Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **60**, 1-32; **http://www.math.u-szeged.hu/ejqtde/**

FRACTAL ANALYSIS OF HOPF BIFURCATION FOR A CLASS OF COMPLETELY INTEGRABLE NONLINEAR SCHRÖDINGER CAUCHY PROBLEMS

JOSIPA PINA MILIŠIĆ, DARKO ŽUBRINIĆ, AND VESNA ŽUPANOVIĆ

Abstract. We study the complexity of solutions for a class of completely integrable, nonlinear integro-differential Schrödinger initial-boundary value problems on a bounded domain, depending on a real bifurcation parameter. The considered Schrödinger problem is a natural extension of the classical Hopf bifurcation model for planar systems into an infinite-dimensional phase space. Namely, the change in the sign of the bifurcation parameter has a consequence that an attracting (or repelling) invariant subset of the sphere in $L^2(\Omega)$ is born. We measure the complexity of trajectories near the origin by considering the Minkowski content and the box dimension of their finite-dimensional projections. Moreover we consider the compactness and rectifiability of trajectories, and box dimension of multiple spirals and spiral chirps. Finally, we are able to obtain the box dimension of trajectories of some nonintegrable Schrödinger evolution problems using their reformulation in terms of the corresponding (not explicitly solvable) dynamical systems in $\mathbb{R}^n$.

Corresponding author: Josipa Pina Milišić
University of Zagreb, Faculty of Electrical Engineering and Computing
Unska 3, 10000 Zagreb, Croatia

pina.milisic@fer.hr

1. Introduction

1.1. **Motivation and formulation of the problem.** Dimension theory for dynamical systems has been rapidly developed due to its important applications in other natural, social and technical sciences. Since the early 1970's scientists have started to estimate and compute fractal dimensions of strange attractors for finite (Lorenz, Henon, Chua, Leonov, etc.), as well as for the infinite-dimensional dynamical systems (Ladyzhenskaya, Foiaş and Temam, Babin and Vishik, Ruelle, Lieb, etc.) Fractal dimensions in dynamics are discussed in a survey article [29]. However, our approach to the question of fractal dimensions in dynamics is different in a way that we are interested in the complexity of *trajectories* of dynamical systems and study the dependence of the computed fractal dimension upon the bifurcation parameter. Knowing the information about fractal dimension of trajectories

---

1991 *Mathematics Subject Classification.* 35Q55, 37G35, 28A12, 34C15.

Key words and phrases. Schrödinger equation, Hopf bifurcation, box dimension, Minkowski content, compactness, rectifiability, bundle of trajectories, oscillation, multiple spiral, spiral chirp.

The authors have been supported by the Austrian-Croatian Project of the Austrian Exchange Service (ÖAD) and the Ministry of Science, Education, and Sports of the Republic of Croatia (MZOS).

enables to measure the complexity of the studied dynamics which we mostly describe by calculating the corresponding box dimension and, in order to describe finer properties, its Minkowski content. Namely, the Hausdorff dimension of trajectories that we are interested in, is always trivial.

The second and third author have undertaken a systematic study of fractal properties of trajectories of vector fields near the weak focus in $\mathbb{R}^2$ and analogously in $\mathbb{R}^3$. Furthermore, they considered a connection between the box dimension of the trajectory and the bifurcation of the related dynamical system. Here we mention that this interesting connection has been studied also in [10, 16] but for one-dimensional discrete dynamical systems.

In this article, using methods developed in [20, 26, 27, 28] we consider the trajectories of vector fields in infinite-dimensional case and we calculate their box dimensions. The original idea of this paper grounds on the connection between nonlocal Schrödinger evolution problems and the corresponding system of ODE's. More precisely, starting from completely integrable nonlinear integro-differential Schrödinger equation we arrive at an equivalent system of infinitely many nonlinear ODE's. Concerning the observed link between Schrödinger problems and vector fields it is interesting to point out that each planar system of ODE's with polynomial right-hand side can be interpreted as a nonlinear Schroedinger equation with an explicit corresponding nonlinear term. Moreover, as it is described in Section 6.3, this consideration is valid even for any dynamical system in $\mathbb{R}^n$. Therefore, it is worth noting that the 16th Hilbert problem about the search of an uniform upper bound for the number of limit cycles in polynomial vector fields can be considered in terms of a Schrödinger equation.

