,,\\
y'&=\epsilon\big((1-y^2\sin^2\phi)y\cos^2\phi+x_3\cos\phi\big)/g_2(y,\phi,x,\epsilon)\,,
\end{aligned}
\tag{3.2}
$$

where $'=\frac{d}{d\phi}$, $x=(x_1,x_2,x_3,x_4,x_5)$ and

$$g_2(y,\phi,x,\epsilon)=1-\epsilon\Big((1-y^2\sin^2\phi)\cos\phi\sin\phi+\frac{x_3}{y}\sin\phi\Big)\,.$$

Of course, we suppose that $y \neq 0$. The unperturbed equation of (3.2) has the form

$$
\begin{aligned}
x_1' &= x_2\,,\\
x_2' &= x_1 - y^2 x_1 (x_1^2 + x_3^2 + x_5)\,,\\
x_3' &= x_4\,,\\
x_4' &= x_3 - y^2 x_3 (x_1^2 + x_3^2 + x_5)\,,\\
x_5' &= (1+y^2) x_5\,.
\end{aligned}
\tag{3.3}
$$

By putting $r(t) = \operatorname{sech} t$, for (3.3) we have [5,6]

$$
\begin{aligned}
\gamma(\theta, y, t) &= \frac{\sqrt{2}}{y}\Big(\sin\theta r(t), \sin\theta \dot r(t), \cos\theta r(t), \cos\theta \dot r(t), 0\Big)\,,\\
u_{\theta 3}^{\perp}(y,t) &= \Big(-\sin\theta \ddot r(t), \sin\theta \dot r(t), -\cos\theta \ddot r(t), \cos\theta \dot r(t), 0\Big)\,,\\
u_{\theta 4}^{\perp}(y,t) &= \Big(-\cos\theta \dot r(t), \cos\theta r(t), \sin\theta \dot r(t), -\sin\theta r(t), 0\Big)\,.
\end{aligned}
\tag{3.4}
$$

Now we consider the equation

$$
\begin{aligned}
y' &= \epsilon \frac{(1-y^2\sin^2\phi) y \cos^2\phi}{1-\epsilon(1-y^2\sin^2\phi)\cos\phi\sin\phi}\\
&= \epsilon\big((1-y^2\sin^2\phi) y\cos^2\phi + O(\epsilon)\big)
\end{aligned}
$$

and its first-order averaging is given by

$$
y' = \epsilon y\Big(\frac{1}{2} - \frac{y^2}{8}\Big)\,.
$$

$y_0 = 2$ is a simple root of $\frac{1}{2} - \frac{y^2}{8} = 0$. Hence we take $y = 2$ in the formulas (3.4).

In the notation of (2.5), we have

$$
\begin{aligned}
& h_1(x, 2, \phi, 0)\\
&= \Big(x_2, x_1 - 4x_1(x_1^2 + x_3^2 + x_5), x_4, x_3 - 4x_1(x_1^2+x_3^2+x_5), 5x_5\Big) g_3(x,\phi)\\
&+ 2\delta\big(0, \cos\phi, 0, 0, 0\big) + \big(0, 0, 0, 0, x_3^2\big)\,,\\
& g_3(x,\phi) = (1-4\sin^2\phi)\sin\phi\cos\phi + \frac{x_3}{2}\sin\phi\,,\\
& g_1(x, 2, \phi, 0) = 2(1-4\sin^2\phi)\cos^2\phi + x_3\cos\phi\,.
\end{aligned}
$$

We see that

$$
\gamma_y(\theta, 2, t) = -\gamma(\theta, 2, t)/2\,.
$$

Since

$$
\begin{aligned}
& h_1(\gamma(\theta,2,t), 2, t+\alpha, 0) = \dot\gamma(\theta, 2, t) g_3(\gamma(\theta,2,t), t+\alpha)\\
& + 2\delta\big(0, \cos(t+\alpha), 0, 0, 0\big) + \frac{1}{2}\big(0,0,0,0,\cos^2\theta r(t)^2\big)\,,\\
& u_{\theta 2}(t) = \dot\gamma(\theta, 2, t), \quad \langle u_{\theta 2}(t), u_{\theta i+2}^{\perp}(t)\rangle = 0, \quad i = 1,\, 2\\
& \langle \gamma_y(\theta, 2, t), u_{\theta 4}^{\perp}(t)\rangle = 0\,,
\end{aligned}
$$

