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Bifurcation analysis of Rössler system with multiple delayed feedback*

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Abstract

In this paper, regarding the delay as parameter, we investigate the effect of delay on the dynamics of a Rössler system with multiple delayed feedback proposed by Ghosh and Chowdhury. At first we consider the stability of equilibrium and the existence of Hopf bifurcations. Then an explicit algorithm for determining the direction and the stability of the bifurcating periodic solutions is derived by using the normal form theory and center manifold argument. Finally, we give a numerical simulation example which indicates that chaotic oscillation is converted into a stable steady state or a stable periodic orbit when the delay passes through certain critical values.
Keywords: Rössler system; delayed feedback control; Hopf bifurcation

1 Introduction

The study of chaotic systems has increasingly gained interest of many researchers since the pioneering work of Lorenz [8]. For a quite long period of time, people thought that chaos was neither predictable nor controllable. Recently the trend of analyzing and understanding chaos has been extended to controlling and utilizing

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chaos. The main goal of chaos control was to eliminate chaotic behavior and to stabilize the chaotic system at one of the system's equilibrium points. More specially, when it is useful, we want to generate chaos intentionally. Until now, many advanced theories and methodologies have been developed for controlling chaos. Many scientists have more concerns with delayed control (see Guan, Chen and Peng [5], Shu et al.[16], Zhang and Su [21]). The existing control method can be classified, mainly, into two categories. The first one, the OGY method developed by Ott, Grebogi and Yorke [9] in 1990s of the last century has completely changed the chaos research topic. The second one, proposed by Pyragas [10, 11], using time-delayed controlling forces. Compared with the first one, it is much simpler and more convenient on controlling chaos in continuous dynamics system. Here, we mainly study the Rössler system with delayed controlling method developed by Pyragas. Rössler system is described by the following three-dimensional smooth autonomous system (see Rössler [13])

$$\begin{aligned} &\dot{x}(t) = -y(t) - z(t), \\ &\dot{y}(t) = x(t) + \beta_1 y(t), \\ &\dot{z}(t) = \beta_2 + z(t)\left(x(t) - \gamma\right), \end{aligned} \tag{1}$$

which is chaotic when $\beta_1 = \beta_2 = 0.2, \quad \gamma = 5.7$.

Rössler system is a quite simple set of differential equations with chaos to simplify the Lorenz model of turbulence that contains just one (second order) nonlinearity in one variable. Due to its simplicity, the Rössler system has become a standard one to issue the effectiveness of the chaos control strategy. Recently, many literatures adopted controlling strategy for the Rössler system. In the last years there are many studies on Rössler system. For example, Pyragas [10] stabilized unstable periodic orbits of a Rössler system to a desired periodic orbit by self-controlling feedback. Tao et al. [18] used the speed feedback control such that the controlled Rössler system will gradually converge to unsteadily equilibrium point. Tian et al. [19] used a nonlinear open-plus-closed-loop (NOPCL) control to Rössler system. Yang et. al. [20] presented an impulsive control to achieve periodic motions for the Rössler system. Moreover there are extensive study, for example Amhed et al. [1], Chang et al. [2], Chen et al. [3], Rasussen et al. [12]. Recently, Ghosh et al. [4] have proposed the multiple delayed system in the following form:

$$\begin{aligned} &\dot{x}(t) = -y(t) - z(t) + \alpha_1 x(t-\tau_1) + \alpha_2 x(t-\tau_2), \\ &\dot{y}(t) = x(t) + \beta_1 y(t), \\ &\dot{z}(t) = \beta_2 + z(t)\left(x(t) - \gamma\right), \end{aligned} \tag{2}$$

where α_i, β_i $(i = 1, 2)$ and γ are all positive constants. They studied the system (2) in numerical simulations mainly. The purpose of the present paper is to investigate system (2) analytically and numerically. Our analytical results show that the stability changes as the delays vary. Meanwhile, our numerical simulations indicate that chaotic oscillation is converted into a stable steady state or a stable periodic orbit when the delay passes through certain critical values. This shows that the chaos property changes as the delay varies.

The rest of the paper is organized as follows. In Section 2, we study the distribution of the eigenvalues by using the result due to Ruan and Wei [14, 15] on the analysis of distribution of the zeros of exponential polynomial. Hence the stability and existence of Hopf bifurcations are obtained. In Section 3, the direction and stability of the Hopf bifurcation are determined by using the center manifold and normal forms theory. Some numerical simulations are carried out for supporting the analysis results in Section 4. Conclusions and discussions are given in Section 5.

2 Analysis of stability and bifurcation

In this section, we shall study the stability of the interior equilibrium and the existence of local Hopf bifurcations. For convenience, denote

$$A = 1 + (\alpha_1 + \alpha_2)\,\beta_1.$$

Proposition 2.1. *(i) If $\gamma^2 A < 4\beta_1\beta_2$, then system (2) has no equilibrium, and if $\gamma^2 A = 4\beta_1\beta_2$, the system has only one equilibrium given by*

$$\left(\frac{\gamma}{2},\ \frac{-\gamma}{2\beta_1},\ \frac{2\beta_2}{\gamma}\right).$$

(ii) If

$$(\mathbf{H_0}) \qquad \gamma^2 > \frac{4\beta_1\beta_2}{A}$$

holds, then the system (2) has two equilibria (x_0, y_0, z_0) and (x_1, y_1, z_1), where

$$x_0 = -\beta_1 X_+,\ y_0 = X_+,\ z_0 = \frac{\beta_2}{\beta_1 X_+ + \gamma},\ x_1 = -\beta_1 X_-,\ y_1 = X_-,\ z_1 = \frac{\beta_2}{\beta_1 X_- + \gamma},$$

and

$$X_\pm = \frac{1}{2\beta_1}[-\gamma \pm \sqrt{\gamma^2 - \frac{4\beta_1\beta_2}{A}}].$$

Thorough out this paper, we always assume that $(\mathbf{H_0})$ is satisfied and only consider the dynamics of system (2) near the equilibrium (x_0, y_0, z_0).

Let $u_1 = x - x_0$, $u_2 = y - y_0$ and $u_3 = z - z_0$. Then system (2) can be written in the following form

$$\begin{aligned}
\dot{u}_1(t) &= -u_2(t) - u_3(t) + \alpha_1 u_1(t-\tau_1) + \alpha_2 u_1(t-\tau_2),\\
\dot{u}_2(t) &= u_1(t) + \beta_1 u_2(t),\\
\dot{u}_3(t) &= u_1(t)u_3(t) + u_3(t)(x_0 - \gamma) + z_0 u_1(t).
\end{aligned} \tag{3}$$

The characteristic equation of system (3) at the equilibrium $(0,0,0)$ is

$$\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 - \alpha_1(\lambda^2 + a_2\lambda + b_0)e^{-\lambda\tau_1} - \alpha_2(\lambda^2 + a_2\lambda + b_0)e^{-\lambda\tau_2} = 0, \tag{4}$$

where

$$\begin{aligned}
a_0 &= \gamma - x_0 - \beta_1 z_0,\\
a_1 &= -\beta_1\gamma + \beta_1 x_0 + 1 + z_0,\\
a_2 &= \gamma - x_0 - \beta_1,\\
b_0 &= -\beta_1\gamma + \beta_1 x_0.
\end{aligned}$$

Now we employ the method due to Ruan and Wei [14, 15] to investigate the distribution of roots of Eq.(4).

