AN EXTENSION OF MILLOUX'S THEOREM TO HALF-LINEAR DIFFERENTIAL EQUATIONS

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Abstract. A theorem of Milloux (1934) concerning the Sturm–Liouville differential equations is extended to the so–called half–linear differential equations.

1. Introduction.

By Sonine-Pólya theorem [16] it is well-known that the local maxima of $|y(t)|$ of a solution $y(t)$ of

$$y'' + q(t)y = 0\,, \quad t \geq 0\,, \quad q(t) > 0\,, \quad {}' = \frac{d}{dt}\,, \tag{1.1}$$

are non-increasing if $q(t)$ is non-decreasing and continuous. Clearly, all the solutions of (1.1) are oscillatory. It is a longstanding problem to decide what happens if the coefficient $q(t)$ tends to ∞ as $t \to \infty$. Milloux [13] was the first who proved that there is at least one solution of (1.1) satisfying the relation

$$\lim_{t\to\infty} y(t) = 0 \tag{1.2}$$

(see also Bihari [3], Hartman [7], Prodi [14], Trevisan [18]). Under a more stringent condition on $q(t)$, namely if $q(t)$ "regularly" tends to ∞ (see for definition in [15]), Armellini [1], Tonelli [17] and Sansone [15] proved that every solution of (1.1) satisfies (1.2).

In [4], Bihari succeeded in generalizing this result of Armellini, Tonelli, Sansone with the same restriction on $q(t)$ to the so-called *half-linear* differential equations

$$y''|y'|^{n-1} + q(t)\,|y|^{n-1}y = 0\,, \quad t \geq 0\,, \quad q(t) > 0\,, \quad n > 0\,, \tag{1.3}$$

where n is real. These differential equations are non-linear but they have the important property that if $y(t)$ is a solution, then $cy(t)$ is also a solution where c is a constant and the term "half-linear" just refers to this property. Clearly, (1.3) reduces to (1.1) if $n = 1$.

Our aim here is to extend the theorem of Milloux to (1.3).

Definition. A solution $y(t)$ of (1.1) is *small* if it satisfies (1.2); otherwise it is *large*.

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Theorem. *Let $q(t)$ be non-decreasing and continuously differentiable function and satisfy*

$$\lim_{t\to\infty} q(t) = \infty. \tag{1.4}$$

Then the differential equation (1.3) *has at least one non-trivial small solution.*

Remark. The hypotheses can be slightly weakened. The "non-decreasing" requirement may be replaced by a condition of limited decrease on $\log q(t)$. The differentiability of $q(t)$ may be weakened to continuity, or even piece-wise continuity. We discuss these points at the end of the paper.

The first version of this paper was written nearly ten years ago and then circulated among collegues. The proof was based on the observation that (1.3) is equivalent to the Hamiltonian system

$$\begin{aligned} y' &= \frac{\partial H(y,z)}{\partial z}, \\ z' &= -\frac{\partial H(y,z)}{\partial y}, \end{aligned} \tag{1.5}$$

where $H(y,z) = \frac{n}{n+1}\big(q(t)|y|^{n+1} + |z|^{\frac{1}{n}+1}\big)$ and $z = |y'|^{n-1}y'$. System (1.5) implies the area–preserving property of the half–linear differential equation (1.3) and this property was used explicitely in our earlier version. Here we give instead an essentially simpler, almost "elementary" proof. However, the geometric aspect of (1.3) or (1.5) has already caused some attention (see [8], [9], [10], [11], [12]) and we think this concept deserves more discussion to which we intend to return later.

