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Generalized Quasilinearization Method for Nonlinear Boundary Value Problems with Integral Boundary Conditions *

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Abstract

The quasilinearization method coupled with the method of upper and lower solutions is used for a class of nonlinear boundary value problems with integral boundary conditions. We obtain some less restrictive sufficient conditions under which corresponding monotone sequences converge uniformly and quadratically to the unique solution of the problem. An example is also included to illustrate the main result.

Keywords Quasilinearization, integral boundary value problem, upper and lower solutions, quadratic convergence

2000 Mathematics Subject Classification 34A45, 34B15

1. Introduction

In this paper, we shall consider the following boundary value problem

$$\begin{cases} x'' = f(t,x), \ t \in I = [0,1], \\ g_1(x(0)) - k_1 x'(0) = \displaystyle\int_0^1 h_1(x(s))\mathrm{d}s, \\ g_2(x(1)) + k_2 x'(1) = \displaystyle\int_0^1 h_2(x(s))\mathrm{d}s. \end{cases} \tag{1}$$

where $f : I \times \mathbb{R} \to \mathbb{R}$, $g_i, h_i : \mathbb{R} \to \mathbb{R}$ are continuous and k_i are nonnegative constants, $i = 1, 2$.

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It is well known (see [8, 9]) that the method of quasilinearization offers an approach for obtaining approximate solutions to nonlinear differential problems. Recently, it was generalized and extended using less restrictive assumptions so as to apply to a large class of differential problems, for details see [1, 2, 3, 4, 5, 6, 7, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20].

The purpose of this paper is to continue the recent ideas for problems of type (1). Concretely, we apply the quasilinearization method coupled with the method of upper and lower solutions to obtain approximate solutions to nonlinear BVP (1) assuming some appropriate properties on f, g_i and h_i $(i = 1, 2)$. Then, we can show that some monotone sequences converge monotonically and quadratically to the unique solution of BVP (1) in the closed set generated by lower and upper solutions. In this work, we define the less restrictive assumptions to make it applicable to a large class of initial and boundary value problems. As far as we know, the problem has not been studied in the available reference materials. Because of our nonlinear and integral boundary conditions, we generalize and extend some existing results. Boundary value problems with nonlinear boundary conditions have been studied by some authors, for example [2, 5, 6, 10] and the references therein. For example, in [10], the authors studied a class of boundary value problems with the following boundary conditions

$$\begin{cases} g(x(a), x(b), px'(a)) = 0, \\ h(x(a), x(b), px'(b)) = 0, \end{cases}$$

and presented a quasilinearization method of the problem under a very smart assumption (see Theorem 5 of [10]). For boundary value problems with integral boundary conditions and comments on their importance, we refer the readers to the papers [3, 4, 7, 13, 15] and the references therein. Especially, in [4], Ahmad, Alsaedi and Alghamdi considered the following forced equation with integral boundary conditions

$$\begin{cases} x''(t) + \sigma x'(t) - f(t, x) = 0, \\ x(0) - \mu_1 x'(0) = \int_0^1 q_1(x(s))\mathrm{d}s, \qquad x(1) + \mu_2 x'(1) = \int_0^1 q_2(x(s))\mathrm{d}s. \end{cases}$$

It should be pointed out that in this paper, we not only *quasilinearize* the function f but also *quasilinearize* the nonlinear boundary conditions, while in [10] the nonlinear boundary conditions are not *quasilinearized.* Furthermore, in this paper, the convexity assumption of f is relaxed and even $f \in C^2$ is not necessary in our framework.

The paper is organized as follows. In section 2, we give some basic concepts and some preparative theorems. Then we present and prove the main result about the quasilinearization method. This is the content of Section 3.

2. Preliminaries

In this section, we will present some basic concepts and some preparative results for later use.

Lemma 2.1. *Consider the following boundary value problem*

$$\begin{cases} x'' = \sigma(t), \quad t \in [0,1], \\ g_1(x(0)) - k_1 x'(0) = \int_0^1 \rho_1(s)\mathrm{d}s, \\ g_2(x(1)) + k_2 x'(1) = \int_0^1 \rho_2(s)\mathrm{d}s. \end{cases} \tag{2}$$

Assume that

(1) $\sigma(t),\ \rho_i(s) \in C[0,1],\ k_i \geqslant 0\ (i = 1,2)$;

(2) $g_i \in C^1(\mathbb{R}),\ g_i(s) \to +\infty$ if $s \to +\infty,\ g_i(s) \to -\infty$ if $s \to -\infty,\ g_i'(s) > 0,\ i = 1,2.$

Then BVP (2) has a unique solution in the segment [0,1].