When $\tau_1 = 0$ and $\tau_2 = 0$, Eq.(4) becomes

$$\lambda^3 + (a_2 - \alpha_1 - \alpha_2)\lambda^2 + (a_1 - \alpha_1 a_2 - \alpha_2 a_2)\lambda + a_0 - \alpha_1 b_0 - \alpha_2 b_0 = 0. \tag{5}$$

For convenience, we make the following hypothesis:

$$(\mathbf{H_1}) \quad \begin{cases} a_0 - \alpha_1 b_0 - \alpha_2 b_0 > 0,\ a_2 - \alpha_1 - \alpha_2 > 0,\\ (a_2 - \alpha_1 - \alpha_2)(a_1 - \alpha_1 a_2 - \alpha_2 a_2) - a_0 + \alpha_1 b_0 + \alpha_2 b_0 > 0. \end{cases}$$

By Routh-Hurwitz criterion we know that if $(\mathbf{H_1})$ holds, then all roots of Eq.(5) have negative real parts. Let $i\nu(\tau_1)(\nu > 0)$ be a root of Eq.(4) with $\tau_2 = 0$. Then it follows that

$$\begin{cases} -a_2\nu^2 + a_0 + \alpha_2\nu^2 - \alpha_2 b_0 = (\alpha_1 b_0 - \alpha_1\nu^2)\cos\nu\tau_1 + \alpha_1 a_2\nu\sin\nu\tau_1,\\ -\nu^3 + a_1\nu - \alpha_2 a_2\nu = \alpha_1 a_2\nu\cos\nu\tau_1 - (\alpha_1 b_0 - \alpha_1\nu^2)\sin\nu\tau_1, \end{cases} \tag{6}$$

which leads to

$$\nu^6 + p\nu^4 + q\nu^2 + r = 0, \tag{7}$$

where

$$\begin{aligned}
p &= (\alpha_2 - a_2)^2 + 2(\alpha_2 a_2 - a_1) - \alpha_1^2,\\
q &= 2(\alpha_2 - a_2)(a_0 - \alpha_2 b_0) - (\alpha_2 a_2 - a_1)^2 + 2b_0\alpha_1^2 - \alpha_1^2 a_2^2,\\
r &= (a_0 - \alpha_2 b_0)^2 - \alpha_1^2 b_0^2.
\end{aligned}$$

If we set $z = \nu^2$, then Eq.(7) becomes

$$z^3 + pz^2 + qz + r = 0. \tag{8}$$

Denote

$$h(z) = z^3 + pz^2 + qz + r.$$

Hence

$$\frac{dh(z)}{dz} = 3z^2 + 2pz + q.$$

Set

$$\triangle = p^2 - 3q.$$

When $r \geq 0$ and $\triangle > 0$, the following equation

$$3z^2 + 2pz + q = 0$$

has two real roots

$$z_1^* = \frac{-p + \sqrt{\triangle}}{3} \quad \textit{and} \quad z_2^* = \frac{-p - \sqrt{\triangle}}{3}.$$

Applying the Claim 3 and Lemma 2.1 of Ruan and Wei [14], we have the following conclusions.

Lemma 2.1. *For Eq.(8), we have following conclusions:*

(i) If $r < 0$, then Eq.(8) has at least one positive root.

(ii) If $r \geq 0$ and $\triangle \leq 0$, then Eq.(8) has no positive root.

(iii) If $r \geq 0$ and $\triangle > 0$, then Eq.(8) has one positive roots if and only if $z_1^* > 0$ and $h(z_1^*) \leq 0$.

Without loss of generality, we assume that Eq.(8) has three positive roots, denoted by x_1, x_2 and x_3, respectively. Then Eq.(7) has three positive roots $\nu_k = \sqrt{x_k}$, $k = 1, 2, 3$. Substituting ν_k into (6) gives

$$\cos \nu_k \tau_1 = \frac{(b_0 - \nu_k^2)[(\alpha_2 - a_2)\nu_k^2 + a_0 - \alpha_2 b_0] + a_2\nu_k[(a_1 - \alpha_2 a_2)\nu_k - \nu_k^3]}{\alpha_1(b_0 - \nu_k^2)^2 + \alpha_1 a_2^2 \nu_k^2}.$$

Let

$$\tau_{1k}^{(j)} = \frac{1}{\nu_k}[\arccos \frac{(b_0 - \nu_k^2)[(\alpha_2 - a_2)\nu_k^2 + a_0 - \alpha_2 b_0] + a_2\nu_k[(a_1 - \alpha_2 a_2)\nu_k - \nu_k^3]}{\alpha_1(b_0 - \nu_k^2)^2 + \alpha_1 a_2^2 \nu_k^2} + 2j\pi],$$

where $k = 1,2,3; j = 0,1,\cdots$, and $\tau_{1k}^{(0)}\nu_k \in (0, 2\pi]$ is determined by the sign of $\sin(\nu_k\tau_1^{(0)})$. These show that $(\nu_k, \tau_{1k}^{(j)})$ is a root of Eq.(6). This shows that $\pm i\nu_k$ is a pair of purely imaginary roots of Eq.(4) with $\tau_2 = 0$ when $\tau_1 = \tau_{1k}^{(j)}$.

Define

$$\tau_1^0 = \min_{1\le k\le 3, j\ge 0}\{\tau_{1k}^{(j)}\}.$$

Let $\lambda(\tau_1) = \alpha(\tau_1) + i\nu(\tau_1)$ be the root of Eq.(4) with $\tau_2 = 0$ satisfying $\alpha(\tau_{1k}^{(j)}) = 0,\ \nu(\tau_{1k}^{(j)}) = \nu_k$. Then the following transversality condition holds.

Lemma 2.2. *Suppose that $z_k = \nu_k^2$, and $h'(z_k) \neq 0$. Then $\frac{d(Re\lambda(\tau_{1k}^{(j)}))}{d\tau_1} \neq 0$ and the sign of $\frac{d(Re\lambda(\tau_{1k}^{(j)}))}{d\tau_1}$ is coincident with that of $h'(z_k)$.*

Proof. Substituting $\lambda(\tau_1)$ into Eq(4) with $\tau_2 = 0$ and differentiating both sides with respect to τ_1, it follows that

$$\left(\frac{d\lambda}{d\tau_1}\right)^{-1} = -\frac{[3\lambda^2 + 2(a_2-\alpha_2)\lambda + a_1 - \alpha_2 a_2]e^{\lambda\tau_1}}{\alpha_1\lambda(\lambda^2 + a_2\lambda + b_0)} + \frac{2\lambda + a_2}{\lambda(\lambda^2 + a_2\lambda + b_0)} + \frac{\tau_1}{\lambda}.$$