2. The generalized Prüfer transformation.

We define (as in [5]) the generalized sine function $S(\theta)$ as the solution of

$$S''|S'|^{n-1} + S|S|^{n-1} = 0, \qquad S(0) = 0,\ S'(0) = 1, \tag{2.1}$$

and note the identity

$$|S'(\theta)|^{n+1} + |S(\theta)|^{n+1} = 1. \tag{2.2}$$

This function has period $2\hat{\pi}$, where

$$\hat{\pi} = 2\frac{\frac{\pi}{n+1}}{\sin\frac{\pi}{n+1}},$$

which reduces to π in the ordinary case $n = 1$. Other properties following the pattern of the ordinary case are:

$$S(\theta) > 0, \quad S'(0) > 0, \qquad 0 < \theta < \frac{\hat{\pi}}{2}, \tag{2.3}$$

$$S(\theta) > 0, \quad S'(0) < 0, \qquad \frac{\hat{\pi}}{2} < \theta < \hat{\pi}, \tag{2.4}$$

furthermore

$$S(\theta + \hat{\pi}) \equiv -S(\theta). \tag{2.5}$$

For a non-trivial solution $y(t)$ of (1.1) the generalized polar coordinates $\varrho(t) > 0$, $\theta(t)$ are introduced by

$$y(t) = \rho(t)\, S(\theta(t)), \qquad y'(t) = \rho(t)\, S'(\theta(t))\, q^{\frac{1}{n+1}}(t). \tag{2.6}$$

Thus, in particular, ϱ is uniquely determined by means of (2.2), in fact

$$\varrho = \{|y|^{n+1} + \frac{1}{q}|y'|^{n+1}\}^{\frac{1}{n+1}}, \tag{2.7}$$

while $\theta(t)$ may be fixed as a continuous function, subject to an arbitrary additive multiple of $2\hat{\pi}$. The differential equations for ϱ and θ are found to be

$$\theta' = q^{\frac{1}{n+1}} + \frac{q'}{q} f(\theta), \tag{2.8}$$

$$\frac{\varrho'}{\varrho} = -\frac{q'}{q} g(\theta), \tag{2.9}$$

where

$$f(\theta) = \frac{1}{n+1}|S'(\theta)|^{n-1} S'(\theta)\, S(\theta), \tag{2.10}$$

$$g(\theta) = \frac{1}{n+1}|S'(\theta)|^{n+1}. \tag{2.11}$$

The right-hand side of (2.8–9) are Lipschitzian in θ. In fact we have, using (2.1-2),

$$f'(\theta) = |S'(\theta)|^{n+1} - \frac{n}{n+1}, \tag{2.12}$$

$$g'(\theta) = S'(\theta)\, |S(\theta)|^n \operatorname{sgn}\big(S(\theta)\big). \tag{2.13}$$

Thus the equations (2.8–9) satisfy the Cartheodory conditions and the functions $f(\theta)$ and $g(\theta)$ are periodic with period $\hat{\pi}$.

We obtain all non–trivial solutions of (1.3) by considering solutions of (2.8–9) with general initial data $\varrho(0) > 0$, and real $\theta(0)$. Moreover, in view of (2.5), we see that if $\theta(t)$, $\varrho(t)$ is a solution then so also is $\theta(t) + \hat{\pi}$, $\varrho(t)$: this corresponds to the fact that (1.3) has solutions $y(t)$ and $-y(t)$ simultaneously. We obtain essentially all solutions if we consider a range of values for $\theta(0)$ where the range is of length $\hat{\pi}$. The value for $\rho(0) > 0$ will not be important.

We accordingly consider the solutions of (2.8–9) with initial data $\theta(0) = \varphi$, $\varrho(0) = 1$. We denote these by $\theta(t, \varphi)$, $\varrho(t, \varphi)$, respectively. Since by (2.9)

$$\varrho(t, \varphi) = \exp\Big(-\int_0^t \frac{q'(s)}{q(s)} g\big(\theta(s, \varphi)\big)\, ds\Big),$$

and $g(\theta) \geq 0$, the function $\varrho(t, \varphi)$ is monotone non–increasing, $\varrho(t, \varphi)$ tends to a limit $\varrho(\infty, \varphi) \geq 0$ as $t \to \infty$. It is clear that $\varrho(\infty, \varphi) = 0$ implies that $y(t) \to 0$ as $t \to \infty$. The converse is also true because $y(t)$ is oscillatory.