Proof. It is easy to see that a solution of BVP (2) is

$$x(t) = c_1 + c_2 t + \varphi(t),$$

where $\varphi(t) \equiv \int_0^t \int_0^s \sigma(v)\mathrm{d}v\mathrm{d}s$, and (c_1, c_2) is determined by

$$\begin{cases} g_1(c_1) - k_1 c_2 = \int_0^1 \rho_1(s)\mathrm{d}s, \\ g_2(c_1 + c_2 + \varphi(1)) + k_2(c_2 + \varphi'(1)) = \int_0^1 \rho_2(s)\mathrm{d}s. \end{cases}$$

From the assumptions and using standard arguments, we may see that (c_1, c_2) exists uniquely. In fact, if $k_1 = 0$, the strict monotonicity of the function g_1 implies that there is a unique c_1 such that $g_1(c_1) = \int_0^1 \rho_1(s)\mathrm{d}s$, and then the strict monotonicity of the function g_2 implies that there is a unique c_2 such that $g_2(c_1 + c_2 + \varphi(1)) + k_2(c_2 + \varphi'(1)) = \int_0^1 \rho_2(s)\mathrm{d}s$. If $k_1 \neq 0$, one can get

$$c_2 = \frac{1}{k_1}\left(g_1(c_1) - \int_0^1 \rho_1(s)\mathrm{d}s\right)$$

and

$$g_2\left(c_1 + \frac{1}{k_1}\big(g_1(c_1) - \int_0^1 \rho_1(s)\mathrm{d}s\big) + \varphi(1)\right) + k_2\left(\frac{1}{k_1}\big(g_1(c_1) - \int_0^1 \rho_1(s)\mathrm{d}s\big) + \varphi'(1)\right) = \int_0^1 \rho_2(s)\mathrm{d}s.$$

Using the strict monotonicity of g_1, g_2, the left is an strictly increasing function in c_1 which implies that c_1 exists uniquely. And then the existence and uniqueness of c_2 can be obtained. Thus the proof is completed.

In BVP (2), if taking $g_1(s) = g_2(s) = s$, then the condition (2) in this lemma is satisfied. The boundary conditions considered here are general. But for this general boundary value problem, we will need the existence and uniqueness of solutions in the next parts of this paper. The role of condition (2) is just to ensure that the unique solution exists.

Lemma 2.2. *Under the assumptions of Lemma 2.1, BVP (2) can be rewritten as*

$$x(t) = P(t) + \int_0^1 G(t,s)\sigma(s)\,ds,$$

where

$$\begin{aligned} P(t) = \ & \frac{1}{1+\frac{k_1}{g_1'(x(0))}+\frac{k_2}{g_2'(x(1))}} \left[\left(1-t+\frac{k_2}{g_2'(x(1))}\right)\left(x(0)-\frac{g_1(x(0))}{g_1'(x(0))}+\frac{1}{g_1'(x(0))}\int_0^1 \rho_1(s)\,ds\right)\right. \\ & \left. + \left(t+\frac{k_1}{g_1'(x(0))}\right)\left(x(1)-\frac{g_2(x(1))}{g_2'(x(1))}+\frac{1}{g_2'(x(1))}\int_0^1 \rho_2(s)\,ds\right)\right] \end{aligned}$$

and

$$G(t,s) = \begin{cases} -\frac{1}{\Delta}\big(k_1+g_1'(x(0))t\big)\big(g_2'(x(1))+k_2-g_2'(x(1))s\big), & 0 \leqslant t < s \leqslant 1; \\ -\frac{1}{\Delta}\big(k_1+g_1'(x(0))s\big)\big(g_2'(x(1))+k_2-g_2'(x(1))t\big), & 0 \leqslant s < t \leqslant 1, \end{cases}$$

in which

$$\Delta = \begin{vmatrix} g_1'(x(0)) & -k_1 \\ g_2'(x(1)) & g_2'(x(1))+k_2 \end{vmatrix}.$$

We note that $G(t,s) < 0$ on $(0,1)\times(0,1)$.

Proof. Clearly, it follows from $g_1' > 0, g_2' > 0$ that the homogenous problem

$$\begin{cases} y'' = 0, \ t \in [0,1], \\ g_1'(x(0)) \cdot y(0) - k_1 y'(0) = 0, \\ g_2'(x(1)) \cdot y(1) + k_2 y'(1) = 0 \end{cases}$$

has only the solution $y \equiv 0$. Then by the Green's functions method (see for instance Theorem 3.2.1 in [19]), the associate nonhomogeneous problem

$$\begin{cases} x'' = \sigma(t), \ t \in [0,1], \\ g_1'(x(0)) \cdot x(0) - k_1 x'(0) = (g_1'(x(0)) \cdot x(0) - g_1(x(0))) + \int_0^1 \rho_1(s)\mathrm{d}s, \\ g_2'(x(1)) \cdot x(1) + k_2 x'(1) = (g_2'(x(1)) \cdot x(1) - g_2(x(1))) + \int_0^1 \rho_2(s)\mathrm{d}s \end{cases}$$