And hence,

$$\begin{aligned}
\left[\frac{d(Re\lambda(\tau_1))}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{1k}^{(j)}} &= Re\left[\frac{(3\lambda^2 + 2(a_2-\alpha_2)\lambda + a_1 - \alpha_2 a_2)e^{\lambda\tau_1}}{-\alpha_1\lambda(\lambda^2 + a_2\lambda + b_0)}\right]_{\tau_1=\tau_{1k}^{(j)}} \\
&\quad + Re\left[\frac{2\lambda + a_2}{\lambda(\lambda^2 + a_2\lambda + b_0)}\right]_{\tau_1=\tau_{1k}^{(j)}} + Re\left(\frac{\tau_1}{\lambda}\right)_{\tau_1=\tau_{1k}^{(j)}} \\
&= \frac{1}{\Lambda}\{3\nu_k^6 + 2[(\alpha_2 - a_2)^2 + 2(\alpha_2 a_2 - a_2) - \alpha_1^2]\nu_k^4 \\
&\quad + [(a_1 - \alpha_2 a_2)^2 + 2(\alpha_2 - a_2)(a_0 - \alpha_2 b_0) - \alpha_1^2 a_2^2 + 2b_0\alpha_1^2]\nu_k^2\} \\
&= \frac{1}{\Lambda}(3\nu_k^6 + 2p\nu_k^4 + q\nu_k^2) \\
&= \frac{z_k}{\Lambda}h'(z_k),
\end{aligned}$$

where

$$\Lambda = \alpha_1^2[a_2^2\nu_k^4 + (\nu_k^2 - b_0)^2\nu_k^2].$$

Notice that $\Lambda > 0$ and $z_k > 0$, we conclude that

$$\mathrm{sign}\left[\frac{d(Re\lambda(\tau_1))}{d\tau_1}\right]^{-1}_{\tau_1=\tau_{1k}^{(j)}} = \mathrm{sign}\{h'(z_k)\}.$$

This completes the proof. $\square$

By the Lemmas 2.1, 2.2 and applying the Hopf bifurcation theorem for functional differential equations (see Hale [6], Chapter 11, Theorem 1.1), we can conclude the existence of Hopf bifurcation as stated in the following theorem.

Theorem 2.3. *Suppose that* $(\mathbf{H_1})$ *is satisfied and* $\tau_2 = 0$.

(i) If $r > 0$ *and* $\triangle = p^2 - 3q \leq 0$*, then all roots of Eq.(4) have negative real parts for all* $\tau_1 \geq 0$*, and hence the equilibrium* (x_0, y_0, z_0) *of system (2) is asymptotically stable for all* $\tau_1 \geq 0$.

(ii) If either $r < 0$ *or* $r \geq 0$ *and* $\triangle > 0$, $z_1^* > 0$ *and* $h(z_1^*) \leq 0$ *hold, then* $h(z)$ *has at least a positive root* z_k*, all roots of Eq.(4) have negative real parts for* $\tau_1 \in [0, \tau_1^0)$*. Hence the equilibrium* (x_0, y_0, z_0)*of system (2) is asymptotically stable when* $\tau_1 \in [0, \tau_1^0)$.

(iii) If all conditions as stated in (ii) and $h'(z_k) \neq 0$ *hold, then system (2) undergoes a Hopf bifurcation at the equilibrium* (x_0, y_0, z_0)*, when* $\tau_1 = \tau_{1k}^{(j)} (j = 0, 1, 2, \cdots)$.

We have known that $(\mathbf{H_1})$ ensures that all roots of Eq.(5) have negative real parts. Now we consider $(\mathbf{H_1})$ is not satisfied. For convenience, denote

$$a = (a_2 - \alpha_1 - \alpha_2),\ b = (a_1 - \alpha_1 a_2 - \alpha_2 a_2),\ c = a_0 - \alpha_1 b_0 - \alpha_2 b_0.$$

Then Eq.(5) becomes

$$\lambda^3 + a\lambda^2 + b\lambda + c = 0.$$

Let $\lambda = X - a/3$. Then it reduces to

$$X^3 + p_1 X + q_1 = 0, \tag{9}$$

where $p_1 = b - \frac{1}{3}a^2$ and $q_1 = \frac{2}{27}a^3 - \frac{1}{3}ab + c$.

Denote

$$\Delta_1 = (\frac{p_1}{3})^3 + (\frac{q_1}{2})^2,\ \varepsilon = \frac{-1}{2} + \frac{\sqrt{3}}{2}i,$$
$$\alpha = \sqrt[3]{-\frac{q_1}{2} + \sqrt{\Delta_1}} \text{ and } \beta = \sqrt[3]{-\frac{q_1}{2} - \sqrt{\Delta_1}}.$$

Then from Cardano's formula for the third degree algebra equation we have the followings.

Proposition 2.2. *If* $\Delta_1 > 0$*, then Eq.(9) has a real root* $\alpha+\beta$ *and a pair of conjugate complex roots* $-\frac{\alpha+\beta}{2} \pm i\frac{\sqrt{3}}{2}(\alpha - \beta)$*, that is, that Eq.(5) has a real root* $\alpha + \beta - \frac{a}{3}$ *and a pair of conjugate complex roots* $-(\frac{\alpha+\beta}{2} + \frac{a}{3}) \pm i\frac{\sqrt{3}}{2}(\alpha - \beta)$.

We make the following assumptions:

$$(\mathbf{H_2}) \qquad \Delta_1 > 0,\ \alpha+\beta-\frac{a}{3}<0,\ \ \frac{\alpha+\beta}{2}+\frac{a}{3}<0,\ \ \alpha-\beta\neq 0.$$

Theorem 2.4. *Suppose that* $(\mathbf{H_2})$ *is satisfied and* $\tau_2 = 0$.

(i) If $r > 0$ *and* $\triangle = p^2 - 3q \leq 0$, *then at least one of the roots of Eq.(4) has positive real parts for all* $\tau_1 \geq 0$, *and hence the equilibrium* (x_0, y_0, z_0) *of system (2) is unstable for all* $\tau_1 \geq 0$.

(ii) If either $r < 0$ *or* $r \geq 0$ *and* $\triangle > 0$, $z_1^* > 0$ *and* $h(z_1^*) \leq 0$ *hold, then* $h(z)$ *has at least a positive root* z_k, *at least one of the roots of Eq.(4) has positive real parts for* $\tau_1 \in [0, \tau_1^0)$. *Hence the equilibrium* (x_0, y_0, z_0) *of system (2) is unstable when* $\tau_1 \in [0, \tau_1^0)$. *Additionally, if* $\frac{d(Re\lambda(\tau_1))}{d\tau_1}|_{\tau_1=\tau_1^0} < 0$, *then the equilibrium* (x_0, y_0, z_0)*of system (2) is asymptotically stable when* $\tau_1 \in (\tau_1^0, \tau_1^1)$, *where* τ_1^1 *is the second critical value.*

(iii) If all conditions as stated in (ii) and $h'(z_k) \neq 0$ *hold, then system (2) undergoes a Hopf bifurcation at the equilibrium* (x_0, y_0, z_0), *when* $\tau_1 = \tau_{1k}^{(j)}(j = 0, 1, 2, \cdots)$.