In view of (2.9), we have the following characterizations of the two possibile solutions:

(i) $\varrho(\infty, \varphi) = 0$, the corresponding solution $y(t) \to 0$, and

$$\int_0^\infty \frac{q'(t)}{q(t)} g(\theta(t, \varphi))\, dt = \infty, \tag{2.14}$$

(ii) $\varrho(\infty, \varphi) > 0$, the solution $y(t)$ oscillates, its amplitude tends to a positive limit, and

$$\int_0^\infty \frac{q'(t)}{q(t)} g(\theta(t, \varphi))\, dt < \infty. \tag{2.15}$$

3. Outline the proof.

This is based on two lemmas concerning the behaviour as $t \to \infty$ of the function

$$\psi(t, \varphi_1, \varphi_2) = \theta(t, \varphi_2) - \theta(t, \varphi_1), \tag{3.1}$$

where, to begin with,

$$\varphi_1 < \varphi_2 < \varphi_1 + \hat{\pi}. \tag{3.2}$$

We have in this case

$$0 < \psi(t, \varphi_1, \varphi_2) < \hat{\pi}, \qquad 0 \le t < \infty, \tag{3.3}$$

by uniqueness properties. Clearly, $\psi(t, \varphi_1, \varphi_2)$ is a strictly increasing function of φ_2, and a strictly decreasing function of φ_1. If $\psi(t, \varphi_1, \varphi_2)$ tends to a limit as $t \to \infty$, we denote this by $\psi(\infty, \varphi_1, \varphi_2)$.

We denote by $\mathcal{X}$ the set of real φ such that (2.15) holds, that is to say such that the corresponding solution $y(t)$ does not tend to zero. We have, of course, to show that $\mathcal{X}$ is a proper subset of $\mathbb{R}$. In the next section we will prove

Lemma 1. *Let $\varphi_1, \varphi_2 \in \mathcal{X}$ and satisfy* (3.2). *Then $\psi(\infty, \varphi_1, \varphi_2)$ exists and equals either 0 or $\hat{\pi}$.*

We deal also with a perturbation property for elements of $\mathcal{X}$.

Lemma 2. *Let $\varphi_0 \in \mathcal{X}$. Then for any $\delta > 0$ there is an $\eta \in (0, \hat{\pi})$ such that if φ satisfies $|\varphi - \varphi_0| < \eta$, then*

$$|\psi(t, \varphi_0, \varphi)| < \delta \qquad \textit{for all } t \ge 0. \tag{3.4}$$

Outlining now our proof of Theorem 1, based on these two lemmas, we assume the contrary, namely that $\mathcal{X} = \mathbb{R}$. We have then that the function of φ given by $\psi(\infty, 0, \varphi)$ is non–decreasing as φ increases in the interval $[0, \hat{\pi}]$. It must go from 0 to $\hat{\pi}$, taking only these values, by Lemma 1, but remaining continuous, by Lemma 2, which is impossible.

4. Proof of Lemma 1.

We write for brevity $\theta_j(t) = \theta(t, \varphi_j)$, $j = 1, 2$, and use the fact that (2.15) holds for $\varphi = \varphi_1, \varphi_2$, so that

$$\int_0^\infty \frac{q'(t)}{q(t)} \left\{ g(\theta_1(t)) + g(\theta_2(t)) \right\} dt < \infty. \tag{4.1}$$

Suppose first that $\psi(t, \varphi_1, \varphi_2)$ does not tend to a limit as $t \to \infty$. Then there exist α and β with $0 < \alpha < \beta < \hat{\pi}$ and sequences t_{1m}, t_{2m} tending to ∞ such that

$$\psi(t_{1m}, \varphi_1, \varphi_2) = \alpha, \qquad \psi(t_{2m}, \varphi_1, \varphi_2) = \beta, \tag{4.2}$$

$$\alpha < \psi(t, \varphi_1, \varphi_2) < \beta, \quad \text{for } t_{1m} < t < t_{2m}. \tag{4.3}$$