(obviously, it is an equivalent form of BVP (2)) has a unique solution given by

$$x(t) = P(t) + \int_0^1 G(t,s)\sigma(s)\mathrm{d}s,$$

where $P(t), G(t,s)$ are specified in this lemma. In fact, $P(t)$ is the unique solution of the problem

$$\begin{cases} y'' = 0,\ t \in [0,1], \\ g_1'(x(0)) \cdot y(0) - k_1 y'(0) = (g_1'(x(0)) \cdot x(0) - g_1(x(0))) + \displaystyle\int_0^1 \rho_1(s)\mathrm{d}s, \\ g_2'(x(1)) \cdot y(1) + k_2 y'(1) = (g_2'(x(1)) \cdot x(1) - g_2(x(1))) + \displaystyle\int_0^1 \rho_2(s)\mathrm{d}s, \end{cases}$$

and $G(t,s)$ is the Green's function of the problem

$$\begin{cases} y'' = \sigma(t),\ t \in [0,1], \\ g_1'(x(0)) \cdot y(0) - k_1 y'(0) = 0, \\ g_2'(x(1)) \cdot y(1) + k_2 y'(1) = 0. \end{cases}$$

Definition 2.1. *Let $\alpha, \beta \in C^2[0,1]$. The function α is called a lower solution of BVP (1) if*

$$\begin{cases} \alpha''(t) \geqslant f(t, \alpha(t)),\ t \in I = [0,1], \\ g_1(\alpha(0)) - k_1 \alpha'(0) \leqslant \displaystyle\int_0^1 h_1(\alpha(s))ds, \\ g_2(\alpha(1)) + k_2 \alpha'(1) \leqslant \displaystyle\int_0^1 h_2(\alpha(s))ds. \end{cases}$$

Similarly, β is called an upper solution of the BVP (1), if β satisfies similar inequalities in the reverse direction.

Now, we state and prove the existence and uniqueness of solutions in an ordered interval generated by the lower and upper solutions of the boundary value problem (1).

Theorem 2.1. *Assume that*

(1) $\alpha, \beta \in C^2[0,1]$ are lower and upper solutions of BVP (1) respectively, such that $\alpha(t) \leqslant \beta(t),\ t \in [0,1]$;

(2) $g_i \in C^1(\mathbb{R}),\ g_i(s) \to +\infty$ if $s \to +\infty,\ g_i(s) \to -\infty$ if $s \to -\infty,\ g_i'(s) > 0,\ i = 1,2$;

(3) $h_i'(s) \geqslant 0,\ i = 1,2$.

Then there exists a solution $x \in C^2[0,1]$ of BVP (1) such that

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \qquad t \in [0,1].$$

Proof. Define

$$\tilde{x} = \delta(\alpha, x, \beta) = \begin{cases} \alpha, & x < \alpha, \\ x, & x \in [\alpha, \beta], \\ \beta, & x > \beta. \end{cases}$$

Consider the following modified problem

$$\begin{cases} x'' = F(t,x) \equiv F^*(t), \\ g_1(x(0)) - k_1 x'(0) = \displaystyle\int_0^1 H_1(x(s))\mathrm{d}s \equiv \int_0^1 H_1^*(s)\mathrm{d}s, \\ g_2(x(1)) + k_2 x'(1) = \displaystyle\int_0^1 H_2(x(s))\mathrm{d}s \equiv \int_0^1 H_2^*(s)\mathrm{d}s, \end{cases} \tag{3}$$

where

$$F(t,x) = f(t,\tilde{x}) + h(x),$$

$$h(x) = \begin{cases} \dfrac{x-\beta}{1+|x-\beta|}, & x > \beta, \\ 0, & x \in [\alpha, \beta], \\ \dfrac{x-\alpha}{1+|x-\alpha|}, & x < \alpha \end{cases}$$

and

$$H_i(x) \equiv h_i(\tilde{x}), \qquad i = 1, 2.$$

We note that $h_i(\alpha) = \min H_i(x)$, $h_i(\beta) = \max H_i(x)$. Noticing that the assumptions of Lemma 2.1 are satisfied for BVP (3), by Lemma 2.2, BVP (3) may be rewritten as an integral equation. Since F^* and H_i^* $(i = 1, 2)$ are continuous and bounded, employing the standard arguments (cf. for example [21]), it follows that the integral equation has at least one solution $x(t) \in C^2[0, 1]$ on the set

$$\Omega = \{x(t) : \|x^{(i)}(t)\| < K,\ i = 0, 1,\ K \text{ is some sufficientlly large constant},\ \forall \mathrm{t} \in [0, 1]\},$$

where $\|\cdot\|$ is the usual maximum norm.

We now argue that each solution $x(t)$ of BVP (3) satisfies $\alpha(t) \leqslant x(t) \leqslant \beta(t)$, $\forall t \in [0, 1]$. We shall show that $\alpha(t) \leqslant x(t)$, $\forall t \in [0, 1]$. Denote $R(t) \equiv \alpha(t) - x(t)$, $t \in [0, 1]$. Assume, for the sake of contradiction, that there exists some $t_0 \in [0, 1]$ such that

$$R(t_0) = \max_{t\in[0,1]} R(t) = \max_{t\in[0,1]} (\alpha(t) - x(t)) > 0.$$

Case 1: Suppose that $t_0 \in (0, 1)$. Then $R(t_0) > 0, R'(t_0) = 0, R''(t_0) \leqslant 0$. Hence

$$\begin{aligned} 0 \geqslant R''(t_0) &= \alpha''(t_0) - x''(t_0) \\ &\geqslant f(t_0, \alpha(t_0)) - F(t_0, x(t_0)) \\ &= f(t_0, \alpha(t_0)) - [f(t_0, \tilde{x}(t_0)) + h(x(t_0))] \\ &= -h(x(t_0)) > 0, \end{aligned}$$

a contradiction.