From the discussions above, we know that there may exist stability switches as τ_1 varies for system (2) with $\tau_2 = 0$. So denote I as stable interval of τ_1, that is the equilibrium $(x_0,\ y_0,\ z_0)$ of Eq.(2) is asymptotically stable when $\tau_1 \in I$ and $\tau_2 = 0$. Let $\tau_1 \in I$, and $\lambda = i\omega(\tau_2)$ $(\omega > 0)$ be a root of Eq.(4). Then we obtain

$$\begin{cases} -a_2\omega^2 + a_0 - \alpha_1(b_0-\omega^2)\cos\omega\tau_1 - \alpha_1 a_2\omega\sin\omega\tau_1 = \alpha_2(b_0-\omega^2)\cos\omega\tau_2 + \alpha_2 a_2\omega\sin\omega\tau_2, \\ a_1\omega - \omega^3 - \alpha_1 a_2\omega\cos\omega\tau_1 + \alpha_1(b_0-\omega^2)\sin\omega\tau_1 = \alpha_2 a_2\omega\cos\omega\tau_2 - \alpha_2(b_0-\omega^2)\sin\omega\tau_2. \end{cases}$$

Then we have

$$\begin{aligned} &\omega^6 + (a_2^2 - 2a_1 - \alpha_2^2 + \alpha_1^2)\omega^4 + [a_1^2 - 2a_0a_2 + (2b_0 - a_2^2)(\alpha_2^2 - \alpha_1^2)]\omega^2 + a_0^2 \\ &-b_0^2(\alpha_2^2 - \alpha_1^2) - 2[\alpha_1(a_0 - a_2\omega^2)(b_0 - \omega^2) + \alpha_1 a_2\omega(a_1\omega - \omega^3)]\cos\omega\tau_1 \\ &-2[\alpha_1 a_2\omega(a_0 - a_2\omega^2) - \alpha_1(a_1\omega - \omega^3)(b_0 - \omega^2)]\sin\omega\tau_1 = 0, \end{aligned} \tag{10}$$

and

$$\begin{aligned} \cos\omega\tau_2 = \tfrac{1}{\alpha_2[a_2^2\omega^2+(b_0-\omega^2)^2]}\{&(b_0-\omega^2)[-a_2\omega^2 + a_0 - \alpha_1(b_0-\omega^2)\cos\omega\tau_1 - \alpha_1 a_2\omega\sin\omega\tau_1] \\ &+ a_2\omega[a_1\omega - \omega^3 - \alpha_1 a_2\omega\cos\omega\tau_1 + \alpha_1(b_0-\omega^2)\sin\omega\tau_1]\}. \end{aligned} \tag{11}$$

We know that Eq.(10) has at most finite positive roots. Without loss of generality, we assume that Eq.(10) has N positive roots, denoted by ω_k $(k = 1, 2, \cdots, N)$. According to (11), define

$$\tau_{2k}^{(j)} = \frac{1}{\omega_k}[\pm \arccos(A) + 2j\pi],\ k = 1, 2, \cdots, N;\ j = 0, 1, 2, \cdots,$$

where A is the value of right hand of (11) with $\omega = \omega_k$ and the sign of arc-cosine function are determined by $\sin \omega\tau_2$. Then $\pm i\omega_k$ is a pair of purely imaginary roots of Eq.(4) with $\tau_{2k}^{(j)}$. Denote

$$\tau_2^0 = \tau_{2k_0}^{(0)} = \min_{k\in\{1,2,\cdots,N\}} \{\tau_{2k}^{(0)}\}, \quad \omega_0 = \omega_{k_0}.$$

Let $\lambda(\tau_2) = \alpha(\tau_2) + i\omega(\tau_2)$ be the root of Eq.(4) satisfying $\alpha(\tau_{2k}^{(j)}) = 0, \omega(\tau_{2k}^{(j)}) = \omega_k$. By computation, we obtain

$$\begin{aligned}\alpha'(\tau_2^0) = \Lambda_1\{&RS - TU - 2\alpha_2^2\omega_0^2 + 2b_0\alpha_2^2\omega_0^2 - a_2^2\alpha_2^2\omega_0^2 + (RB\omega_0 - AT\omega_0)\sin\omega_0\tau_2^0 \\ &+ (-QB - PA)\omega_0 \cos\omega_0(\tau_1 + \tau_2^0) + (-AQ + PB)\omega_0 \sin\omega_0(\tau_1 + \tau_2^0) \\ &- (AR\omega_0 + BT\omega_0)\cos\omega_0\tau_2^0 + (SP + UQ)\cos\omega_0\tau_1 + (QS - PU)\sin\omega_0\tau_1\}^{-1},\end{aligned}$$

where

$$P = \alpha_1(-a_2 + b_0\tau_1 - \tau_1\omega_0^2), Q = \alpha_1(-2\omega_0 + a_2\tau_1\omega_0), R = -3\omega_0^2 + a_1,$$

$$\Lambda_1 = a_2^2\alpha_2^2\omega_0^4 + (b_0^2 - \omega_0^2)^2\alpha_2^2\omega_0^2, A = \alpha_1 a_2\omega_0, B = \alpha_1(b_0 - \omega_0),$$

$$S = a_1\omega_0^2 - \omega_0^4, T = 2a_2\omega_0, U = -a_2\omega_0^3 + a_0\omega_0.$$

Summarizing the discussions above, we have the following conclusions.

Theorem 2.5. *Suppose that either (H_1) or (H_2) is satisfied, $\tau_1 \in I$ and Eq.(10) has positive roots. Then all roots of Eq.(4) have negative real parts for $\tau_2 \in [0,\ \tau_2^0)$. Furthermore, the equilibrium (x_0, y_0, z_0) of system (2) is asymptotically stable when $\tau_2 \in [0,\ \tau_2^0)$. Additionally, if $\alpha'(\tau_2^0) \neq 0$, then system (2) undergoes a Hopf bifurcation at the equilibrium (x_0, y_0, z_0) when $\tau_2 = \tau_2^0$.*

3 Stability and direction of the Hopf bifurcation

In the previous section, we obtained conditions for Hopf bifurcation to occur when $\tau_2 = \tau_2^0$. In this section we study the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions when $\tau_2 = \tau_2^0$, using techniques from normal form and center manifold theory (see e.g. Hassard et al.[7]).

We assume $\tau_1 > \tau_2^0$ and denote $\tau_2 = \tau_2^0 + \mu$. Then the system (3) can be written as an FDE in $C = C([-\tau_1,\ 0], R^3)$ as

$$\dot{u}(t) = L_\mu(u_t) + F(\mu,\ u_t), \tag{12}$$

where $u = (u_1, u_2, u_3)^T$, $u_t(\theta) = u(t+\theta) \in C$, and $L_\mu : C \to R^3, F : R \times C \to R^3$ are given, respectively, by

$$L_\mu\varphi = A\varphi(0) + B_1\varphi(-\tau_1) + B_2\varphi(-(\tau_2^0 + \mu)), \tag{13}$$

where

$$A = \begin{pmatrix} 0 & -1 & -1 \\ 1 & \beta_1 & 0 \\ z_0 & 0 & x_0 - \gamma \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} \alpha_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t))^T,$$

and

$$F(\mu,\ \varphi) = \begin{pmatrix} 0 \\ 0 \\ \varphi_1(0)\varphi_3(0) \end{pmatrix}.$$

From the discussion in Section 2, we know that system (2) undergoes a Hopf bifurcation at $(0,\ 0,\ 0)$ when $\mu = 0$, and the associated characteristic equation of system (2) with $\mu = 0$ has a pair of simple imaginary roots $\pm i\omega_0$.

