ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF SOME DIFFERENTIAL EQUATIONS WITH AN UNBOUNDED DELAY

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Abstract. We investigate the asymptotic properties of all solutions of the functional differential equation

$$\dot{x}(t) = p(t)[x(t) - kx(t - \tau(t))] + q(t), \qquad t \in I = [t_0, \infty),$$

where $k \neq 0$ is a scalar and $\tau(t)$ is an unbounded delay. Under certain restrictions we relate the asymptotic behaviour of the solutions $x(t)$ of this equation to the behaviour of a solution $\varphi(t)$ of the auxiliary functional nondifferential equation

$$\varphi(t) = |k|\, \varphi(t - \tau(t)), \qquad t \in I.$$

1. Introduction

We discuss the asymptotic properties of solutions of the functional differential equation (FDE)

$$\dot{x}(t) = p(t)[x(t) - kx(t - \tau(t))] + q(t), \qquad t \in I = [t_0, \infty), \tag{1.1}$$

where $k \neq 0$ is a real constant, $p(t) \in C^0(I)$, $p(t) > 0$, $p(t)$ is nondecreasing on I, $\tau(t) \in C^1(I)$, $\tau(t) > 0$, $0 < \delta \le \dot{\tau}(t) < 1$ on I and $q(t) \in C^0(I)$, $q(t) = O\left(p(t)\exp\{\int_{t_0}^{t-\tau(t)} p(s)\,\mathrm{d}s\}\right)$ as $t \to \infty$. We note that our method is applicable also for delays $\tau(t)$ vanishing at the initial point t_0.

Such equations form a wide and natural class of FDEs involving especially equations with constant coefficients.

Example 1. The equation

$$\dot{x}(t) = a\,x(\lambda t) + b\,x(t) + q(t) \tag{1.2}$$

is a nonhomogeneous pantograph equation. The corresponding homogeneous equation arises in [11] as a mathematical model of the motion of a pantograph head on the electric locomotive. Equation (1.2) and its modifications has been studied, e.g., by A. Iserles [7], E. B. Lim [9], [10], L. Pandolfi [12] and in [2].

1991 *Mathematics Subject Classification.* 34K15, 34K25; Secondary 39B99.

Key words and phrases. Functional differential equation, functional nondifferential equation, asymptotic behaviour.

This paper is in final form and no version of it will be submitted for publication elsewhere. The research was supported by the grant # A101/99/02 of the Grant Agency of the Academy of Sciences of the Czech Republic.

Example 2. A more general equation is the differential equation with the linearly transformed argument of the form

$$\dot{x}(t) = a\,x(\lambda t + \mu) + b\,x(t) + q(t). \tag{1.3}$$

Equations of this type have been investigated, e.g., by G. Derfel [4] and have diverse applications in areas ranging from number theory to astrophysics.

Example 3. The equation

$$\dot{x}(t) = a\,x(t^{\gamma}) + b\,x(t) + q(t) \tag{1.4}$$

has been studied in the homogeneous case by M. L. Heard [6].

Asymptotic properties of solutions of homogeneous equations with constant coefficients and unbounded delays have been investigated also in [3]. Our aim is to unify and generalize some results concerning (1.2), (1.3), (1.4) and other related equations.

Example 4. Many authors paid attention to the study of the equation of the form

$$\dot{x}(t) = p(t)[x(t) - x(t - \tau(t))] + q(t). \tag{1.5}$$

This equation is usually studied under the assumption of the boundedness of the delay $\tau(t)$ (for results and references see, e.g., F. V. Atkinson, J. R. Haddock [1] or J. Diblík [5]). We wish to extend some of these results to equation (1.5) with an unbounded $\tau(t)$.

2. Preliminaries

By a solution $x(t)$ of (1.1) we mean a real valued continuous function $x(t)$ defined in some interval $[\sigma - \tau(\sigma), \infty)$, $\sigma \geq t_0$ and satisfying (1.1) for all $t \in [\sigma, \infty)$. The existence of a solution $x(t)$ of (1.1) defined in $[\sigma - \tau(\sigma), \infty)$ can be shown in the usual way by the step method.