Choose now $\varepsilon \in (0, \frac{\hat{\pi}}{2})$ such that $\varepsilon < \alpha, \beta < \hat{\pi} - \varepsilon$, so that, by (4.3),

$$\varepsilon < \theta_2(t) - \theta_1(t) < \hat{\pi} - \varepsilon \tag{4.4}$$

for $t_{1m} < t < t_{2m}$. Hence for every such t there exists $m' \in \mathbb{N}$ such that either

$$|\theta_1(t) - (m' + \frac{1}{2})\hat{\pi}| \leq \frac{1}{2}\varepsilon \quad \text{and} \quad (m' + \frac{1}{2})\hat{\pi} + \frac{1}{2}\varepsilon < \theta_2(t) < (m' + \frac{3}{2})\hat{\pi} - \frac{1}{2}\varepsilon,$$

or

$$(m' + \frac{1}{2})\hat{\pi} + \frac{1}{2}\varepsilon < \theta_1(t) < (m' + \frac{3}{2})\hat{\pi} - \frac{1}{2}\varepsilon$$

is true. This implies by (2.11), (2.13) that

$$g(\theta_1(t)) + g(\theta_2(t)) > g(\frac{1}{2}\hat{\pi} - \frac{1}{2}\varepsilon) \tag{4.5}$$

in these intervals, and so, by (4.1), that

$$\sum_{m=1}^{\infty} \{\log q(t_{2m}) - \log q(t_{1m})] < \infty. \tag{4.6}$$

We now use (2.8), which shows that

$$\psi'(t, \varphi_1, \varphi_2) = \frac{q'(t)}{q(t)} \{f(\theta_2(t)) - f(\theta_1(t))\}.$$

By (4.6), we thus have

$$\sum_{m=1}^{\infty} \{\psi(t_{2m}, \varphi_1, \varphi_2) - \psi(t_{1m}, \varphi_1, \varphi_2)\} < \infty. \tag{4.7}$$

Now (4.7) contradicts (4.2), and so we conclude that $\psi(\infty, \varphi_1, \varphi_2)$ exists.

Suppose next that $\psi(\infty, \varphi_1, \varphi_2) = \gamma \in (0, \hat{\pi})$, and let ε be such that $0 < \varepsilon < \gamma < \hat{\pi} - \varepsilon$. Then for sufficiently large t we have (4.4) then also (4.5), which gives a contradiction with (4.1). This completes the proof of Lemma 1.

5. Proof of Lemma 2.

We may suppose δ is suitable small, and will assume that $\delta < \hat{\pi}/8$, and also that δ is such that

$$f'(\theta) < 0 \quad \text{if } |\theta - \frac{\hat{\pi}}{2}| \leq 2\delta. \tag{5.1}$$

Here we remark that, by (2.12), $f'(\frac{\hat{\pi}}{2}) = -\frac{n}{n+1}$, so that indeed $f'(\theta) < 0$ in some neighbourhood of $\frac{\hat{\pi}}{2}$. It follows that also

$$f'(\theta) < 0 \quad \text{if } |\theta - (m + \frac{1}{2})\hat{\pi}| \leq 2\delta \tag{5.2}$$

for any integer m. We take first the case of φ satisfying

$$\varphi_0 < \varphi < \varphi_0 + \eta, \tag{5.3}$$

where η is about to be specified. For brevity write $\theta_0(t)$ instead of $\theta(t, \varphi_0)$. We choose T so large that

$$\int_T^\infty \frac{q'(t)}{q(t)} g(\theta_0(t))\, dt < \frac{1}{4}\delta\, g(\frac{1}{2}\hat{\pi} - \delta). \tag{5.4}$$

Relaying on continuous dependence on initial data, we then choose $\eta > 0$ so that $\psi(t, \varphi_0, \varphi) < \delta$ for $0 \le t \le T$ and, in addition, $\psi(T, \varphi_0, \varphi) < \delta/4$ holds if φ satisfies (5.3). Now fix a value φ. Let T' be defined as

$$T' = \sup\{t \mid \psi(\tau, \varphi_0, \varphi) < \delta,\ T < \tau < t\}, \tag{5.5}$$

and we need to show that $T' = \infty$.