By the Riesz Representation Theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-\tau_1, 0]$, such that

$$L_\mu\varphi = \int_{-\tau_1}^{0} d\eta(\theta,\mu)\varphi(\theta), \quad for \quad \varphi \in C. \tag{14}$$

In fact, we can choose

$$\eta(\theta,\mu) = \begin{cases} A + B_1 + B_2, & \theta = 0, \\ B_1 + B_2, & \theta \in (-\tau_2, 0), \\ B_1, & \theta \in (-\tau_1, -\tau_2], \\ 0, & \theta = -\tau_1. \end{cases}$$

For $\varphi \in C^1([-\tau_1, 0], R^3)$, we set

$$A(\mu)\varphi = \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & \theta \in [-\tau_1, 0), \\ \int_{-\tau_1}^{0} d\eta(\xi, \mu)\varphi(\xi), & \theta = 0, \end{cases}$$

and

$$R(\mu)\varphi = \begin{cases} 0, & \theta \in [-\tau_1, 0), \\ f(\mu, \varphi), & \theta = 0. \end{cases}$$

Then system (12) can be rewritten as

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t, \tag{15}$$

where

$$u_t(\theta) = u(t + \theta)$$

for $\theta \in [-\tau_1, 0]$.
For $\psi \in C^1([0, \tau_1], (R^3)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, \tau_1], \\ \int_{-\tau_1}^{0} \psi(-t)d\eta(t, 0), & s = 0. \end{cases}$$

For $\varphi \in C^1([-\tau_1, 0], R^3)$ and $\psi \in C^1([0, \tau_1], (R^3)^*)$, using the bilinear form

$$< \psi, \varphi >= \bar{\psi}(0)\varphi(0) - \int_{-\tau_1}^{0} \int_{\xi=0}^{\theta} \bar{\psi}(\xi - \theta)\mathrm{d}\eta(\theta)\varphi(\xi)\mathrm{d}\xi, \tag{16}$$

where $\eta(\theta) = \eta(\theta, 0)$, we know that A^* and $A = A(0)$ are adjoint operators. By the discussion in Section 2, we know that $\pm i\omega_0$ are eigenvalues of $A(0)$. Thus they are eigenvalues of A^*.

We know that the eigenvector of $A(0)$ corresponding to the eigenvalue $i\omega_0$ denoted by $q(\theta)$ satisfies $A(0)q(\theta) = i\omega_0 q(\theta)$. By the definition of $A(0)$ we obtain

$$q(\theta) = (1, \frac{1}{i\omega_0 - \beta_1}, \frac{z_0}{i\omega_0 - x_0 + \gamma})e^{i\omega_0\theta} \tag{17}$$

Similarly, it can be verified that

$$q^*(s) = D(1, \ \frac{1}{i\omega_0 + \beta_1}, \ \frac{1}{i\omega_0 + x_0 - \gamma})e^{i\omega_0 s}$$

is a eigenvector of A^* corresponding to $-i\omega_0$, where D is a constant such that $< q^*(s), q(\theta) >= 1$. Denote

$$a = \frac{1}{i\omega_0 - \beta_1}, \ b = \frac{z_0}{i\omega_0 - x_0 + \gamma}, \ a^* = \frac{1}{i\omega_0 + \beta_1}, \ and\ b^* = \frac{1}{i\omega_0 + x_0 - \gamma}.$$

By (16) we have

$$
\begin{aligned}
< q^*(s), q(\theta) > &= \overline{D}(1, \overline{a^*}, \overline{b^*})(1, a, b)^T - \int_{-\tau_1}^{0} \int_{\xi=0}^{\theta} (1, \overline{a^*}, \overline{b^*}) e^{-i(\xi-\theta)\omega_0} d\eta(\theta)(1, a, b)^T e^{i\xi\omega_0} d\xi \\
&= \overline{D}[1 + a\overline{a^*} + b\overline{b^*} - \int_{-\tau_1}^{0} (1, \overline{a^*}, \overline{b^*}) \theta e^{i\theta\omega_0} d\eta(\theta)(1, a, b)^T] \\
&= \overline{D}[1 + a\overline{a^*} + b\overline{b^*} + \alpha_1 \tau_1 e^{-i\omega_0\tau_1} + \alpha_2 \tau_2^0 e^{-i\omega_0\tau_2^0}].
\end{aligned}
$$

Hence, we can choose

$$
D = \frac{1}{1 + a^*\bar{a} + b^*\bar{b} + \alpha_1 \tau_1 e^{i\omega_0\tau_1} + \alpha_2 \tau_2^0 e^{i\omega_0\tau_2^0}}
$$

so that $< q^*, q >= 1$. Clearly, $< q^*, \bar{q} >= 0$.

Following the algorithms in Hassard et al. [1981] to describe the center manifold C_0 at $\mu = 0$. Let u_t be the solution of Eq.(15) when $\mu = 0$. Define

$$
z(t) =< q^*(s), u_t(\theta) >, \quad W(t, \theta) = u_t(\theta) - 2Re\{z(t)q(\theta)\}. \tag{18}
$$

On the center manifold C_0 we have

$$
W(t, \theta) = W(z(t), \overline{z}(t), \theta),
$$

where

$$
W(z, \overline{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta) z\overline{z} + W_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots,
$$

z and $\overline{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\overline{q^*}$. Note that W is real if u_t is real. We consider only real solution.

For solution $u_t \in C_0$ of (13), since $\mu = 0$,

$$
\begin{aligned}
\dot{z}(t) &= i\omega_0 z + < q^*(\theta), f(0, W(z, \overline{z}, \theta) + 2Re\{zq(\theta)\}) > \\
&= i\omega_0 z + \overline{q^*} f(0, W(z, \overline{z}, 0) + 2Re\{zq(0)\}) \\
&\stackrel{\text{def}}{=} i\omega_0 z + \overline{q^*}(0) f_0(z, \overline{z}).
\end{aligned}
$$

We rewrite this equation as

$$
\dot{z}(t) = i\omega_0 z(t) + g(z, \overline{z}), \tag{19}
$$

where

$$
g(z, \overline{z}) = \overline{q^*}(0) f_0(z, \overline{z}) = g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots. \tag{20}
$$

By (18), we have

$$
u_t(\theta) = (u_{1t}, u_{2t}, u_{3t}) = W(t, \theta) + zq(\theta) + \overline{zq}(\theta),
$$

and then

$$u_{1t}(0)=z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+\cdots,$$

$$u_{3t}(0)=bz+\overline{bz}+W_{20}^{(3)}(0)\frac{z^2}{2}+W_{11}^{(3)}(0)z\overline{z}+W_{02}^{(3)}(0)\frac{\overline{z}^2}{2}+\cdots.$$

From(20), we have

$$\begin{aligned}g(z,\overline{z})&=\bar{q^*}(0)f_0(z,\overline{z})=\bar{D}\bar{b^*}u_{1t}(0)u_{3t}(0)\\&=\bar{D}\bar{b^*}(z+\overline{z}+W_{20}^{(1)}(0)\frac{z^2}{2}+W_{11}^{(1)}(0)z\overline{z}+W_{02}^{(1)}(0)\frac{\overline{z}^2}{2}+\cdots)(bz+\overline{bz}+W_{20}^{(3)}(0)\frac{z^2}{2}\\&+W_{11}^{(3)}(0)z\overline{z}+W_{02}^{(3)}(0)\frac{\overline{z}^2}{2}+\cdots).\end{aligned} \tag{21}$$

Comparing the coefficients with (20), we have

$$\begin{aligned}g_{20}&=2\bar{D}\bar{b^*}b,\\g_{11}&=\bar{D}\bar{b^*}(\overline{b}+b),\\g_{02}&=2\bar{D}\bar{b^*}\overline{b},\\g_{21}&=2\overline{D}\bar{b^*}(\tfrac{1}{2}W_{20}^{(1)}(0)\overline{b}+\tfrac{1}{2}W_{20}^{(3)}(0)+W_{11}^{(3)}(0)+bW_{11}^{(1)}(0)).\end{aligned}$$