In the next section we shall use the following results.

Proposition 1 [13, Corollary 1]. *Consider the FDE*

$$\dot{x}(t) = f(t, x(t - \tau(t))), \qquad t \in I, \tag{2.1}$$

where $\tau(t) \in C^0(I)$, $\tau(t) \geq 0$ on I and $\inf\{t - \tau(t) : t \in I\} > -\infty$. Further, let $f(t,x)$ be a continuous function for which there exist a constant C and a continuous function $r(t)$ such that

$$|f(t, x_1) - f(t, x_2)| \leq r(t)|x_1 - x_2| \tag{2.2}$$

and

$$|f(t, 0)| \leq Cr(t) \tag{2.3}$$

for any $t \in I$ and $x_1,\, x_2 \in \mathbb{R}$. If

$$\int_{t_0}^{\infty} r(t)\,\mathrm{d}t < \infty, \tag{2.4}$$

then for every solution $x(t)$ of (2.1) there exists $L \in \mathbb{R}$ such that

$$x(t) \to L \qquad \text{as } t \to \infty.$$

Moreover, if

$$\int_{t_0}^{\infty} r(t)\,\mathrm{d}t < 1, \tag{2.5}$$

then for every $L \in \mathbb{R}$ there exists a solution $x_L(t)$ of (2.1) defined on $[t_0, \infty)$ such that

$$x_L(t) \to L \qquad \text{as } t \to \infty.$$

Proposition 2. *Consider the functional equation*

$$\varphi(t) = s\,\varphi(t - \tau(t)), \qquad t \in I, \tag{2.6}$$

where $s > 0$ is a real constant, $\tau(t) \in C^r(I)$, $r \ge 0$, $\tau(t) > 0$, $t - \tau(t)$ is increasing on I. Let $\varphi_0(t) \in C^r(I_0)$, where $I_0 = [t_0 - \tau(t_0), t_0]$ be a positive function satisfying

$$\varphi_0^{(j)}(t_0) = s\,(\varphi_0 \circ (\mathrm{id} - \tau))^{(j)}(t_0), \qquad j = 0, 1, \ldots, r.$$

Then there exists a unique positive solution $\varphi(t) \in C^r(I)$ of (2.6) such that $\varphi(t) = \varphi_0(t)$ for every $t \in I_0$.

Proof. The statement can be proved by the step method (see also [8]). □

3. Asymptotic behaviour of solutions

We start off with the following

Lemma 1. *Consider equation (1.1), where $k \ne 0$, $p(t) \in C^0(I)$, $p(t) > 0$, $p(t)$ is nondecreasing on I, $\tau(t) \in C^1(I)$, $\tau(t) > 0$, $0 < \delta \le \dot{\tau}(t) < 1$ on I and $q(t) \in C^0(I)$, $q(t) = O\left(p(t)\exp\{\int_{t_0}^{t-\tau(t)} p(s)\,\mathrm{d}s\}\right)$ as $t \to \infty$. If $x(t)$ is a solution of (1.1), then there exists $L \in \mathbb{R}$ such that*

$$\exp\left\{-\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\} x(t) \to L \qquad \text{as } t \to \infty.$$

Conversely, for every $L \in \mathbb{R}$ there exists a solution $x_L(t)$ of (1.1) such that

$$\exp\left\{-\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\} x_L(t) \to L \qquad \text{as } t \to \infty. \tag{3.1}$$

Proof. We set $z(t) = \exp\{-\int_{t_0}^{t} p(s)\,\mathrm{d}s\}x(t)$ in (1.1) to obtain the equation

$$\dot{z}(t) = g(t)z(t - \tau(t)) + h(t), \qquad t \in I,$$

where $g(t) = -kp(t)\exp\{-\int_{t-\tau(t)}^{t} p(s)\,\mathrm{d}s\}$, $h(t) = q(t)\exp\{-\int_{t_0}^{t} p(s)\,\mathrm{d}s\}$. We verify the validity of the assumptions of Proposition 1 for this equation.