We denote by I_1 the subset of $[T, T']$ such that for all integer m

$$|\theta_0(t) - (m + \frac{1}{2})\hat{\pi}| \ge \delta, \tag{5.6}$$

and by I_2 the complementary subset such that for some integer m,

$$|\theta_0(t) - (m + \frac{1}{2})\hat{\pi}| < \delta. \tag{5.7}$$

On the set I_1 we have

$$g(\theta_0(t)) \ge g(\frac{1}{2}\hat{\pi} - \delta),$$

and so, by (5.4),

$$\int_{I_1} \frac{q'(t)}{q(t)}\, dt < \frac{1}{4}\delta.$$

Since $|\psi'| \le 2\frac{q'}{q} \sup|f| = \frac{2}{n+1}\frac{q'}{q}$, we have

$$\int_{I_1} |\psi'(t, \varphi_0, \varphi)|\, dt \le \frac{\delta}{2(n+1)}. \tag{5.8}$$

In the set I_2 for each t and some integer m, by (5.5) and (5.7),

$$(m + \frac{1}{2})\hat{\pi} - 2\delta < \theta(t, \varphi_0) < \theta(t, \varphi) \le (m + \frac{1}{2})\hat{\pi} + 2\delta. \tag{5.9}$$

By (5.2) we then have $f(\theta(t, \varphi)) < f(\theta_0(t))$ in I_2, and so

$$\int_{I_2} \psi'(t, \varphi_0, \varphi)\, dt \le 0.$$

Hence

$$\int_T^{T'} \psi'(t, \varphi_0, \varphi)\, dt \le \frac{\delta}{2(n+1)} < \frac{\delta}{2}. \tag{5.10}$$

By (5.5), we thus have $T' = \infty$. This completes the proof in the case (5.3).

The proof is very similar in the case $\varphi_0 - \eta < \varphi < \varphi_0$. In place of (5.5) we define

$$T' = \sup\{t \mid \psi(\tau, \varphi_0, \varphi) > -\delta,\ T < \tau < t\}. \tag{5.11}$$

With the same definitions of I_1, I_2, (5.8) remains in force, while in (5.9) and (5.10) the middle inequality is reversed. In place of (5.10) we get

$$\int_T^{T'} \psi'(t, \varphi_0, \varphi)\, dt \geq -\frac{\delta}{2(n+1)} > -\frac{\delta}{2}.$$

The proof is then completed as before. This also completes the proof of Theorem 1.

6. Distribution of initial data for small solutions.

In the linear case ($n = 1$), Theorem 1 can be made more precise: *either* all solutions are small, *or else* there is just one linearly independent small solution. If we topologize the set of real solutions $y(t)$ by means of their initial data $y(0), y'(0)$, then the set of non–trivial small solutions has just two connected components. This last statement extends to the general case.

