Exponential Stability for Singularly Perturbed Systems with State Delays

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Abstract

In this paper the problem of stability of the zero solution of singularly perturbed system of linear differential equation with state delays is investigated.

We show that if the zero solution of reduced subsystem and the one of the fast subsystem are exponentially stable, then the zero solution of the given singularly perturbed system of differential equations is also exponentially stable.

Estimates of the block components of the fundamental matrix solution are derived. These estimates are used to obtain asymptotic expansions on unbounded interval for the solutions of this class of singularly perturbed systems.

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[1]This paper is in the final form and no version of it will be submitted for publication elsewhere.

1 Problem formulation

Consider the singularly perturbed system of functional differential equations

$$\begin{aligned} \dot{x}(t) &= L_{11}x_t + L_{12}y_t \\ \varepsilon\dot{y}(t) &= L_{21}x_t + L_{22}y_t \end{aligned} \tag{1.1}$$

where $x \in \mathbf{R}^{n_1}, y \in \mathbf{R}^{n_2}, \varepsilon > 0$ is a small parameter;

$$L_{j1}x_t = \sum_{i=0}^{p} A_{j1}^i x(t-\tau_i) + \int_{-\tau_p}^{0} D_{j1}(s)x(t+s)ds$$

$$L_{j2}y_t = \sum_{i=0}^{m} A_{j2}^i y(t-\varepsilon\mu_i) + \int_{-\mu_m}^{0} D_{j2}(s)y(t+\varepsilon s)ds$$

$j = 1, 2, A_{jk}^i$ are constant matrices with apropriate dimensions $D_{jk}(\cdot)$ are integral matrix valued functions and $0 = \tau_0 < \tau_1 < ... < \tau_p, 0 = \mu_0 < \mu_1 < ... < \mu_m$.

Setting $L_\varepsilon = \begin{pmatrix} L_{11} & L_{12} \\ \frac{1}{\varepsilon}L_{21} & \frac{1}{\varepsilon}L_{22} \end{pmatrix}; z = \begin{pmatrix} x \\ y \end{pmatrix}$ the system (1.1) may be written in a compact form as:

$$\dot{z}(t) = L_\varepsilon z_t.$$

It is known (see [3], [4]) that the exponential stability of the zero solution of the system (1.1) is equivalent with the fact that the roots of the equation

$$det(\lambda I - L_\varepsilon(e^{\lambda \cdot} I)) = 0$$

are located in the half plane $Re(\lambda) < 0$.

Usually this condition is difficult to be check when the system is of high dimension.

The goal of this paper is to provide some sufficient conditions assuring the exponential stability for the system of type (1.1).

Such conditions are expressed in term of exponential stability of the zero solution of some subsystems of lower dimensions not depending upon small parameter ε.

Taking $\varepsilon = 0$ in (1.1) we obtain:

$$\begin{aligned} \dot{x}(t) &= L_{11}x_t + \tilde{L}_{12}y(t) \\ 0 &= L_{21}x_t + \tilde{L}_{22}y(t) \end{aligned} \tag{1.2}$$

where

$$\tilde{L}_{j2} = \sum_{i=0}^{m} A_{j2}^i + \int_{-\mu_m}^{0} D_{j2}(s)ds, j = 1, 2.$$

Assuming that the matrix $\tilde{L}_{22}$ is invertible, we may associate the following "reduced subsystem" :

$$\dot{x}(t) = L_r x_t \tag{1.3}$$

with

$$L_r x_t = \sum_{i=0}^{p} [A_r^i x(t-\tau_i) + \int_{-\tau_p}^{0} D_r(s) x(t+s) ds] \tag{1.4}$$

$A_r^i = A_{11}^i - \tilde{L}_{12}\tilde{L}_{22}^{-1} A_{21}^i, i = 0, 1, ..., p, \quad D_r(s) = D_{11}(s) - \tilde{L}_{12}\tilde{L}_{22}^{-1} D_{21}(s).$

Sometimes (1.3) will be called "slow subsystem" associated to (1.1).

Also, we associate the so called "fast subsystem" or "boundary layer subsystem":

$$y'(\sigma) = L_f y_\sigma \tag{1.5}$$

with

$$L_f y_\sigma = \sum_{i=0}^{m} A_{22}^i y(\sigma - \mu_i) + \int_{-\mu_m}^{0} D_{22}(s) y(\sigma + s) ds$$

where $\sigma = \frac{t}{\varepsilon}$.

In this paper we show that the exponential stability of the zero solution of the slow subsystem (1.3) and the one of the fast subsystem (1.5) implies the exponential stability of the zero solution of the system (1.1) for arbitrary $\varepsilon > 0$ small enough.

Our result extends to the class of systems of functional differential equations with state delays of type (1.1) the well known result of Klimusev-Krasovski [6].