The inequality (2.2) is obviously fulfilled with $r(t) \equiv |g(t)|$ on I. Further, with the respect to the O-estimate of $q(t)$ we have

$$\begin{aligned}
&|f(t,0)| = |h(t)| = |q(t)|\exp\left\{-\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\} \le \\
&C|k|p(t)\exp\left\{\int_{t_0}^{t-\tau(t)} p(s)\,\mathrm{d}s\right\}\exp\left\{-\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\} \le \\
&C|k|p(t)\exp\left\{-\int_{t-\tau(t)}^{t} p(s)\,\mathrm{d}s\right\} = Cr(t),
\end{aligned}$$

which implies (2.3). We show the validity of (2.4).

$$\begin{aligned}
&\int_{t_0}^{\infty} r(t)\,\mathrm{d}t = |k|\int_{t_0}^{\infty} p(t)\exp\left\{-\int_{t-\tau(t)}^{t} p(s)\,\mathrm{d}s\right\}\mathrm{d}t = \\
&|k|\int_{t_0}^{\infty} \frac{p(t)}{-p(t)+p(t-\tau(t))(1-\dot\tau(t))}\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left\{-\int_{t-\tau(t)}^{t} p(s)\,\mathrm{d}s\right\}\right]\mathrm{d}t \le \\
&\frac{|k|}{\delta}\exp\left\{-\int_{t_0-\tau(t_0)}^{t_0} p(s)\,\mathrm{d}s\right\} < \infty.
\end{aligned}$$

Finally, considering $\sigma \ge t_0$ large enough it is possible to fulfil (2.5) with t_0 replaced by σ. The statement now follows from Proposition 1. $\square$

Lemma 2. *Consider equation*

$$\dot x(t) = p(t)[x(t) - kx(t-\tau(t))], \qquad t \in I, \tag{3.2}$$

where $k \ne 0$, $p(t) \in C^0(I)$, $p(t) > 0$, $p(t)$ is nondecreasing on I, $\tau(t) \in C^1(I)$, $\tau(t) > 0$, $0 < \delta \le \dot\tau(t) < 1$ on I. Assume that $\varphi(t) \in C^1(I)$ is a positive solution of the functional equation

$$\varphi(t) = |k|\,\varphi(t-\tau(t)), \qquad t \in I. \tag{3.3}$$

If $x(t)$ is a solution of (3.2) satisfying

$$\exp\left\{-\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\}x(t) \to 0 \qquad \text{as } t \to \infty,$$

then

$$x(t) = O(\varphi(t)) \qquad \text{as } t \to \infty.$$

Proof. First we note that the existence of a positive solution $\varphi(t)$ follows from Proposition 2. For the sake of simplicity we assume that $x(t)$ is defined on $[t_0, \infty)$. Multiplying both sides of equation (3.2) by $\exp\{-\int_{t_0}^{t} p(s)\,\mathrm{d}s\}$ we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp\left\{-\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\}x(t)\right] = -kp(t)\exp\left\{-\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\}x(t-\tau(t)).$$

Integrating this over $[t, \infty)$ we get

$$x(t) = k \exp\left\{\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\} \int_{t}^{\infty} p(u) \exp\left\{-\int_{t_0}^{u} p(s)\,\mathrm{d}s\right\} x(u-\tau(u))\,\mathrm{d}u.$$

Denote $\xi(t) \equiv t - \tau(t)$ on I and put $t_n = \xi^{-n}(t_0)$, n=0,1,2, ... , where the symbol $\xi^{-n}(t)$ means the n-th iterate of the inverse function $\xi^{-1}(t)$. Further, let $M > 0$ be such that

$$|x(t)| \le M \exp\left\{\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\}, \qquad t \ge t_0.$$