In Žubrinić and Županović [26, Theorem 9] it has been shown that the box dimension of spiral trajectories of the classical Hopf bifurcation system in the plane described by (4), viewed near the focus, is equal to $d = 4/3$ when the bifurcation parameter a_0 is equal 0. Furthermore, these trajectories are Minkowski nondegenerate, i.e. their d-dimensional Minkowski contents, are different from 0 and ∞. On the other hand, for $a_0 \neq 0$ all trajectories have trivial box dimension equal to 1, due to the fact that we have strong focus in this case. We shall see that it is possible to obtain analogous results in infinite-dimensional case for problem (1). The main results are stated in Theorem 11 (Minkowski content), Theorem 15 (box dimension of multiple spirals), Theorem 17 (box dimension of solutions of NLS Cauchy problems), and Theorem 18 (box dimension of spiral chirps). In Theorem 11 we obtain some surprising relations between the sequence of Minkowski contents of projections of the solution of (1) at Hopf bifurcation, and the Sobolev and Lebesgue norms of the initial function u_0.

We start with the Cauchy problem for the following nonlinear Schrödinger (NLS) initial-boundary value problem:

$$
(1) \qquad \begin{cases} u_t(t,x) &= i\Delta u(t,x) - \gamma\, u(t,x)\left(\int_\Omega |u(t,x)|^2 dx + a_0\right) \\ u(t,x) &= 0 \quad \text{for } x \in \partial\Omega,\ t \in (t_{min}, t_{max}) \\ u(0,x) &= u_0(x) \quad \text{for } x \in \Omega. \end{cases}
$$

where i is the imaginary unit, γ is a fixed nonzero complex number, a_0 a real bifurcation parameter, Ω a bounded domain in $\mathbb{R}^N$, $N \geq 1$, and $u_0 : \Omega \to \mathbb{C}$ is a given initial function, $u_0 \in L^2(\Omega, \mathbb{C})$. Here $\Delta u = \sum_{j=1}^N \frac{\partial^2 u}{\partial x_j^2}$ is the Laplace operator. For a fixed space variable x, the solutions $u : (t_{min}, t_{max}) \to L^2(\Omega, \mathbb{C})$ of the NLS Cauchy problem (1) can be considered as trajectories in the Hilbert space $L^2(\Omega, \mathbb{C})$ where $L^2(\Omega, \mathbb{C}) = L^2(\Omega) + iL^2(\Omega)$ is the complexification of the real space $L^2(\Omega)$. We point out that t_{min} and t_{max} depend on γ, a_0 and u_0, and $0 \in (t_{min}, t_{max})$. In this article, we are dealing with unbounded time intervals (t_{min}, t_{max}) either of the form (t_{min}, ∞), or $(-\infty, \infty)$, or $(-\infty, t_{max})$.

Due to the integral term in (1), this NLS Cauchy problem is of nonlocal type. Similar nonlocal problems with the same integral term have been considered for the modified cubic wave equation,

$$
u_{tt} - \Delta u + c\left(\int_\Omega u(t,x)^2 dx\right) u = 0,
$$

with homogeneous Dirichlet boundary condition, in Cazenave, Haraux and Weissler [3, 4, 5], in their study of completely integrable abstract wave equations. We consider the Schrödinger equation (1) for $a_0 = 0$ as an approximation of the equation $u_t = i\Delta u - \gamma u|u|^2$ following the approach of [5, p. 130].

The same integral term as in (1) can be seen in Christ [8, pp. 132 and 133]. An analogous one appears in an equation of fourth order arising from the theory of aeroelasticity,

$$
u_{xxxx} + \left(\alpha - \beta \int_0^1 u_x^2 dx\right) u_{xx} + \gamma u_x + \delta u_t + \varepsilon u_{tt} = 0
$$

see Chicone [7, p. 310]. Other integro-differential Schrödinger problems involving different integral operators on the right-hand side have been studied in numerous papers, see for example Chen and Guo [6].