Systems of differential equations of type (1.1) with $m = 0, p = 1$ and $D_{jk}(s) = 0$ were considered in [5], where asymptotic structure of solutions was studied.

In [2] the system (1.1) with $p = 0$ was considered and the separation of time scales was proposed.

2 The main result

We make the following assumptions:

H_1) The roots of the equation

$$det[\lambda I_{n_2} - \sum_{i=0}^{m} A_{22}^i e^{-\lambda \mu_i} - \int_{-\mu_m}^{0} D_{22}(s) e^{\lambda s} ds] = 0 \tag{2.1}$$

are located in the half plane $Re(\lambda) < -\gamma_f < 0$ for some positive constant γ_f.

$H_2)$ The roots of the equation

$$det\left[\lambda I_{n_1} - \sum_{i=0}^{p} A_r^i e^{-\lambda\tau_i} - \int_{-\tau_p}^{0} D_r(s)D_r(s)e^{\lambda s}ds\right] = 0 \tag{2.2}$$

are located in the half plane $Re(\lambda) \leq -\gamma_r < 0$ for some positive constant γ_r.

Remark:

a) Under the assumption $H_1)$ it follows that $\lambda = 0$ is not a root of the equation (2.1) and hence the matrix $\tilde{L}_{22}$ is invertible.

Thus the matrices $A_r^i, i = 0, 1, ..., p$ and $D_r(s)$ are well defined.

b) If $0 < \alpha_f < \gamma_f$ there exists $\beta_f \geq 1$ such that the solutions of the system (1.5) satisfy:

$$|y(\sigma)| \leq \beta_f e^{-\alpha_f\sigma}||y_0|| \tag{2.3}$$

for all $\sigma \geq 0$; $||y_0|| = \sup_{s\in[-\mu_m,0]} |y(s)|$.

c) If $0 < \alpha_r < \gamma_r$ there exists $\beta_r \geq 1$ such that for any solutions of the system (1.3) we have

$$|x(t)| \leq \beta_r e^{-\alpha_r t}||x_0||, (\forall)\ t \geq 0. \tag{2.4}$$

The main result of this paper is:

Theorem 2.1 *Under the assumptions H_1), H_2) there exists $\varepsilon_0 > 0$ such that for arbitrary $\varepsilon \in (0, \varepsilon_0)$ the zero solution of the system (1.1) is exponentially stable.*

Moreover, if $\begin{pmatrix} \Phi_{11}(t,\varepsilon) & \Phi_{12}(t,\varepsilon) \\ \Phi_{21}(t,\varepsilon) & \Phi_{22}(t,\varepsilon) \end{pmatrix}$ *is the partition of the fundamental matrix solution, we have*

$$
\begin{aligned}
|\Phi_{11}(t,\varepsilon)| &\leq \beta_1 e^{-\alpha_1 t}\\
|\Phi_{12}(t,\varepsilon)| &\leq \varepsilon\beta_1 e^{-\alpha_1 (t)}\\
|\Phi_{21}(t,\varepsilon)| &\leq \beta_2 e^{-\alpha_1 t}\\
|\Phi_{22}(t,\varepsilon)| &\leq \beta_2 (e^{-\alpha_2 \frac{t}{\varepsilon}} + \varepsilon e^{-\alpha_1 t}), \quad (\forall) t \geq 0, \varepsilon \in (0,\varepsilon_0)
\end{aligned}
$$

$\alpha_j, \beta_j, j = 1,2$ *are positive constants which depend by* $\alpha_f, \alpha_r, \beta_f, \beta_r$, *respectively.*

Proof:

Let $\Phi_r(t)$ be the fundamental matrix solution of the reduced system (1.3) and $\Phi_f(\sigma)$ be the fundamental matrix solution of the fast system (1.5).

It is easy to see that $t \to \Phi_f(\frac{t}{\varepsilon}$ is the fundamental matrix solution of the system

$$\varepsilon \dot{y}(t) = L_{22} y_t, t \geq 0.$$

Let $\begin{pmatrix} x(t,\varepsilon) \\ y(t,\varepsilon) \end{pmatrix}$ be a solution of the system (1.1) with the initial conditions:

$$
\begin{aligned}
x(s,\varepsilon) &= 0, -\tau_p \leq s < 0\\
y(s,\varepsilon) &= 0, -\varepsilon\mu_m \leq s < 0.
\end{aligned}
$$

Using the variation of constants formula we obtain:

$$y(t,\varepsilon) = \Phi_f(\frac{t}{\varepsilon}) y(0) + \frac{1}{\varepsilon}\int_0^t \Phi_f(\frac{t-s}{\varepsilon}) L_{21} x_s ds \tag{2.5}$$

$$x(t,\varepsilon) = \Phi_r(t) x(0) + \int_0^t \Phi_r(t-s) L_{12} y_s ds + \int_0^t \Phi_r(t-s)\tilde{L}_{12}\tilde{L}_{22}^{-1} L_{21} x_s ds \tag{2.6}$$