We still need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (15) and (18), we have

$$\begin{aligned}\dot{W}=\dot{u}_t-\dot{z}q-\dot{\overline{z}}\overline{q}&=\begin{cases}AW-2Re\{\overline{q^*}(0)f_0q(\theta)\}, & \theta\in[-\tau_1^*,0),\\AW-2Re\{\overline{q^*}(0)f_0q(0)\}+f_0, & \theta=0,\end{cases}\\&\stackrel{\text{def}}{=}AW+H(z,\overline{z},\theta),\end{aligned} \tag{22}$$

where

$$H(z,\overline{z},\theta)=H_{20}(\theta)\frac{z^2}{2}+H_{11}(\theta)z\overline{z}+H_{02}(\theta)\frac{\overline{z}^2}{2}+\cdots, \tag{23}$$

$$A(0)W(t,\theta)=\frac{1}{2}A(0)W_{20}(\theta)z^2+A(0)W_{11}(\theta)z\overline{z}+\cdots, \tag{24}$$

and

$$\begin{aligned}\dot{W}&=W_z\dot{z}+W_{\overline{z}}\dot{\overline{z}}=(W_{20}(\theta)z+W_{11}(\theta)\overline{z}+W_{02}(\theta)\tfrac{\overline{z}^2}{2}+\cdots)(i\omega_0z+g(z,\overline{z}))\\&+(W_{02}(\theta)\overline{z}+W_{11}(\theta)z+\cdots)(i\omega_0\overline{z}+\overline{g}(z,\overline{z}))\\&=2i\omega_0W_{20}(\theta)z\overline{z}+\cdots,\end{aligned}$$

$$A(0)W(t,\theta)-\dot{W}=(A(0)-2i\omega_0)W_{20}(\theta)\tfrac{\overline{z}^2}{2}+A(0)W_{11}()z\overline{z}+\cdots.$$

Hence

$$(A(0)-2i\omega_0)W_{20}(\theta)\frac{\overline{z}^2}{2}+A(0)W_{11}(\theta)z\overline{z}+\cdots=-H_{20}(\theta)\frac{z^2}{2}-H_{11}(\theta)z\overline{z}-H_{02}(\theta)\frac{\overline{z^2}}{2}+\cdots. \tag{25}$$

Comparing the coefficients, we obtain

$$(A-2i\omega_0 I)W_{20}(\theta) = -H_{20}(\theta),\ AW_{11}(\theta) = -H_{11}(\theta), \cdots . \tag{26}$$

By (22), we know that for $\theta \in [-\tau_1, 0)$,

$$H(z,\overline{z},\theta) = -\overline{q^*}(0)f_0 q(\theta) - q^*(0)\overline{f_0}\overline{q}(\theta) = -gq(\theta) - \overline{gq}(\theta). \tag{27}$$

Comparing the coefficients with (23) gives that

$$\begin{aligned} H_{20}(\theta) &= -g_{20}q(\theta) - \overline{g}_{02}\overline{q}(\theta), \\ H_{11}(\theta) &= -g_{11}q(\theta) - \overline{g}_{11}\overline{q}(\theta). \end{aligned} \tag{28}$$

From (26), (27) and the definition of A, we obtain

$$\dot{W}_{20} = 2i\omega_0 W_{20}(\theta) + g_{20}q(\theta) + \overline{g}_{02}\overline{q}(\theta).$$

Solving for $W_{20}(\theta)$, we obtain

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\overline{g}_{02}}{3\omega_0}\overline{q}(0)e^{-i\omega_0\theta} + E_1 e^{2i\omega_0\theta},$$

and similarly

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0}q(0)e^{i\omega_0\theta} + \frac{i\overline{g}_{11}}{\omega_0}\overline{q}(0)e^{-i\omega_0\theta} + E_2,$$

where E_1 and E_2 are both three-dimensional vectors, and can be determined by setting $\theta = 0$ in H. In fact, since

$$H(z,\overline{z},0) = -2Re\{\overline{q^*}(0)f_0 q(0)\} + f_0,$$

we have

$$H_{20}(0) = -g_{20}q(0) - \overline{g}_{02}\overline{q}(0) + f_{z^2},$$

and

$$H_{11}(0) = -g_{11}q(0) - \overline{g}_{11}\overline{q}(0) + f_{z\overline{z}},$$

where

$$f_0 = f_{z^2}\frac{z^2}{2} + f_{z\overline{z}}z\overline{z} + f_{\overline{z}^2}\frac{\overline{z}^2}{2} + \cdots .$$

Hence, combining with the definition of A, we obtain

$$\int_{-\tau_1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_0 W_{20}(0) + g_{20}q(0) + \overline{g}_{02}\overline{q}(0) - f_{z^2},$$

and

$$\int_{-\tau_1}^{0} d\eta(\theta)W_{11}(\theta) = g_{11}q(0) + \overline{g}_{11}\overline{q}(0) - f_{z\overline{z}}.$$

Notice that

$$\left(i\omega_0 I - \int_{-\tau_1}^{0} e^{i\omega_0\theta}d\eta(\theta)\right) q(0) = 0, \left(-i\omega_0 I - \int_{-\tau_1}^{0} e^{-i\omega_0\theta}d\eta(\theta)\right) \overline{q}(0) = 0$$

we have

$$\left(2i\omega_0 I - \int_{-\tau_1}^{0} e^{2i\omega_0\theta}d\eta(\theta)\right) E_1 = f_{z^2}.$$

Similarly, we have

$$-\left(\int_{-\tau_1}^{0} d\eta(\theta)\right) E_2 = f_{z\overline{z}}.$$

Hence, we get

$$\begin{pmatrix} 2i\omega_0 - \alpha_1 e^{-2i\omega_0\tau_1} - \alpha_2 e^{-2i\omega_0\tau_2^0} & 1 & 1 \\ -1 & 2i\omega_0 - \beta_1 & 0 \\ -z_0 & 0 & 2i\omega_0 - x_0 + \gamma \end{pmatrix} E_1 = \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix}$$

and

$$\begin{pmatrix} -\alpha_1 - \alpha_2 & 1 & 1 \\ -1 & -\beta_1 & 0 \\ -z_0 & 0 & -x_0 + \gamma \end{pmatrix} E_2 = \begin{pmatrix} 0 \\ 0 \\ b + \overline{b} \end{pmatrix}.$$

Then g_{21} can be expressed by the parameters.

Based on the above analysis, we can see that each g_{ij} can be determined by the parameters. Thus we can compute the following quantities:

$$\begin{aligned} C_1(0) &= \frac{i}{2\omega_0}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2}, \\ \mu_2 &= -\frac{Re(C_1(0))}{Re(\lambda'(\tau_2^0))}, \\ \beta_2 &= 2Re(C_1(0)), \\ T_2 &= -\frac{Im(C_1(0)) + \mu_2 Im(\lambda'(\tau_2^0))}{\omega_0}. \end{aligned}$$

Theorem 3.1. *For system (2)*

(i) μ_2 *determines the direction of the Hopf bifurcation: if* $\mu_2 > 0$ *(resp.*$\mu_2 < 0$*), then the bifurcating periodic solutions exist for* τ_2 *in a right hand side neighborhood* $(\tau_2^0, \tau_2^0 + \epsilon)$ *(resp. left hand side neighborhood* $(\tau_2^0 - \epsilon, \tau_2^0)$*) of the bifurcation value* τ_2^0*.*

(ii) β_2 *determines the stability of bifurcating periodic solutions: the bifurcating periodic solutions are orbitally asymptotically stable (resp. unstable) if* $\beta_2 < 0$ *(resp.*$\beta_2 > 0$*);*

(iii) T_2 *determines the period of the bifurcating periodic solutions: the period increases (resp. decreases) if* $T_2 > 0$ *(resp.* $T_2 < 0$*).*

4 Numerical examples

In this section, we shall use MATLAB to perform some numerical simulations on system (2).