1.2. **Interpretation in ℓ_2.** In order to write down problem (1) as an infinite system of nonlinear ordinary differential equations we exploit a well known Fourier series expansion using the decomposition with respect to a Hilbert basis in $L^2(\Omega)$. Let (φ_j) be the orthonormal basis of eigenfunctions of the operator $-\Delta$ with zero boundary data and with eigenvalues (λ_j) such that $0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \ldots$ (see e.g. Brezis [1, ersatz (12) on p. 209]). The basis (φ_j) is contained in the real Lebesgue space $L^2(\Omega)$, and therefore it is also an orthonormal base in $L^2(\Omega,\mathbb{C})$. By writing the solution $u(t,x)$ in the form:

$$u(t,x) = \sum_{j=1}^{\infty} z_j(t)\varphi_j(x), \tag{2}$$

the NLS Cauchy problem (1) formally reduces to a lattice Schrödinger equation on the Hilbert space of quadratically summable sequences of complex numbers, which we denote by $\ell_2(\mathbb{C})$. More precisely, we obtain an infinite system of nonlinear ODE's:

$$\dot{z}_j = -i\lambda_j z_j - \gamma z_j(\|z\|^2 + a_0),\ j = 1, 2, \ldots, \tag{3}$$

with initial condition $z_j(0) = z_{j0}$, $j \geq 1$, where $(z_{j0}) \in \ell_2(\mathbb{C})$ and $z(t) = (z_j(t))_j \in \ell_2(\mathbb{C})$ for each $t \in (t_{min}, t_{max})$. Clearly, $z_{j0} = \langle u_0, \varphi_j\rangle = \int_\Omega u_0\varphi_j dx$ and $\|\cdot\|$ is a standard Euclid norm. For given $u_0 \in L^2(\Omega,\mathbb{C})$, we interpret NLS equation $(1)_1$ as the lattice Schrödinger equation (3). To any $v \in L^2(\Omega,\mathbb{C})$ one can assign $z = (z_j)_j \in \ell_2(\mathbb{C})$, where $v = \sum_j z_j\varphi_j$ is the Fourier expansion of v, and this assignment is an isometric isomorphism.

Here, we note that the NLS Cauchy problem (1) is a natural extension of the Hopf bifurcation model for planar systems. The Hopf bifurcation is connected to 1-parameter families of vector fields where a limit cycle surrounding a singular point is born. It is well known that the Hopf bifurcation occurs at an equilibrium point $\mathbf{x}_0$ of a planar system $\dot{\mathbf{x}} = f(\mathbf{x}, a_0)$ depending on a parameter $a_0 \in \mathbb{R}$ when the matrix $Df(\mathbf{x}_0, a_0)$ has a pair of pure imaginary eigenvalues, see [21, pg. 314.]. In this sense we say that the point $\mathbf{x}_0$ is the *weak focus* of the system. On the other side, if the eigenvalues are such that both their real and imaginary parts differ from zero, we are speaking about the *strong focus*.

Namely, when $z = (z_1, 0, 0, \ldots)$, and assuming that $\lambda_1 = 1$, $\gamma = 1$, the system (3) reduces to the classical Hopf bifurcation system in the plane:

$$\begin{cases} \dot{x} &= \quad y - x(x^2 + y^2 + a_0) \\ \dot{y} &= -x - y(x^2 + y^2 + a_0), \end{cases} \tag{4}$$

where we denote $x(t) = \operatorname{Re} z(t)$, $y(t) = \operatorname{Im} z(t)$, real and imaginary part of the complex number $z(t)$, respectively. System (4) written using the polar coordinates (r, θ) reads

$$\dot{r} = r(r^2 + a_0), \quad \dot{\theta} = -1. \tag{5}$$

and can be explicitly solved. Since $\dot{\theta} \neq 0$, the origin $r = 0$ is the only critical point. Since $\dot{\theta} < 0$ the flow is always clockwise. The phase portrait of (4) for $a_0 < 0$, $a_0 = 0$ and $a_0 > 0$ is shown by Figure 1. Viewing a_0 as the bifurcation parameter, we point out the following cases in which the stability of the origin changes, i.e. the bifurcation occurs:

(i) For $a_0 > 0$ the origin is unstable strong focus and for all trajectories we have $(t_{min}, t_{max}) = (-\infty, t_0)$ with $t_{max} = t_0 > 0$.

(ii) For $a_0 = 0$ the origin is unstable weak focus, and $(t_{min}, t_{max}) = (-\infty, t_0)$, $t_0 > 0$.