Case 2: Suppose that $t_0 = 0$. Then $R(0) > 0, R'(0) \leqslant 0$. Hence

$$0 < (g_1(\alpha(0)) - k_1\alpha'(0)) - (g_1(x(0)) - k_1x'(0)) \leqslant \int_0^1 h_1(\alpha(s))\mathrm{d}s - \int_0^1 H_1(x(s))\mathrm{d}s \leqslant 0,$$

a contradiction.

Case 3: Suppose that $t_0 = 1$. Then $R(1) > 0, R'(1) \geqslant 0$. Hence

$$0 < (g_2(\alpha(1)) + k_1\alpha'(1)) - (g_2(x(1)) + k_2x'(1)) \leqslant \int_0^1 h_2(\alpha(s))\mathrm{d}s - \int_0^1 H_2(x(s))\mathrm{d}s \leqslant 0,$$

a contradiction.

To sum up, $x(t) \geqslant \alpha(t)$ holds. A similar proof shows that $x(t) \leqslant \beta(t)$. The proof is completed.

Theorem 2.2. *Assume that*

(1) $\alpha, \beta \in C^2[0,1]$ are lower and upper solutions of BVP (1), respectively;

(2) $f_x(t,x) > 0,\ (t,x) \in [0,1] \times \mathbb{R}$;

(3) $0 < l_{i1} \leqslant g_i'(x)$, each l_{i1} is a constant, $i = 1,2,\ x \in \mathbb{R}$;

(4) $0 \leqslant h_i'(x) \leqslant \lambda_i$, each λ_i is a constant such that $\lambda_i < l_{i1},\ i = 1,2,\ x \in \mathbb{R}$.

Then $\alpha(t) \leqslant \beta(t),\ t \in [0,1]$.

Proof. Denote $S(t) \equiv \alpha(t) - \beta(t)$, $t \in [0,1]$. As in the proof of Theorem 2.1, assume for the sake of contradiction that there exists some $t_0 \in [0,1]$ such that

$$S(t_0) = \max_{t\in[0,1]} S(t) = \max_{t\in[0,1]} (\alpha(t) - \beta(t)) > 0.$$

Case 1: Suppose that $t_0 \in (0,1)$. Then $S(t_0) > 0, S'(t_0) = 0, S''(t_0) \leqslant 0$. Hence

$$\begin{aligned} 0 \geqslant S''(t_0) &= \alpha''(t_0) - \beta''(t_0) \\ &\geqslant f(t_0, \alpha(t_0)) - f(t_0, \beta(t_0)) > 0, \end{aligned}$$

a contradiction.

Case 2: Suppose that $t_0 = 0$. Then $S(0) > 0, S'(0) \leqslant 0$. Hence

$$\begin{aligned} l_{11}S(0) \leqslant g_1'(\xi)S(0) &\leqslant (g_1(\alpha(0)) - k_1\alpha'(0)) - (g_1(\beta(0)) - k_1\beta'(0)) \\ &\leqslant \int_0^1 h_1(\alpha(s))\mathrm{d}s - \int_0^1 h_1(\beta(s))\mathrm{d}s \\ &= \int_0^1 h_1'(\eta(s))(\alpha(s) - \beta(s))\mathrm{d}s \leqslant \int_0^1 h_1'(\eta(s))S(0)\mathrm{d}s \leqslant \lambda_1 S(0), \end{aligned}$$

where $\xi \in [\beta(0), \alpha(0)]$, and η is between α and β. Thus, we get a contradiction.

Case 3: Suppose that $t_0 = 1$. Then $S(1) > 0, S'(1) \geqslant 0$. A similar proof shows that this case cannot hold.

To sum up, $\alpha(t) \leqslant \beta(t),\ t \in [0,1]$.

Corollary 2.1 *Assume that the conditions of Theorem 2.1 and Theorem 2.2 hold. Then BVP (1) has a unique solution.*

3. Main Result

Now, we develop the approximation scheme and show that under suitable conditions on f, g and h, there exists a monotone sequence of solutions of linear problems that converges uniformly and quadratically to a solution of the original nonlinear problem.

Theorem 3.1. *Assume that the conditions of Theorem 2.1 and Theorem 2.2 hold. And assume that $g_i, h_i \in C^2(\mathbb{R})$ satisfy $g_i''(s) \leqslant 0, h_i''(s) \geqslant 0, s \in \mathbb{R}$. Then, there exists a monotone sequence $\{\alpha_n\}$ which converges uniformly to the unique solution x of BVP (1) and the convergence is in a quadratic manner.*

Proof. In view of the assumptions, by Corollary 2.1, BVP (1) has a unique solution $x(t) \in C^2[0,1]$, such that