Further we write:

$$
\begin{aligned}
\int_0^t \Phi_r(t-s) L_{12} y_s ds &= \sum_{i=0}^m \int_0^t \Phi_r(t-s) A_{12}^i y(s-\varepsilon\mu_i,\varepsilon) ds\\
&+ \int_0^t \Phi_r(t-s)\int_{-\mu_m}^0 D_{12}(\theta) y(s+\varepsilon\theta,\varepsilon) d\theta ds\\
&= \sum_{i=0}^m \int_{\varepsilon\mu_i}^t \Phi_r(t-s) A_{12}^i y(s-\varepsilon\mu_i,\varepsilon) ds\\
&+ \int_{-\mu_m}^0 \int_\theta^t \Phi_r(t-s) D_{12}(\theta) y(s+\varepsilon\theta,\varepsilon) ds d\theta \quad (2.7)\\
&= \sum_{i=0}^m \int_0^t \Phi_r(t-s-\varepsilon\mu_i) A_{12}^i y(s,\varepsilon) ds\\
&+ \int_0^t \int_{-\mu_m}^0 \Phi_r(t-s+\theta) D_{12}(\theta) d\theta y(s,\varepsilon) ds.
\end{aligned}
$$

Substituting (2.5) in (2.7) we get:

$$\sum_{i=0}^{m}\int_0^t \Phi_r(t-s-\varepsilon\mu_i)A_{12}^i y(s,\varepsilon)ds$$
$$=\sum_{i=0}^{m}\int_0^t \Phi_r(t-s-\varepsilon\mu_i)A_{12}^i \Phi_f(\frac{s}{\varepsilon})y(0)ds$$
$$+\frac{1}{\varepsilon}\sum_{i=0}^{m}\int_0^t \Phi_r(t-s-\varepsilon\mu_i)A_{12}^i \int_0^s \Phi_f(\frac{s-\sigma}{\varepsilon})L_{21}y_\sigma d\sigma ds. \tag{2.8}$$

Applying Fubini theorem we have:

$$\frac{1}{\varepsilon}\int_0^t \Phi_r(t-s-\varepsilon\mu_i)A_{12}^i \int_0^s \Phi_f(\frac{s-\sigma}{\varepsilon})L_{21}x_\sigma d\sigma ds$$
$$=\frac{1}{\varepsilon}\int_0^t\int_s^t \Phi_r(t-\sigma-\varepsilon\mu_i)A_{12}^i\Phi_f(\frac{\sigma-s}{\varepsilon})d\sigma L_{21}x_s ds. \tag{2.9}$$

On the other hand:

$$\frac{1}{\varepsilon}\int_s^t \Phi_r(t-\sigma-\varepsilon\mu_i)A_{12}^i\Phi_f(\frac{\sigma-s}{\varepsilon})d\sigma$$
$$=\int_0^{\frac{t-s}{\varepsilon}} \Phi_r(t-s-\varepsilon\sigma-\varepsilon\mu_i)A_{12}^i\Phi_f(\sigma)d\sigma$$
$$=\Phi_r(t-s)A_{12}^i\int_0^\infty \Phi_f(\sigma)+\Psi_1(t,s,\varepsilon)$$

where

$$\begin{aligned}\Psi_1(t,s,\varepsilon) &= \int_0^\infty [\Phi_r(t-s-\varepsilon\sigma-\varepsilon\mu_i)-\Phi_r(t-s)]A_{12}^i\Phi_f(\sigma)d\sigma \\ &- \int_{\frac{t-s}{\varepsilon}}^\infty \Phi_r(t-s-\varepsilon\sigma-\varepsilon\mu_i)A_{12}^i\Phi_f(\sigma)d\sigma.\end{aligned}$$

Based on (2.3) and (2.4) we deduce:

$$|\Psi_1(t,s,\varepsilon)|\le\tilde{\beta}_1[e^{-\alpha_2\frac{t-s}{\varepsilon}}+\varepsilon e^{-\alpha_1(t-s)}]$$