(iii) For $a_0 < 0$ the origin is a stable strong focus. The limit cycle is the circle of radius $r = \sqrt{-a_0}$, which is unstable. For trajectories inside the circle the solutions

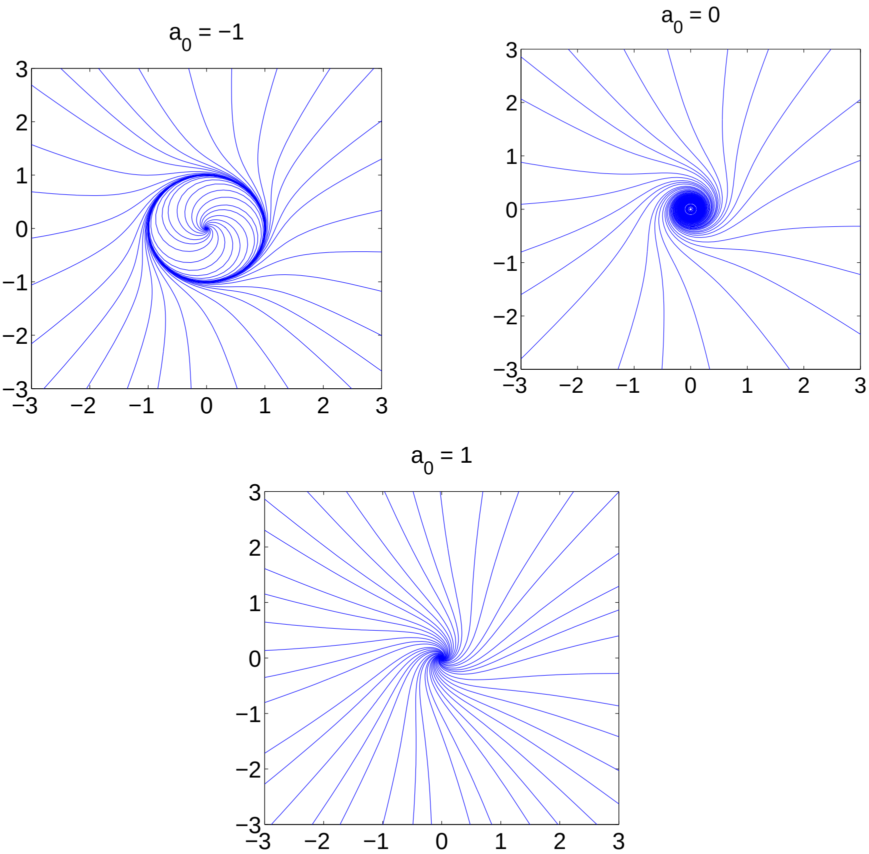

FIGURE 1. Phase portrait of the classical Hopf bifurcation system (4).

are global, i.e. $(t_{min}, t_{max}) = (-\infty, \infty)$, while for those outside the circle we have $(t_{min}, t_{max}) = (t_0, \infty)$ with $t_0 < 0$.

Next, let us consider the Hopf bifurcation in a complex system (3), but in finite-dimensional phase space. We start with two-dimensional complex system

$$\dot{z}_j = -i\lambda_j z_j - \gamma z_j(|z_1|^2 + |z_2|^2 + a_0),\ \ j = 1, 2, \tag{6}$$

with the corresponding initial conditions $z_j(0) = z_{j0}$, where a_0 is the bifurcation parameter. To simplify, for the moment we assume that $\gamma \in \mathbb{R}$ and $\gamma > 0$. Near $a_0 = 0$ we have the qualitative change of the behavior of the system. Indeed, using the polar coordinates the system (6) can be written as

$$\dot{r}_j = \gamma r_j(r_1^2 + r_2^2 + a_0), \quad \dot{\theta}_j = -\lambda_j,\ \ j = 1, 2,$$

where $z_j = r_j \exp(i\theta_j)$. For $a_0 < 0$ an invariant set, 3-sphere

$$\sqrt{-a_0}S^3 = \{(z_1, z_2) \in \mathbb{R}^4 \colon |z_1|^2 + |z_2|^2 = -a_0, \} \tag{7}$$

in $\mathbb{R}^4$ is born, where S^3 is the unit sphere in $\mathbb{R}^4$. In analogy with the case of $\mathbb{R}^2$, we call this phenomenon the Hopf bifurcation in $\mathbb{R}^4$. Clearly, if $a_0 \to 0^-$, then the sphere shrinks to the origin.