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \qquad t \in [0,1].$$

Set

$$\Phi(t,x) \equiv F(t,x) - f(t,x) \ \text{ on } [0,1] \times \mathbb{R},$$

where function $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ is selected to be such that $F(t,x), F_x(t,x), F_{xx}(t,x)$ are continuous on $[0,1] \times \mathbb{R}$ and

$$F_{xx}(t,x) \leqslant 0, \ (t,x) \in [0,1] \times \mathbb{R}.$$

Obviously, the function satisfying the above conditions is very easily found. F and Φ are two auxiliary functions in this proof. Using the mean value theorem and the assumptions, we obtain

$$\begin{array}{rcl} f(t,x) & \leqslant & f(t,y) + F_x(t,y)(x-y) - [\Phi(t,x) - \Phi(t,y)] \equiv \bar{F}(t,x;y), \\ g_i(x) & \leqslant & g_i(y) + g_i'(y)(x-y) \equiv \bar{G}_i(x;y), \\ h_i(x) & \geqslant & h_i(y) + h_i'(y)(x-y) \equiv \bar{H}_i(x;y) \end{array}$$

for any $(t,x,y) \in [0,1] \times \mathbb{R}^2$, $i = 1,2$. In particular, we consider the proof only on the set $\Omega = \{(t,x) : t \in [0,1], x \in [\alpha,\beta]\}$.

We divide the proof into two steps.

Step 1. Construction of a convergent sequence

Now, set $\alpha_0 = \alpha$ and consider the following BVP

$$\begin{cases} x'' = \bar{F}(t,x;\alpha_0(t)), \\ \bar{G}_1(x(0);\alpha_0(0)) - k_1 x'(0) = \displaystyle\int_0^1 \bar{H}_1(x(s);\alpha_0(s))\mathrm{d}s, \\ \bar{G}_2(x(1);\alpha_0(1)) + k_2 x'(1) = \displaystyle\int_0^1 \bar{H}_2(x(s);\alpha_0(s))\mathrm{d}s. \end{cases} \tag{4}$$

Then

$$\begin{array}{rcl} \alpha_0''(t) & \geqslant & f(t,\alpha_0(t)) = \bar{F}(t,\alpha_0(t);\alpha_0(t)), \\ \bar{G}_1(\alpha_0(0);\alpha_0(0)) - k_1\alpha'(0) & = & g_1(\alpha_0(0)) - k_1\alpha'(0) \\ & \leqslant & \displaystyle\int_0^1 h_1(\alpha_0(s))\mathrm{d}s = \int_0^1 \bar{H}_1(\alpha_0(s);\alpha_0(s))\mathrm{d}s, \\ \bar{G}_2(\alpha_0(1);\alpha_0(1)) + k_2\alpha_0'(1) & = & g_2(\alpha_0(1)) + k_2\alpha_0'(1) \\ & \leqslant & \displaystyle\int_0^1 h_2(\alpha_0(s))\mathrm{d}s = \int_0^1 \bar{H}_2(\alpha_0(s);\alpha_0(s))\mathrm{d}s \end{array}$$

and

$$
\begin{aligned}
\beta''(t) &\leqslant f(t,\beta(t)) \leqslant \bar F(t,\beta(t);\alpha_0(t)),\\
\bar G_1(\beta(0);\alpha_0(0)) - k_1\beta'(0) &\geqslant g_1(\beta(0)) - k_1\beta'(0)\\
&\geqslant \int_0^1 h_1(\beta(s))\mathrm{d}s \geqslant \int_0^1 \bar H_1(\beta(s);\alpha_0(s))\mathrm{d}s,\\
\bar G_2(\beta(1);\alpha_0(1)) + k_2\beta'(1) &\geqslant g_2(\beta(1)) + k_2\beta'(1)\\
&\geqslant \int_0^1 h_2(\beta(s))\mathrm{d}s \geqslant \int_0^1 \bar H_2(\beta(s);\alpha_0(s))\mathrm{d}s,
\end{aligned}
$$

which implies that α_0 and β are lower and upper solutions of BVP (4), respectively. Also, it is easy to see that $\bar F, \bar G_i$ and $\bar H_i$ $(i=1,2)$ are such that the assumptions of Corollary 2.1. Hence, by Corollary 2.1, BVP (4) has a unique solution $\alpha_1 \in C^2[0,1]$, such that

$$\alpha_0(t) \leqslant \alpha_1(t) \leqslant \beta(t), \qquad t\in[0,1].$$

Furthermore, we note that

$$
\begin{aligned}
\alpha_1''(t) &= \bar F(t,\alpha_1(t);\alpha_0(t)) \geqslant f(t,\alpha_1(t)),\\
g_1(\alpha_1(0)) - k_1\alpha_1'(0) &\leqslant \bar G_1(\alpha_1(0);\alpha_0(0)) - k_1\alpha_1'(0)\\
&= \int_0^1 \bar H_1(\alpha_1(s);\alpha_0(s))\mathrm{d}s \leqslant \int_0^1 h_1(\alpha_1(s))\mathrm{d}s,\\
g_2(\alpha_1(1)) + k_2\alpha_1'(1) &\leqslant \bar G_2(\alpha_1(1);\alpha_0(1)) + k_2\alpha_1'(1)\\
&= \int_0^1 \bar H_2(\alpha_1(s);\alpha_0(s))\mathrm{d}s \leqslant \int_0^1 h_2(\alpha_1(s))\mathrm{d}s
\end{aligned}
$$

which implies that α_1 is a lower solution of BVP (1).