Then

$$\begin{aligned}
&|x(t)| \le M|k| \exp\left\{\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\} \int_{t}^{\infty} p(u) \exp\left\{-\int_{u-\tau(u)}^{u} p(s)\,\mathrm{d}s\right\} \mathrm{d}u = \\
&M|k| \exp\left\{\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\} \times \\
&\int_{t}^{\infty} \frac{p(u)}{-p(u) + p(u-\tau(u))(1-\dot{\tau}(u))} \frac{\mathrm{d}}{\mathrm{d}u}\left[\exp\left\{-\int_{u-\tau(u)}^{u} p(s)\,\mathrm{d}s\right\}\right] \mathrm{d}u \le \\
&M|k| \exp\left\{\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\} \frac{1}{\delta} \exp\left\{-\int_{t-\tau(t)}^{t} p(s)\,\mathrm{d}s\right\} = \\
&\frac{M|k|}{\delta} \exp\left\{\int_{t_0}^{t-\tau(t)} p(s)\,\mathrm{d}s\right\}, \qquad t \ge t_1.
\end{aligned}$$

Repeating this we have

$$|x(t)| \le \frac{M|k|^n}{\delta.(1-(1-\delta)^2)\dots(1-(1-\delta)^n)} \exp\left\{\int_{t_0}^{\xi^n(t)} p(s)\,\mathrm{d}s\right\}, \qquad t \ge t_n,$$

$n = 1, 2, \dots$. Here $\xi^n(t)$ means the n-th iterate of $\xi(t) \equiv t - \tau(t)$. Since

$$\exp\left\{\int_{t_0}^{\xi^n(t)} p(s)\,\mathrm{d}s\right\} \le \exp\left\{\int_{t_0}^{t_1} p(s)\,\mathrm{d}s\right\}, \qquad t \le t_{n+1},$$

$n = 1, 2, \dots,$ we can deduce that

$$|x(t)| \le M_1 |k|^n, \qquad t_n \le t \le t_{n+1}, \tag{3.4}$$

where

$$M_1 = \frac{M}{\prod_{j=1}^{\infty}(1-(1-\delta)^j)} \exp\left\{\int_{t_0}^{t_1} p(s)\,\mathrm{d}s\right\}.$$

On the other hand,

$$|\varphi(t)| \ge M_2 |k|^n, \qquad t_n \le t \le t_{n+1}, \tag{3.5}$$

where $M_2 > 0$ is as suitable constant. The statement now follows from (3.4) and (3.5). $\square$

Remark. It is easy to check that functional equation (3.3) occuring in Lemma 2 can be replaced by the functional inequality

$$\varphi(t) \geq |k|\, \varphi(t-\tau(t)), \qquad t \in I.$$

This replacement can be used, e.g., if we are not able to solve equation (3.3) explicitly.

Summarizing these results we can derive an asymptotic formula for all solutions of (1.1). Under the assumptions of previous statements we can, for any $L \in \mathbb{R}$, denote by $x_L(t)$ a particular solution of (1.1) possessing the property (3.1) and by $\varphi(t)$ a positive solution of (3.3).

Theorem. *Consider equation (1.1), where $k \neq 0$, $p(t) \in C^0(I)$, $p(t) > 0$, $p(t)$ is nondecreasing on I, $\tau(t) \in C^1(I)$, $\tau(t) > 0$, $0 < \delta \leq \dot{\tau}(t) < 1$ on I and $q(t) \in C^0(I)$, $q(t) = O\left(p(t) \exp\{\int_{t_0}^{t-\tau(t)} p(s)\,\mathrm{d}s\}\right)$ as $t \to \infty$. Then for every solution $x(t)$ of (1.1) there exists $L \in \mathbb{R}$ such that*

$$x(t) = x_L(t) + O(\varphi(t)) \qquad \text{as } t \to \infty. \tag{3.6}$$

Proof. Let $x(t)$ be a solution of (1.1). By Lemma 1, the limit

$$L = \lim_{t\to\infty} \exp\left\{-\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\} x(t)$$

is finite. Then $y(t) = x(t) - x_L(t)$ is a solution of (3.2) with the property

$$\exp\left\{-\int_{t_0}^{t} p(s)\,\mathrm{d}s\right\} y(t) \to 0 \qquad \text{as } t \to \infty.$$

Consequently, by Lemma 2, $y(t) = O(\varphi(t))$ as $t \to \infty$. $\square$

4. Applications

In this section we apply the presented results to equations (1.2) - (1.5). As it was remarked above, the functional equation (3.3) can be solved by the step method. Nevertheless, for special types of delays occuring in (1.2) - (1.4) we can give an explicit form of the required solution $\varphi(t)$ of (3.3).