Moreover, if $r_1^2 + r_2^2 < -a_0$, then $\dot{r}_j < 0$, and $r(t)$ is decreasing as $t \to \infty$. The α-limit set of the trajectory is the origin, while the ω-limit set is a subset of the sphere (7). More precisely, the corresponding limit set is the subset of the 2-*torus*, contained in the 3-sphere (7). The torus has the form

$$r_1 S^1 \times r_2 S^1 \subset \sqrt{-a_0}S^3,$$

where $r_{1,2}$ are positive scaling numbers depending on the initial point, $r_1^2 + r_2^2 = -a_0$, and S^1 is the unit circle in the complex plane $\mathbb{C}$. Clearly, if $r_1^2 + r_2^2 > -a_0$, and $r(t)$ monotonically increases, then the corresponding trajectories converge to the sphere and their ω-limit set is a subset of a 2-torus contained in the 3-sphere (7).

In the similar way the Hopf bifurcation for the following system of k complex equations:

$$\dot{z}_j = -i\lambda_j z_j - \gamma z_j(|z_1|^2 + \cdots + |z_k|^2 + a_0), \ j = 1, 2, \ldots, k, \tag{8}$$

can be understood. An *invariant* $(2k-1)$*-sphere is born* in $\mathbb{C}^k$ when $a_0 < 0$, defined by $|z_1|^2 + \cdots + |z_k|^2 = -a_0$. For the general theory of bifurcation problems in finite-dimensional dynamical systems see Guckenheimer and Holmes [13].

1.3. **Hopf bifurcation in an infinite-dimensional phase space.** Since we are interested in the phenomenon of the Hopf bifurcation in an infinite-dimensional complex system (3), we start with the description of the Hopf bifurcation for an ODE defined in a Banach space X (real or complex):

$$\dot{u} = F(u, a_0), \quad u(0) = u_0. \tag{9}$$

Here $F : D \times \mathbb{R} \to X$ is a given map, where D is a dense subspace of X, and the initial value $u_0 \in D$ is prescribed. For our purposes the mapping F is usually of the form

$$F(u, a_0) = Au + f(u, a_0),$$

where $A : D \subseteq X \to X$ is a second order differential operator, $f \colon X \times \mathbb{R} \to X$ continuous such that $f(u, 0) = o(u)$ as $u \to 0$, and a_0 is the bifurcation parameter. Additionally we assume that for all a_0 we have $F(0, a_0) = 0$ so that $u = 0$ is an equilibrium point. For each initial point u_0 in a neighbourhood of $0 \in D$ let a trajectory

$$\Gamma(u_0) = \{u(t) \in X \colon t \in (t_{min}, t_{max}), \ u(0) = u_0\} \tag{10}$$

of (9) be defined on an unbounded interval (t_{min}, t_{max}) containing the origin.

Definition 1. We say that $a_0 = 0$ is the point of *Hopf bifurcation* for the system (9) if the following conditions are fulfilled:

(i) For $a_0 \geq 0$ small enough the system is unstable near the origin, and for u_0 from a neighbourhood of the origin we have $(t_{min}, t_{max}) = (-\infty, t_0)$, $t_{max} = t_0 > 0$.
(ii) For $a_0 < 0$ with $|a_0|$ small enough, the origin is stable, and an unstable invariant set $S(a_0) \subset X$ for (9) is born near the origin. There exists an open neighbourhood $U(a_0)$ of the origin such that its boundary is $S(a_0)$, and for any $u_0 \in \overline{U(a_0)}$ the corresponding solution is global, i.e. $(t_{min}, t_{max}) = (-\infty, \infty)$, while for $u_0 \in B \setminus \overline{U(a_0)}$ we have that $(t_{min}, t_{max}) = (-\infty, t_0)$ with $t_0 > 0$.

Definition 1 is analogous if the signs of a_0 are reversed, in which case the stability should be reversed as well as semi-infinite time intervals. Somewhat loosely we say that the Hopf bifurcation occurs if an invariant set is born near the origin when a_0 passes through the value of 0, and the origin changes from stable to unstable or vice versa.