Now, consider the following BVP

$$
\begin{cases}
x'' = \bar F(t,x;\alpha_1(t)),\\
\bar G_1(x(0);\alpha_1(0)) - k_1x'(0) = \displaystyle\int_0^1 \bar H_1(x(s);\alpha_1(s))\mathrm{d}s,\\
\bar G_2(x(1);\alpha_1(1)) + k_2x'(1) = \displaystyle\int_0^1 \bar H_2(x(s);\alpha_1(s))\mathrm{d}s.
\end{cases}
\tag{5}
$$

Again, we find that α_1 and β are lower and upper solutions of BVP (5), respectively. Also, it is easy to see that $\bar F, \bar G_i$ and $\bar H_i$ $(i=1,2)$ are such that the assumptions of Corollary 2.1. Hence, by Corollary 2.1, BVP (5) has a unique solution $\alpha_2 \in C^2[0,1]$, such that

$$\alpha_1(t) \leqslant \alpha_2(t) \leqslant \beta(t), \qquad t\in[0,1].$$

Employing the same arguments successively, we conclude that for all n and $t\in[0,1]$,

$$\alpha = \alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_n \leqslant \beta,$$

where the elements of the monotone sequence $\{\alpha_n\}$ are the unique solutions of the BVP

$$
\begin{cases}
x'' = \bar F(t,x;\alpha_{n-1}),\\
\bar G_1(x(0);\alpha_{n-1}(0)) - k_1x'(0) = \displaystyle\int_0^1 \bar H_1(x(s);\alpha_{n-1}(s))\mathrm{d}s,\\
\bar G_2(x(1);\alpha_{n-1}(1)) + k_2x'(1) = \displaystyle\int_0^1 \bar H_2(x(s);\alpha_{n-1}(s))\mathrm{d}s.
\end{cases}
$$

Consider the following Robin type BVP

$$\begin{cases} x'' = \bar{F}(t, \alpha_n; \alpha_{n-1}), \\ \bar{G}_1(x(0); \alpha_{n-1}(0)) - k_1 x'(0) = \displaystyle\int_0^1 \bar{H}_1(\alpha_n(s); \alpha_{n-1}(s)) \mathrm{d}s, \\ \bar{G}_2(x(1); \alpha_{n-1}(1)) + k_2 x'(1) = \displaystyle\int_0^1 \bar{H}_2(\alpha_n(s); \alpha_{n-1}(s)) \mathrm{d}s. \end{cases} \tag{6}$$

From Lemma 2.1, BVP (6) has a unique solution. It is easy to see that α_n is the unique solution. Thus, we may conclude that

$$\alpha_n(t) = \bar{P}(t) + \int_0^1 \bar{G}(t,s)\bar{F}(s, \alpha_n(s); \alpha_{n-1}(s)) \mathrm{d}s, \tag{7}$$

where

$$\bar{P}(t) = \frac{1}{1 + \frac{k_1}{g_1'(\alpha_{n-1}(0))} + \frac{k_2}{g_2'(\alpha_{n-1}(1))}} \left[\left(1 - t + \frac{k_2}{g_2'(\alpha_{n-1}(1))}\right) \cdot \right.$$
$$\left(\alpha_{n-1}(0) - \frac{g_1(\alpha_{n-1}(0))}{g_1'(\alpha_{n-1}(0))} + \frac{1}{g_1'(\alpha_{n-1}(0))} \int_0^1 \bar{H}_1(\alpha_n(s); \alpha_{n-1}(s)) \mathrm{d}s \right)$$
$$\left. + \left(t + \frac{k_1}{g_1'(\alpha_{n-1}(0))}\right) \left(\alpha_{n-1}(1) - \frac{g_2(\alpha_{n-1}(1))}{g_2'(\alpha_{n-1}(1))} + \frac{1}{g_2'(\alpha_{n-1}(1))} \int_0^1 \bar{H}_2(\alpha_n(s); \alpha_{n-1}(s)) \mathrm{d}s \right) \right]$$

and

$$\bar{G}(t,s) = \begin{cases} -\frac{1}{\Delta}(k_1 + g_1'(\alpha_{n-1}(0))t)(g_2'(\alpha_{n-1}(1)) + k_2 - g_2'(\alpha_{n-1}(1))s), & 0 \leqslant t < s \leqslant 1, \\ -\frac{1}{\Delta}(k_1 + g_1'(\alpha_{n-1}(0))s)(g_2'(\alpha_{n-1}(1)) + k_2 - g_2'(\alpha_{n-1}(1))t), & 0 \leqslant s < t \leqslant 1 \end{cases}$$

with

$$\Delta = \begin{vmatrix} g_1'(\alpha_{n-1}(0)) & -k_1 \\ g_2'(\alpha_{n-1}(1)) & g_2'(\alpha_{n-1}(1)) + k_2 \end{vmatrix}.$$

By similar arguments to some references, see for instance [4], employing the fact that [0,1] is compact and the monotone convergence is pointwise, it follows by the Ascoli-Arzela Theorem and Dini's Theorem that the convergence of the sequence is uniform. If x is the limit point of the sequence α_n, then passing to the limit $n \to \infty$, (7) gives

$$x(t) = P(t) + \int_0^1 G(t,s) f(s, x(s)) \mathrm{d}s.$$

Thus, $x(t)$ is the solution of the BVP (1).