$\alpha_j>0, \tilde{\beta}_1>0, j=1,2.$

It is easy to check that

$$\int_0^\infty \Phi_f(\sigma)d\sigma=-\tilde{L}_{22}^{-1}.$$

Finally we obtain

$$\begin{aligned}\int_0^t \Phi_r(t-s)L_{12}y_s ds &= -\int_0^t \Phi_r(t-s)\tilde{L}_{12}\tilde{L}_{22}^{-1}L_{21}x_s ds \\ &+ \int_0^t \Psi_0(t,s,\varepsilon)dsy(0)+\int_0^t \Psi_2(t,s,\varepsilon)L_{21}x_s ds\end{aligned} \tag{2.10}$$

where

$$\begin{aligned}
|\Psi_0(t,s,\varepsilon)| &\leq \tilde{\beta}_0 e^{-\alpha_1(t-s)} e^{-\alpha_2 \frac{s}{\varepsilon}} \\
|\Psi_2(t,s,\varepsilon)| &\leq \tilde{\beta}_2 (e^{\alpha_2 \frac{(t-s)}{\varepsilon}} + \varepsilon e^{-\alpha_1(t-s)}).
\end{aligned}$$

Substituting (2.10) into (2.6) we get

$$x(t,\varepsilon) = \Phi_r(t)x(0) + \int_0^t \Psi_0(t,s,\varepsilon)ds y(0) + \int_0^t \Psi_2(t,s,\varepsilon)L_{21}x_s ds.$$

Directly

$$|x(t,\varepsilon)| \leq \tilde{\beta}_3 e^{-\alpha_1 t}(|x(0)| + \varepsilon|y(0)|) + \tilde{\beta}_4 \int_0^t (e^{-\alpha_2 \frac{t-s}{\varepsilon}} + \varepsilon e^{-\alpha_1(t-s)})|L_{21}x_s| ds.$$

By standard techniques in singular perturbation theory [8], [1] we obtain that there exists $\varepsilon_0 > 0$ such that for arbitrary $0 < \varepsilon < \varepsilon_0$ we have

$$|x(t,\varepsilon)| \leq \beta_1 e^{-\alpha_1 t}(|x(0)| + \varepsilon|y(0)|). \tag{2.11}$$

Using (2.11) in (2.5) we deduce

$$|y(t,\varepsilon) \leq \beta_2 e^{-\alpha_2 \frac{t}{\varepsilon}}|y(0)| + \beta_3 e^{-\alpha_1 t}(|x(0)| + \varepsilon|y(0)|). \tag{2.12}$$

From (2.11) and (2.12) we conclude that the zero solution of the system (1.1) is exponentially stable. The estimates of the block components of the fundamental matrix solution follows from (2.11) and (2.12) taking $x(0) = I_{n_1}, y(0) = 0$ and $x(0) = 0, y(0) = I_{n_2}$, respectively. Thus the proof is complete.

3 Asymptotic expansions

Let us consider system (1.1) with $D_{jk}(\theta) = 0$ for all $\theta < 0$.

Based on the result of Theorem 2.1 we have:

Theorem 3.1 *Under the assumptions $H_1 - H_2$ the block components of the fundamental matrix solutions of the system (1.1) have the following asymptotic structure:*

$$\begin{aligned}
\Phi_{11}(t,\varepsilon) &= \Phi_r(t) + \varepsilon\hat{\Phi}_{11}(t,\varepsilon) \\
\Phi_{12}(t,\varepsilon) &= -\varepsilon\Phi_r(t)\tilde{L}_{12}\tilde{L}_{22}^{-1} + \varepsilon\tilde{\Phi}_{12}(\frac{t}{\varepsilon},\varepsilon) + \varepsilon^2\hat{\Phi}_{12}(t,\varepsilon) \\
\Phi_{21}(t,\varepsilon) &= -\tilde{L}_{22}^{-1}L_{21}(\Phi_r)_t + \tilde{\Phi}_{21}(\frac{t}{\varepsilon},\varepsilon) + \varepsilon\hat{\Phi}_{21}(t,\varepsilon) \\
\Phi_{22}(t,\varepsilon) &= \Phi_f(\frac{t}{\varepsilon}) + \varepsilon\hat{\Phi}_{22}(t,\varepsilon)
\end{aligned}$$

where

$$\begin{aligned} |\hat{\Phi}_{jk}(t,\varepsilon)| &\le \hat{\beta}e^{-\hat{\alpha}t}, \quad \forall t \ge 0 \\ |\tilde{\Phi}_{jl}(\sigma,\varepsilon)| &\le \tilde{\beta}e^{-\tilde{\alpha}\sigma} \quad \forall \sigma \ge 0, \end{aligned}$$

$\hat{\alpha}, \tilde{\alpha}, \hat{\beta}, \tilde{\beta}$ *being positive constants not depending upon* ε, t, σ.

The asymptotic formulae in Theorem 3.1 allow us to obtain the asymptotic structure of the solutions of a singularly perturbed affine system of functional differential equations:

$$\begin{aligned} \dot{x}(t) &= L_{11}x_t + L_{12}y_t + f(t) \\ \varepsilon\dot{y}(t) &= L_{21}x_t + L_{22}y_t + g(t) \quad t \ge 0 \end{aligned}$$

where $f(\cdot), g(\cdot)$ are integrable functions.

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