Returning to the NLS Cauchy problem (1), the Hopf bifurcation consists in the *birth of an invariant sphere* in the space $L^2(\Omega)$ when $a_0 < 0$. Here we introduce some notation. For $a_0 < 0$ we define $R = \sqrt{-a_0}$ and let

$$B_R(0) = \{v \in L^2(\Omega) \colon \|v\|_{L^2(\Omega)} < R\}, \quad S_R(0) = \partial B_R(0), \tag{11}$$

be the ball and the sphere in $L^2(\Omega)$, respectively. In the same way, when $a_0 < 0$ for the corresponding system (3) in $\ell_2(\mathbb{C})$, the Hopf bifurcation consists in the *birth of an invariant sphere* in the space $\ell_2(\mathbb{C})$. Since (8) is also a special case of the NLS Cauchy problem (1), corresponding to the case when $u_0 \in \operatorname{span}\{\varphi_1, \ldots, \varphi_k\}$, we see that we can expect trajectories of (1) in the Hilbert space $L^2(\Omega, \mathbb{C})$, which oscillate at infinitely many scales. We can achieve this by choosing u_0 so that $\langle u_0, \varphi_j \rangle \neq 0$ for infinitely many j's.

Furthermore, we are interested in Hopf bifurcations from the point of view of fractal geometry. For that purpose, in the next subsection firstly we review some standard notation and definitions from fractal geometry and Sobolev spaces.

1.4. **Notation and definitions.** Let A be a bounded set in $\mathbb{R}^k$, and let $d(x, A)$ be Euclidean distance from x to A. Then the *Minkowski sausage* A_ε is $A_\varepsilon := \{y \in \mathbb{R}^k d(y, A) < \varepsilon\}$, a term coined by B. Mandelbrot. By *lower s-dimensional Minkowski content* of A, $s \geq 0$, we mean the following:

$$\mathcal{M}_*^s(A) := \liminf_{\varepsilon \to 0} \frac{|A_\varepsilon|}{\varepsilon^{k-s}}, \tag{12}$$

where $|\cdot|$ is N-dimensional Lebesgue measure. Analogously for the *upper s-dimensional Minkowski content* of A. The lower box dimension of A is defined by

$$\underline{\dim}_B A = \inf\{s > 0 : \mathcal{M}_*^s(A) = 0\},$$

and analogously the upper box dimension $\overline{\dim}_B A$. For various properties of fractal dimensions see Falconer [11].

If A is such that $\underline{\dim}_B A = \overline{\dim}_B A$, then the common value is denoted by $d := \dim_B A$, and is called the box dimension of A. Furthermore, if both the upper and lower d-dimensional Minkowski contents of A are different from 0 and ∞, we say that the set A is *Minkowski nondegenerate*. If in addition to this we have $\mathcal{M}^d_*(A) = \mathcal{M}^{*d}(A) =: \mathcal{M}^d(A) \in (0, \infty)$, then A is said to be Minkowski measurable. The notion of Minkowski content appears for example in the study of fractal drums and fractal strings, see He and Lapidus [14], Lapidus and Frankenhuysen [18]. Furthermore, Minkowski content is essential for understanding some singular integrals, see [25].

If A is a subset of an infinite-dimensional vector space X, we say that $\dim_B A = \infty$ if there exists an increasing sequence of finite-dimensional subspaces X_k of X such that $\dim_B(A \cap X_k) \to \infty$ as $k \to \infty$. We need this definition in Theorem 6(b).

We deal with the Sobolev spaces $H^1_0(\Omega, \mathbb{C})$ and $H^1_0(\Omega, \mathbb{C}) \cap H^2(\Omega, \mathbb{C})$ (in the sequel we omit $\mathbb{C}$) equipped with the corresponding norms defined by

$$\|u_0\|^2_{H^1_0} = \sum_j \lambda_j |\langle u_0, \varphi_j \rangle|^2, \quad \|u_0\|^2_{H^1_0 \cap H^2} = \sum_j \lambda_j^2 |\langle u_0, \varphi_j \rangle|^2. \tag{13}$$

See Henry [1, 15].

All the results of this paper hold if in (1) we have $-\Delta$ instead of Δ. The j-th component of trajectory, viewed as a spiral in $\mathbb{C}$, only changes the orientation for each j from negative to positive, see (18) below.

2. Well-posedness and stability of solutions

2.1. **Explicit solutions of the NLS problem.** We consider the NLS initial-boundary value problem (1). For some special values of parameters γ and a_0 it is possible to find explicit solutions of problem (1) and then to consider their qualitative properties. In order to calculate directly the explicit solutions of problem (1), we need the following lemma.