Step 2. Quadratic convergence

To show the quadratic rate of convergence, define the error function

$$e_n(t) \equiv x(t) - \alpha_n(t) \geqslant 0, \qquad t \in [0,1].$$

Then

$$
\begin{aligned}
e_n''(t) &= x''(t) - \alpha_n''(t) \\
&= f(t, x(t)) - f(t, \alpha_{n-1}(t)) - F_x(t, \alpha_{n-1}(t))(\alpha_n(t) - \alpha_{n-1}(t)) \\
&\quad +[(\Phi(t, \alpha_n(t)) - \Phi(t, \alpha_{n-1}(t))] \\
&= F(t, x(t)) - F(t, \alpha_{n-1}(t)) - F_x(t, \alpha_{n-1}(t))(\alpha_n(t) - \alpha_{n-1}(t)) \\
&\quad +[\Phi(t, \alpha_n(t)) - \Phi(t, x(t))] \\
&= F_x(t, \xi_1)(x(t) - \alpha_{n-1}(t)) - F_x(t, \alpha_{n-1}(t))(\alpha_n(t) - \alpha_{n-1}(t)) \\
&\quad +[\Phi(t, \alpha_n(t)) - \Phi(t, x(t))] \\
&= (F_x(t, \xi_1) - F_x(t, \alpha_{n-1}(t)))(x(t) - \alpha_{n-1}(t)) + F_x(t, \alpha_{n-1}(t))(x(t) - \alpha_n(t)) \\
&\quad +[\Phi(t, \alpha_n(t)) - \Phi(t, x(t))] \\
&= F_{xx}(t, \xi_2)(\xi_1 - \alpha_{n-1})(x(t) - \alpha_{n-1}(t)) + F_x(t, \alpha_{n-1}(t))(x(t) - \alpha_n(t)) \\
&\quad -\Phi_x(t, \xi_3)(x(t)) - \alpha_n(t)) \\
&= F_{xx}(t, \xi_2)(\xi_1 - \alpha_{n-1})(x(t) - \alpha_{n-1}(t)) + [F_x(t, \alpha_{n-1}(t)) - \Phi_x(t, \xi_3)](x(t)) - \alpha_n(t)),
\end{aligned}
$$

where $\alpha_{n-1}(t) \leqslant \xi_1 \leqslant \xi_2 \leqslant x(t)$ and $\alpha_n(t) \leqslant \xi_3 \leqslant x(t)$. Since $F_{xx} \leqslant 0$ and $f_x > 0$, it follows that there exists $\gamma > 0$ and an integer N such that

$$F_x(t, \alpha_{n-1}(t)) - \Phi_x(t, \xi_3) \geqslant \gamma, \quad t \in [0, 1], \quad n \geqslant N.$$

Hence, we obtain

$$e_n''(t) \geqslant \gamma e_n(t) - M \parallel e_{n-1} \parallel^2, \tag{8}$$

where $M \geqslant |F_{xx}(t, s)|$, for $s \in [\alpha_{n-1}(t), x(t)]$, $t \in [0, 1]$. Furthermore,

$$
\begin{aligned}
& (g_1(x(0)) - k_1 x'(0)) - (\bar{G}_1(\alpha_n(0); \alpha_{n-1}(0)) - k_1 \alpha_n'(0)) \\
= \ & g_1(x(0)) - \bar{G}_1(\alpha_n(0); \alpha_{n-1}(0)) - k_1 e_n'(0) \\
= \ & g_1(x(0)) - g_1(\alpha_{n-1}(0)) - g_1'(\alpha_{n-1}(0))(\alpha_n(0) - \alpha_{n-1}(0)) - k_1 e_n'(0) \\
= \ & g_1'(\alpha_{n-1}(0)) e_n(0) + \frac{g_1''(\xi_4)}{2} e_{n-1}^2(0) - k_1 e_n'(0) \\
= \ & \int_0^1 [h_1(x(s)) - \bar{H}_1(\alpha_n(s); \alpha_{n-1}(s))] \mathrm{d}s \\
= \ & \int_0^1 \left[h_1'(\alpha_{n-1}(s)) e_n(s) + \frac{h_1''(\xi_5)}{2} e_{n-1}^2(s) \right] \mathrm{d}s \\
\leqslant \ & \lambda_1 \int_0^1 e_n(s) \mathrm{d}s + \frac{h_1''(\xi_5)}{2} \|e_{n-1}\|^2,
\end{aligned}
$$

where $\alpha_{n-1}(0) \leqslant \xi_4 \leqslant x(0)$ and $\alpha_{n-1}(s) \leqslant \xi_5 \leqslant x(s)$. On the other hand, noticing that

$$g_1'(\alpha_{n-1}(0)) e_n(0) + \frac{g_1''(\xi_4)}{2} e_{n-1}^2(0) - k_1 e_n'(0) \geqslant l_{11} e_n(0) + \frac{g_1''(\xi_4)}{2} \|e_{n-1}\|^2 - k_1 e_n'(0),$$

we have

$$l_{11} e_n(0) - k_1 e_n'(0) \leqslant \lambda_1 \int_0^1 e_n(s) \mathrm{d}s + \frac{h_1''(\xi_5) - g_1''(\xi_4)}{2} \|e_{n-1}\|^2.$$