the formula (2.9) has after several calculations [4] now the form

$$\begin{aligned}M_1(\theta,\alpha) &= 2\delta\pi\,\mathrm{sech}\,\frac{\pi}{2}\sin\theta\sin\alpha + \frac{2\pi\sqrt{2}}{3}\,\mathrm{cosech}\,\pi\cos 2\alpha\\ &\quad + \frac{10\pi\sqrt{2}}{3}\,\mathrm{cosech}\,2\pi\cos 4\alpha + \frac{5\pi}{24}\,\mathrm{sech}\,\frac{\pi}{2}\cos\theta\cos\alpha\,,\\ M_2(\theta,\alpha) &= 2\delta\pi\,\mathrm{sech}\,\frac{\pi}{2}\cos\theta\cos\alpha\,.\end{aligned}$$

For finding a simple root of $M(\theta,\alpha)=0$, we suppose $\delta\neq 0$ and take $\theta=-\pi/2$ while $\alpha\neq\pm\pi/2$ must be a simple zero of the equation

$$\delta = \sqrt{2}\,\frac{\mathrm{cosech}\,\pi\cos 2\alpha + 5\,\mathrm{cosech}\,2\pi\cos 4\alpha}{3\,\mathrm{sech}\,(\pi/2)\sin\alpha} = \Omega(\alpha)\,. \tag{3.5}$$

Function $\Omega(\alpha)$ is odd and it is satisfying

$$\Omega(\alpha)=\Omega(\pi-\alpha),\quad \Omega(\alpha)=-\Omega(\pi+\alpha),\quad \lim_{\alpha\to 0_+}\Omega(\alpha)=+\infty\,.$$

Furthermore, $\Omega$ has on $(0,\pi)$ only three critical points $\alpha_1$, $\alpha_2=\pi-\alpha_1$, $\alpha_3=\pi/2$ for some $\alpha_1\simeq 1.378$. Moreover, $\Omega$ attains on $(0,\pi)$ its global minimum at $\alpha_1$, $\alpha_2$ and a local maximum at $\alpha_3$. We note that $\Omega(\alpha_1)=\Omega(\alpha_2)$. Consequently as $\Omega(\pi/2)<0$, (3.5) has a simple zero for any $\delta$.

Summarizing, by applying Theorem 2.1 and results of the papers [2,5], we arrive at the following result.

**Theorem 3.1.** *Let $\delta\neq 0$ be fixed. Equation (3.1) has chaos for any $\epsilon\neq 0$ sufficiently small.*

We note that for any compact interval $[a_1,a_2]\subset\mathbb{R}$, $0\notin[a_1,a_2]$, there is an $\epsilon_0>0$ such that (3.1) has chaos for any $\delta\in[a_1,a_2]$ and $0<|\epsilon|<\epsilon_0$. On the other hand, the function $M_2(\theta,\alpha)$ is vanishing for $\delta=0$, and we should derive higher-degenerate Melnikov mapping to get a reasonable bifurcation result as $\delta$ is crossing 0. We do not follow this line in this paper.

When $w=u=0$ in (3.1), we get the simpler system

$$\begin{aligned}\ddot z &= z-(v^2+\dot v^2)z^3+\epsilon\delta\dot v\,,\\ \ddot v+v &= \epsilon(1-v^2)\dot v\,.\end{aligned} \tag{3.6}$$

Then (3.3) has the form

$$x_1'=x_2,\quad x_2'=x_1-y^2x_1^3\,. \tag{3.7}$$

(3.7) has a homoclinic $\gamma(y,t)=\frac{\sqrt{2}}{y}\big(r(t),\dot r(t)\big)$. So now we have $d=1$ and $u_2^{\perp}(y,t)=\big(-\ddot r(t),\dot r(t)\big)$. The Melnikov function has now the form

$$M(\alpha)=2\delta\int_{-\infty}^{\infty}\cos(t+\alpha)\dot r(t)\,dt = 2\delta\pi\,\mathrm{sech}\,\frac{\pi}{2}\sin\alpha\,.$$

We see that $\alpha_0=0$ is a simple root of $M(\alpha)=0$ for $\delta\neq 0$. Consequently, (3.6) is chaotic for $\delta\neq 0$ fixed and $\epsilon\neq 0$ sufficiently small. Hence (3.1) has, in addition to Theorem 3.1, also “trivial” chaos of (3.6) with $w=u=0$.

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