Lemma 2. *Let u be a solution of the NLS Cauchy problem* (1) *in the form* (2). *Furthermore, let $\rho(t) = \|u(t)\|_{L^2}$, $z_j(t) = r_j(t)\exp(i\theta_j(t))$, and $V(\rho) = \rho^2 + a_0$. Then*

$$\begin{aligned} \dot\rho &= -\gamma_1 \rho V(\rho) \\ \dot r_j &= -\gamma_1 r_j V(\rho),\ j \in \mathbb{N} \\ \dot\theta_j &= -\lambda_j - \gamma_2 V(\rho),\ j \in \mathbb{N}, \end{aligned} \tag{14}$$

where $\gamma = \gamma_1 + i\gamma_2$. Furthermore, for $u_0 \neq 0$ we have

$$r_j(t) = \frac{|\langle u_0, \varphi_j \rangle|}{\|u_0\|_{L^2}} \rho(t). \tag{15}$$

Proof. In Section 1, we showed that the NLS Cauchy problem (1) can be written in the form on an infinite-dimensional ODE system

$$\dot z_j = -i\lambda_j z_j - \gamma V(\rho) z_j, \quad j \in \mathbb{N}, \tag{16}$$

where $z_j = r_j \exp(i\theta_j)$. The expressions $(14)_2$ and $(14)_3$ follow easily by multiplying the expression (16) by $\exp(-i\theta_j)$. The first equation in (14) we get directly from the formal calculation. Namely,

$$\begin{aligned} 2\rho\dot{\rho} &= \frac{d}{dt}\int_\Omega |u(t)|^2 dx = \int_\Omega \frac{d}{dt}(u(t)\overline{u}(t))\,dx = \int_\Omega (\dot{u}(t)\overline{u}(t) + u(t)\dot{\overline{u}}(t))\,dx \\ &= i\int_\Omega (\Delta u(t)\,\overline{u}(t) - u(t)\,\Delta\overline{u}(t))\,dx - 2\gamma_1 \int_\Omega |u(t)|^2 dx \cdot V(\|u(t)\|_{L^2}) \\ &= -2\gamma_1\rho^2 V(\rho). \end{aligned}$$

By dividing the first two expressions given in (14) we obtain

$$\frac{\dot{r}_j}{r_j} = \frac{\dot{\rho}}{\rho},$$

hence $r_j = C_j\rho$, where C_j is a constant depending on the initial value u_0. Using the expression (2), we obtain that

$$c_j = \frac{|\langle u_0, \varphi_j\rangle|}{\|u_0\|_{L^2}} e^{-i\theta_j(0)},$$

where $\langle u_0, \varphi_j\rangle = \int_\Omega u_0\varphi_j dx \in \mathbb{C}$. □

Next, we use the results given by lemma 2 in order to find the explicit solutions of the NLS Cauchy problem (1) for some special values of the parameters γ and a_0. We use the notation $\rho_0 = \|u_0\|_{L^2}$. For the sake of simplicity we take $\gamma_2 = 0$, i.e. $\gamma \in \mathbb{R}$.

Case 1: $a_0 \in \mathbb{R}$, $\gamma_1 = 0$.

¿From lemma 2 directly follows that for $\gamma_1 = 0$ it holds $\|u(t)\|_{L^2} = C(u_0)$, and moreover, $r_j(t) = C_j(u_0)$ for each $j \in \mathbb{N}$.

Case 2: $a_0 = 0$, $\gamma_1 \neq 0$.

In this case $V(\rho) = \rho^2$ and the explicit solution of equation $(14)_1$ is given by

$$\rho(t) = (2\gamma_1 t + \rho_0^{-2})^{-1/2}. \tag{17}$$

¿From (15) and $(14)_3$ using (17), and noting $\theta_j(0) = \arg\langle u_0, \varphi_j\rangle$ we obtain

$$\begin{cases} r_j(t) = \frac{|\langle u_0,\varphi_j\rangle|}{\rho_0}(2\gamma_1 t + \rho_0^{-2})^{-1/2}, \\ \theta_j(t) = -\lambda_j t - \frac{\gamma_2}{2\gamma_1}\ln|2\gamma_1 t + \rho_0^{-2}| + \arg\langle u_0, \varphi_j\rangle, \end{cases} \tag{18}$$

where $z_j(t) = r_j(t)\exp(i\theta_j(t))$.