Formulating it differently, we continue to keep the notations of $\mathcal{X}$ and $\mathcal{Y}$ as in Section 3, i.e. we denote by $\mathcal{X}$ the set of $\varphi \in \mathbb{R}$ such that the corresponding solution $y(t)$ does not tend to zero, and denote by $\mathcal{Y}$ the complementary set. Thus, as we have just shown, $\mathcal{Y}$ is not empty, though $\mathcal{X}$ may be, in particular in cases of regular growth of $q(t)$. Disregarding such cases, we have

Theorem 2. *Let $\mathcal{X} \neq \emptyset$. Then there exist α, β, with $\alpha \leq \beta \leq \alpha + \hat{\pi}$, such that*

$$\mathcal{Y} = \bigcup_{m=-\infty}^{\infty} [\alpha + m\hat{\pi}, \beta + m\hat{\pi}], \tag{6.1}$$

$$\mathcal{X} = \bigcup_{m=-\infty}^{\infty} (\beta + m\hat{\pi}, \alpha + (m+1)\hat{\pi}), \tag{6.2}$$

where m runs through all the integral values.

In particular, $\mathcal{X}$ is open. In the case $n = 1$, at least, we have $\alpha = \beta$, so that $\mathcal{Y}$ is a periodic set of isolated points. Whether this is true in general is not clear.

For the proof we need two lemmas. The first is a development of Lemma 2, the second is a very simple remark.

Lemma 3. *Let $\varphi_0 \in \mathcal{X}$. Then there is an $\eta > 0$ such that if $|\varphi - \varphi_0| < \eta$, then $\varphi \in \mathcal{X}$, and $\psi(\infty, \varphi_0, \varphi) = 0$.*

Lemma 4. *If $\varphi_1, \varphi_2 \in \mathcal{X}$, $\varphi_1 < \varphi_2$ and $\psi(\infty, \varphi_1, \varphi_2) = 0$, then the whole interval $[\varphi_1, \varphi_2]$ belongs to $\mathcal{X}$.*

7. Proof of Lemmas 3, 4.

We take a fixed δ as in Lemma 2. Determine T, η accordingly as in the proof of Lemma 2. Since now we know that $T' = \infty$, I_1 and I_2 will be complementary subsets of the half–axis $[T, \infty)$. Again we take as typical the case $\varphi_0 < \varphi < \varphi_0 + \eta$. We denote now by k a positive number such that

$$f'(u) < -k \quad \text{for } u \in [\frac{1}{2}\hat{\pi} - 2\delta, \frac{1}{2}\hat{\pi} + 2\delta]. \tag{7.1}$$

We now re–formulate slightly the upper bounds on ψ' found in Section 5; we abbreviate $\psi(t, \varphi_0, \varphi)$ to $\psi(t)$.

In I_1 we have

$$\psi' \leq 2\frac{q'}{q} \leq 2\frac{q'}{q}\,\frac{g(\theta_0)}{g(\frac{\hat{\pi}}{2} - \delta)}. \tag{7.2}$$

In I_2 we have

$$\psi' = \frac{q'}{q}\{f(\theta) - f(\theta_0)\} \leq -k\frac{q'}{q}\psi. \tag{7.3}$$

We combine these in the form

$$\psi' \leq C\frac{q'}{q}g(\theta_0) - k\frac{q'}{q}\psi, \tag{7.4}$$

valid in (T, ∞), for suitable $C > 0$. In the case of (7.3) any such C will do. For (7.2) to be included, it will be sufficient that

$$Cg(\theta_0) \geq 2\frac{g(\theta_0)}{g(\frac{\hat{\pi}}{2} - \delta)} + k\psi.$$

Here $0 < \psi < \delta$, and so $k\psi \leq k\delta g(\theta_0)/g(\hat{\pi}/2 - \delta)$. We may thus take

$$C = \frac{2 + k\delta}{g(\frac{\hat{\pi}}{2} - \delta)}. \tag{7.5}$$

The differential inequality (7.4) may be integrated over $[T, t]$, to yield

$$\psi(t) \leq \psi(T)\left[\frac{q(T)}{q(t)}\right]^k + C\int_T^t \frac{q'(s)}{q(s)}\left[\frac{q(s)}{q(t)}\right]^k g(\theta_0(s))\,ds. \tag{7.6}$$