In the following, we choose a set of parameters as follows::

$$(a) \qquad \alpha_1 = 0.2,\ \alpha_2 = -0.2,\ \beta_1 = 0.2,\ \beta_2 = 0.2,\ \gamma = 5.7.$$

With these parameters, one can find that $(\mathbf{H_2})$ is satisfied. When $\tau_2 = 0$, by a direct computation we obtain that Eq.(7) has two positive roots $\nu_1 \doteq 0.8327$ and $\nu_2 \doteq 1.1506$. Substituting them and the data (a) into Eq.(6) gives, respectively,

$$\tau_{11}^{(j)} \doteq 2.2945 + 7.5456j \ (j = 0, 1, 2, \cdots), \quad \tau_{12}^{(j)} \doteq -1.1036 + 5.4608j \ (j = 1, 2, \cdots).$$

Eq.(4) has pure imaginary roots when $\tau_1 = \tau_{11}^{(j)}$ or $\tau_1 = \tau_{12}^{(j)}$. Further $\alpha'(\tau_{11}^{(j)}) < 0$ and $\alpha'(\tau_{12}^{(j)}) > 0$. By Theorem 2.4 we know that the stability switches exist, that is the equilibrium is unstable when $\tau_1 \in [0, 2.2945)$, and asymptotically stable when $\tau_1 \in (2.2945, 4.3572)$. The results are illustrated in Fig.1-Fig.4.

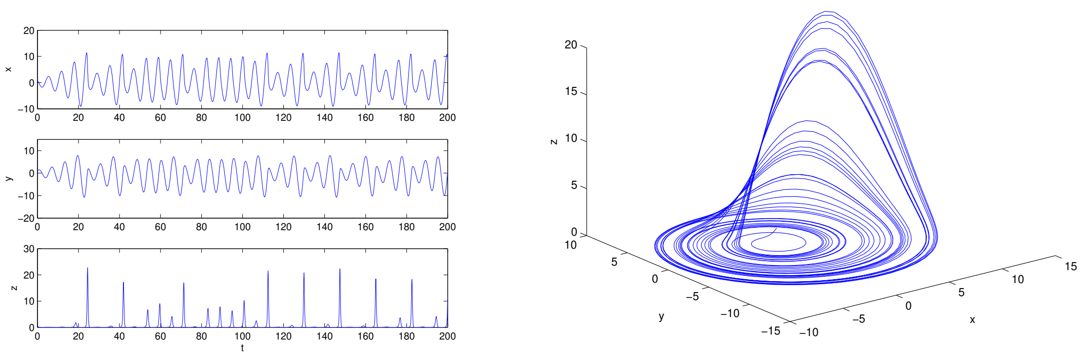

Fig.1. The equilibrium is unstable, and chaos exists for system (2) with the data (a) and $\tau_1 = \tau_2 = 0$.

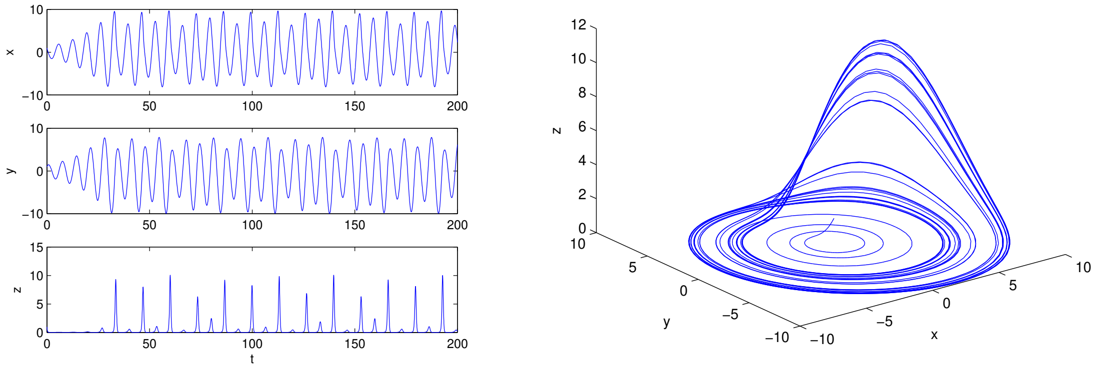

Fig.2. The equilibrium is unstable and chaos phenomenon still exists for system (2) with the data (a), $\tau_1 \in [0, 2.2945)$ *and* $\tau_2 = 0$, where $\tau_1 = 1$.

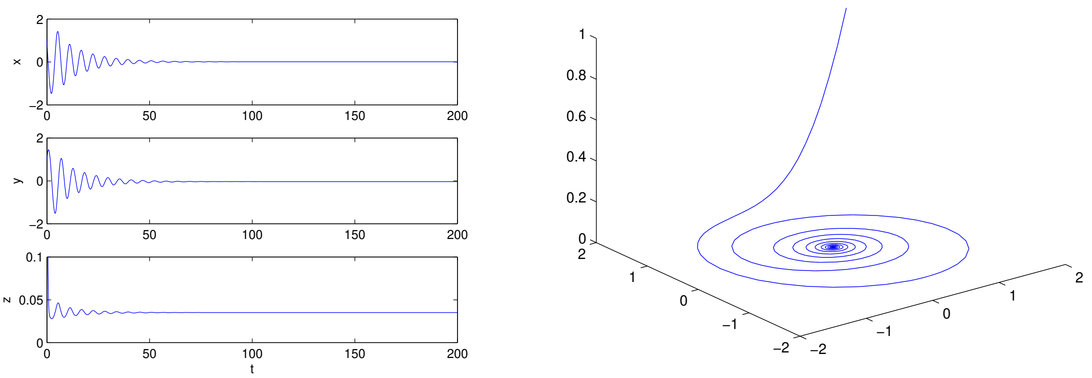

Fig.3. The equilibrium becomes stable and the chaos phenomenon disappears for system (2) with the data (a), $\tau_1 \in (2.2945, 4.3572)$ *and* $\tau_2 = 0$, where $\tau_1 = 3$.

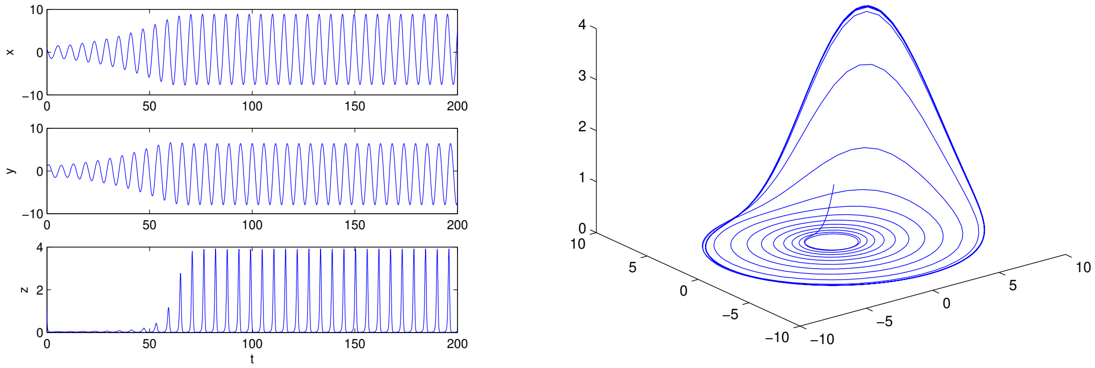

Fig.4. The equilibrium is unstable, and a bifurcating periodic solution appears for system (2) with the data (a) and $\tau_1 > 4.3572$ and close to 4.3572, and $\tau_2 = 0$, where $\tau_1 = 5$.