Example 1. Consider equation (1.2), where $t \in [0,\infty)$, $a \neq 0$, $b > 0$, $0 < \lambda < 1$ and $q(t) \in C^0([0,\infty))$, $q(t) = O(\exp\{b\lambda t\})$ as $t \to \infty$. In accordance with Lemma 1 we can, for any $L \in \mathbb{R}$, denote by $x_L(t)$ a particular solution of (1.2) fulfilling the relation

$$\exp\{-bt\}x_L(t) \to L \qquad \text{as } t \to \infty.$$

Further, it is easy to verify that equation (3.3) becomes

$$|a|\, \varphi(\lambda t) = b\, \varphi(t), \qquad t \in [0,\infty)$$

and admits the solution

$$\varphi(t) = t^{\alpha}, \quad \alpha = \frac{\log \frac{b}{|a|}}{\log \lambda}.$$

Let $x(t)$ be a solution of (1.2). By Theorem, there exists $L \in \mathbb{R}$ such that

$$x(t) = x_L(t) + O(t^{\alpha}) \qquad \text{as } t \to \infty.$$

We note that this formula extends the asymptotic results derived by E. B. Lim in [10] for equation (1.2).

The following two examples can be dealt with quite similarly. Therefore we state only the required solution of the auxiliary equation (3.3).

Example 2. Consider equation (1.3), where $t \in [0,\infty)$, $a \neq 0$, $b > 0$, $0 < \lambda < 1$, $\mu < 0$ and $q(t) \in C^0([0,\infty))$, $q(t) = O(\exp\{b(\lambda t + \mu)\})$ as $t \to \infty$. Equation (3.3) can be read as

$$|a|\,\varphi(\lambda t + \mu) = b\,\varphi(t), \qquad t \in [0,\infty)$$

and has the solution

$$\varphi(t) = (t + \frac{\mu}{\lambda - 1})^{\alpha}, \qquad \alpha = \frac{\log \frac{b}{|a|}}{\log \lambda}.$$

Example 3. Consider equation (1.4), where $t \in [1,\infty)$, $a \neq 0$, $b > 0$, $0 < \gamma < 1$ and $q(t) \in C^0([1,\infty))$, $q(t) = O(\exp\{b\,t^{\gamma}\})$ as $t \to \infty$. The corresponding functional equation is

$$|a|\,\varphi(t^{\gamma}) = b\,\varphi(t), \qquad t \in [1,\infty)$$

with the solution

$$\varphi(t) = (\log t)^{\alpha}, \qquad \alpha = \frac{\log \frac{b}{|a|}}{\log \gamma}.$$

Example 4. Consider equation (1.5), where $t \in I$, $p(t) \in C^0(I)$, $p(t) > 0$, $p(t)$ is nondecreasing on I, $\tau(t) \in C^1(I)$, $\tau(t) > 0$, $0 < \delta \leq \dot{\tau}(t) < 1$ on I and $q(t) \in C^0(I)$, $q(t) = O\left(p(t)\exp\{\int_{t_0}^{t-\tau(t)} p(s)\,\mathrm{d}s\}\right)$ as $t \to \infty$. We denote by $x_L(t)$, $L \in \mathbb{R}$ the set of particular solutions of (1.5) possessing (3.1). The functional equation (3.3) now has the simple form

$$\varphi(t) = \varphi(t - \tau(t)), \qquad t \in I$$

and admits $\varphi(t) \equiv \text{const} > 0$ as the required solution. Consequently, if $x(t)$ is a solution of (1.5), then there exists $L \in \mathbb{R}$ such that

$$x(t) = x_L(t) + O(1) \qquad \text{as } t \to \infty.$$

We note that a similar structure formula has been derived by J. Diblík in [5] for the homogeneous equation (1.5) with a bounded delay $\tau(t)$. Thus we can extend results of this type also to a wider class of FDEs.

5. Acknowledgment

The author thanks the referee for his valuable remarks.

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