Similarly, we get

$$l_{21}e_n(1) + k_2 e_n'(1) \leqslant \lambda_2 \int_0^1 e_n(s)\mathrm{d}s + \frac{h_2''(\xi_7) - g_2''(\xi_6)}{2}\|e_{n-1}\|^2,$$

where $\alpha_{n-1}(1) \leqslant \xi_6 \leqslant x(1)$ and $\alpha_{n-1}(s) \leqslant \xi_7 \leqslant x(s)$. Let

$$C_1 \geqslant \frac{h_1''(\xi_5) - g_1''(\xi_4)}{2} \geqslant 0, \quad C_2 \geqslant \frac{h_2''(\xi_7) - g_2''(\xi_6)}{2} \geqslant 0,$$

then

$$\begin{aligned} l_{11}e_n(0) - k_1 e_n'(0 &\leqslant \lambda_1 \int_0^1 e_n(s)\mathrm{d}s + C_1\|e_{n-1}\|^2, \\ l_{21}e_n(1) + k_2 e_n'(1) &\leqslant \lambda_2 \int_0^1 e_n(s)\mathrm{d}s + C_2\|e_{n-1}\|^2. \end{aligned} \tag{9}$$

Now, we consider the following BVP

$$\begin{cases} y''(t) = \gamma y(t) - M \parallel e_{n-1} \parallel^2, \quad t \in [0,1], \\ l_{11}y(0) - k_1 y'(0) = \lambda_1 \int_0^1 y(s)\mathrm{d}s + C_1\|e_{n-1}\|^2, \\ l_{21}y(1) + k_2 y'(1) = \lambda_2 \int_0^1 y(s)\mathrm{d}s + C_2\|e_{n-1}\|^2. \end{cases} \tag{10}$$

From (8) and (9), it follows that $e_n(t)$ is a lower solution of BVP (10). Let

$$r(t) = \frac{M}{\gamma}\|e_{n-1}\|^2,$$

then it is clear that

$$r''(t) = \gamma r(t) - M \parallel e_{n-1} \parallel^2 \equiv 0. \tag{11}$$

Also, if we let $\gamma > 0$ be sufficiently small, we have

$$\begin{aligned} l_{11}r(0) - k_1 r'(0) &\geqslant \lambda_1 \int_0^1 r(s)\mathrm{d}s + C_1\|e_{n-1}\|^2, \\ l_{21}r(1) + k_2 r'(1) &\geqslant \lambda_2 \int_0^1 r(s)\mathrm{d}s + C_2\|e_{n-1}\|^2. \end{aligned} \tag{12}$$

From (11) and (12), it follows that $r(t)$ is an upper solution of BVP (10). Hence, by Theorem 2.2, we obtain

$$e_n(t) \leqslant r(t) = \frac{M}{\gamma}\|e_{n-1}\|^2, \quad t \in [0,1], \quad n \geqslant N.$$

This establishes the quadratic convergence of the iterates.

Now we will illustrate the main result by the following example (which is a modified version of the example in [4]):

Example. Let

$$f(t,x) = \begin{cases} t\mathrm{e}^{x+1} + 2x, & \text{if } (t,x) \in [0,1] \times (-\infty, 0), \\ \mathrm{e}t + x(\mathrm{e}t + 2), & \text{if } (t,x) \in [0,1] \times [0, +\infty), \end{cases}$$

$$g(x) = \begin{cases} -x^4 + \frac{1}{2}x\sin x + 2x + \cos x, & \text{if } x \in (-\infty, 0), \\ 2x + 1, & \text{if } x \in [0, +\infty). \end{cases}$$

Consider the boundary value problem

$$\begin{cases} x'' = f(t, x), \qquad t \in [0, 1], \\ g(x(0)) - k_1 x'(0) = \int_0^1 \frac{cx(s) - 1}{2} \mathrm{d}s, \\ g(x(1)) + k_2 x'(1) = \int_0^1 (cx(s) + 1) \mathrm{d}s, \end{cases} \tag{13}$$

where $0 \leqslant k_1 \leqslant (3/2 - c/4)$, $0 \leqslant k_2$, $0 \leqslant c < 1$. It can easily be verified that $\alpha(t) = -1$ and $\beta(t) = t$ are the lower and super solutions of BVP (13), respectively. Also the assumptions of Theorem 3.1 are satisfied. Hence we can obtain a monotone sequence of approximate solutions converging uniformly and quadratically to the unique solution of BVP (13).

By a direct calculation, one can see that in the foregoing example, f_{xx} does not exist. However, in many references (see for example [3, 4, 7, 12]), the existence of f_{xx} is an important condition.

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