Let $\tau_1 = 3.0 \in (2.2945, 4.3572)$, we obtain $\tau_2^0 \doteq 1.6295$. By the Theorem 2.4 we know that (x_0, y_0, z_0) is asymptotically stable for $\tau_1 = 3$ and $\tau_2 \in [0, 1.6295)$. Furthermore by direct computation using the algorithm derived in Section 3, we have $C_1(0) \doteq -0.0003 + 0.0002i$, $\beta_2 \doteq -0.0006 < 0$, and $\mu_2 > 0$. We know that, at $\tau_2^0 \doteq 1.6295$, the bifurcating periodic solution is orbitally asymptotically stable, and the direction of the Hopf bifurcation is forward, which are illustrated in Fig.5-Fig.6. On the other hand, the numerical simulations show that the bifurcating periodic solutions disappear when the delay τ_2 is far away $\tau_2^0 = 1.6295$, and chaos occurs again. This is shown in Fig.7.

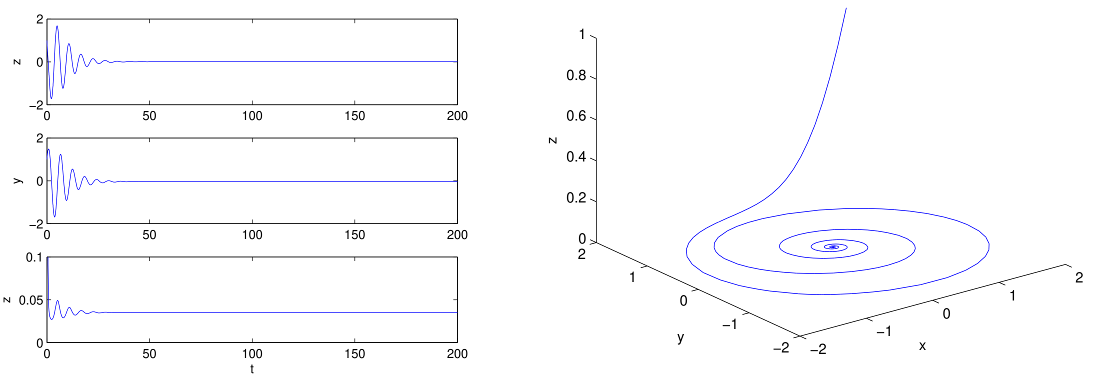

Fig.5. The equilibrium is asymptotically stable for system (2) with the data (a), $\tau_1 \in (2.2945, 4.3572)$ and $\tau_2 \in [0, \tau_2^0)$, where $\tau_1 = 3$, *and* $\tau_2 = 1$.

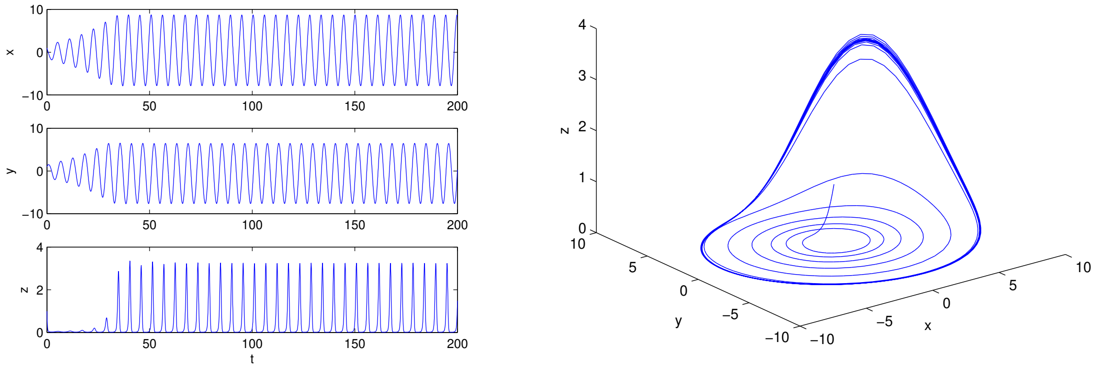

Fig.6. The equilibrium is unstable, and a bifurcating periodic solution appears for system (2) with the data (a), $\tau_1 \in (2.2945, 4.3572)$ and $\tau_2 > \tau_2^0$ is close to τ_2^0, where $\tau_1 = 3$ *and* $\tau_2 = 2 > 1.6295$.

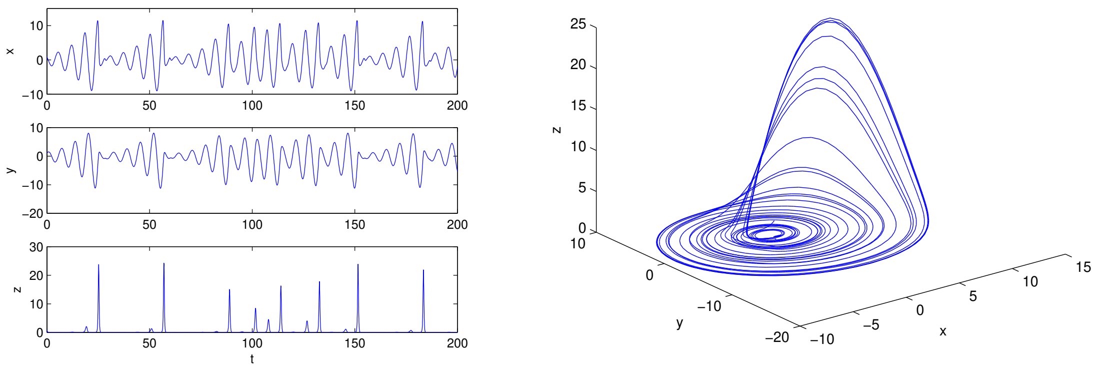

Fig.7. Chaos occurs again for system (2) with the data (a), $\tau_1 \in (2.2945, 4.3572)$ and $\tau_2 > \tau_2^0$ increasing further, where $\tau_1 = 3,\ \tau_2 = 3.5 > 1.6295$.

5 Conclusion

Bifurcation in Rössler system with single delay has been observed by many researchers. However, there are few papers on the bifurcation of Rössler system with multiple delays.

In this paper we have analyzed the Rössler system with multiple delays on two different conditions. We find out that there are stability switches for the interior equilibrium when τ_1 varies in the case of $\tau_2 = 0$. Then for τ_1 in a stability interval, regarding the delay τ_2 as parameter, we show that there exists a first critical value of τ_2 at which the interior equilibrium loses its stability and the Hopf bifurcation occurs. We also investigate the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions, by using the center manifold theory and normal form method.

Our theoretical results and numerical simulations show that, for a Rössler system with chaos phenomena, the chaos oscillation can be controlled by delays. For example, the multiple delayed Rössler system we studied possess chaos oscillation when $\tau_1 = \tau_2 = 0$. The chaos disappears when the delays increase, and the stability of the equilibrium is lost at same time, and the periodic solutions occur from Hopf bifurcation. As the delays increasing further, the numerical simulations show that the periodic solution disappears and the chaos oscillation appears again.

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