The claim that $\psi(t) \to 0$ as $t \to \infty$ now follows from the facts that $q(t) \to \infty$ and that

$$\frac{q'(t)}{q(t)}g(\theta_0(t)) \in \mathcal{L}(T, \infty). \tag{7.7}$$

It remains to be proved that $\varphi \in \mathcal{X}$, i.e. by (2.15)

$$\frac{q'(t)}{q(t)}g(\theta(s)) \in \mathcal{L}(T, \infty). \tag{7.8}$$

Since $\theta - \theta_0 = \psi$ and g' is bounded, namely by (2.13) $|g'(\theta)| \le 1$, we see that it is sufficient to prove that

$$\frac{q'(t)}{q(t)}\psi(t) \in \mathcal{L}(T,\infty). \tag{7.9}$$

For this we use (7.6). As regards the first term on the right of (7.6) we have clearly $(q'/q)q^{-k} \in \mathcal{L}(T,\infty)$, since $k > 0$. It remains to show that

$$\int_T^\infty \frac{q'(t)}{q(t)} \int_T^t \frac{q'(s)}{q(s)} \left[\frac{q(s)}{q(t)}\right]^k g(\theta_0(s))\, dsdt < \infty. \tag{7.10}$$

On evaluating the t-integral this is seen to be equivalent to statement (7.7). This completes the proof of Lemma 3.

Passing to the proof of Lemma 4, we observ first that there is a constant K such that if $u < w < v < u + \frac{1}{2}\hat{\pi}$, then

$$g(w) < K[g(u) + g(v)]. \tag{7.11}$$

The result is true with $K = 1$ if $g(x)$ is monoton in (u, v). This disposes of cases in which (u, v) does not contain any point congruent to 0 or $\frac{\hat{\pi}}{2}$. We deal next with the latter case. We suppose for definiteness that $u < \frac{\hat{\pi}}{2} < v$. Then $g(x)$ is decreasing in $(u, \frac{\hat{\pi}}{2})$ and increasing in $(\frac{\hat{\pi}}{2}, v)$. Thus again (7.11) holds with $K = 1$.

Finally, suppose that 0 lies in (u, v). Then $g(x)$ increases to its maximum value of $\frac{1}{n+1}$ as x increases in $[u, 0]$, and is decreasing in $[0, v]$. Also, at least one of the inequalities $u \ge -\frac{\hat{\pi}}{4}$, $v \le \frac{\hat{\pi}}{4}$ is true. Hence in this case we have $g(u) + g(v) \ge g(\frac{\hat{\pi}}{4})$, $g(w) \le \frac{1}{n+1}$, so that a value of K exists for this case also.

We write as before $\theta_j(t)$ which stands for $\theta(t, \varphi_j)$, $j = 1, 2$. Denote by T_0 a number such that $\psi(t, \varphi_1, \varphi_2) < \frac{\hat{\pi}}{4}$ for $t \ge T_0$. Thus we have

$$\theta_1(t) < \theta_2(t) < \theta_1(t) + \frac{\hat{\pi}}{4}, \qquad t \ge T_0, \tag{7.12}$$

For any $\varphi \in (\varphi_1, \varphi_2)$ we also have $\theta_1(t) < \theta(t, \varphi) < \theta_2(t)$ and so, by (7.11),

$$g(\theta(t, \varphi)) < K\left[g(\theta_1(t)) + g(\theta_2(t))\right], \quad t \ge T_0.$$

We may now appeal to (4.1), which shows that (2.15) holds in this case. This proves Lemma 4.

8. Extensions.

1) As in a number of stability criteria, $q(t)$ may be of limited decrease rather than non–decreasing, in the sense that

$$\max\{-\frac{q'}{q}, 0\} \in \mathcal{L}(0,\infty),$$

while still $q(t) \to \infty$.

2) Th function $q(t)$ can be replaced by $q(t) + r(t)$, where $q(t)$ is as before, and $r(t)$ satisfies some smallness or integral condition, without necessarily being smooth. This permits extensions to at least some discontinuous cases.

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