In this way, using the decomposition (2) we have the explicit solution of NLS problem (1)

$$u(t,x) = (2\gamma_1 t + \|u_0\|_{L^2}^{-2})^{-1/2} \sum_{j=1}^{\infty} \frac{\langle u_0, \varphi_j\rangle}{\|u_0\|_{L^2}}\, e^{-i\lambda_j t}\,\varphi_j(x), \tag{19}$$

where u_0 is a given initial function. Here we notice that the sign of the parameter γ_1 affects the maximal interval of the solution. More precisely, following the terminology introduced in Cazenave [2, Remark 3.1.6(ii)], we say that the solutions given by formula (19) are *positively (negatively) global* if $\gamma_1 > 0$ ($\gamma_1 < 0$). The solutions are global for $\gamma_1 = 0$ for any initial value u_0, while for $\gamma_1 \neq 0$ the global solution exists only when $u_0 = 0$.

Case 3: $a_0 \neq 0$, $\gamma_1 \neq 0$.

In this case one has $V(\rho) = \rho^2 + a_0$ and we obtain the Bernoulli equation $\dot{\rho} = -\gamma_1 \rho V(\rho)$ with the solution

$$\rho(t) = [(\rho_0^{-2} + a_0^{-1})e^{(2\gamma_1 a_0 t)} - a_0^{-1})]^{-1/2}. \tag{20}$$

We obtain the analogous series representation of solution $u(t,x)$ of (1) as in (19), assuming again that $\gamma_2 = 0$:

$$u(t,x) = ((\rho_0^{-2} + a_0^{-1})e^{2\gamma_1 a_0 t} - a_0^{-1})^{-1/2} \sum_{j=1}^{\infty} \frac{\langle u_0, \varphi_j \rangle}{\|u_0\|_{L^2}} e^{-i\lambda_j t} \varphi_j(x). \tag{21}$$

The ability to calculate the explicit solution of NLS boundary-value problem (1), given by formulas (19) and (21), guarantees its uniqueness. More precisely, it is obvious that the following result is true.

Proposition 3. *Let $u_0 \in L^2(\Omega)$ be the initial function for NLS initial-boundary value problem (1), where $a_0 \in \mathbb{R}$ is the bifurcation parameter. For any $u_0 \in L^2(\Omega)$, NLS initial-boundary value problem (1) possesses a unique solution of the form (2), with $z_j(t) \in \ell_2(\mathbb{C})$ and $z_j(\cdot)$ of class C^1. Moreover, the solution is represented by formula* (19) *and* (21) *for $a_0 = 0$ and $a_0 \neq 0$, respectively.*

The explicit formulas (19) and (21) for the solution of the NLS problem (1) enable us to express the corresponding norms of the solution in the case when $u_0 \in H_0^1(\Omega)$ or $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, respectively. Namely, it follows

$$\|u(t)\|_{H_0^1}^2 = \rho(t)^2 \sum_j \lambda_j \frac{|\langle u_0, \varphi_j \rangle|^2}{\|u_0\|_{L^2}^2} = \rho(t)^2 \frac{\|u_0\|_{H_0^1}^2}{\|u_0\|_{L^2}^2}.,$$

where $\rho(t)$ is defined by (17) or (20) if $a_0 = 0$ or $a_0 \neq 0$ respectively. Similarly for $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ we have

$$\|u(t)\|_{H_0^1 \cap H^2} = \rho(t) \frac{\|u_0\|_{H_0^1 \cap H^2}}{\|u_0\|_{L^2}}.$$

Next, let us consider formula (17) again. It is obvious that for $t \to t_{min}^+$ where $t_{min} = -(2\gamma_1 \rho_0^2)^{-1}$ one has $\|u(t)\|_{L^2(\Omega)} \to \infty$, i.e. we have the blow up of the solution. Similar conclusion holds for $a_0 \neq 0$ and the expression (20).

Finally, we note that directly from formulas (19) and (21) follows the invariance property of the solutions of the NLS problem. More precisely, if U_0 is the span of a given subset of $\{\varphi_j : j \geq 1\} \subset L^2(\Omega)$, then the assumption $u_0 \in \overline{U}_0$ implies that $u(t) \in \overline{U}_0$ for all $t > 0$, where the closure is taken in $L^2(\Omega)$. Moreover, the invariance property can be reformulated as follows: each subspace $\overline{U}_0$, where U_0 is spanned by a subset of φ_j-s, is invariant for the nonlinear evolution operator $T(t)$, $T(t)u_0 = u(t)$, see (19), associated with the problem (1): $T(t)\overline{U}_0 \subseteq \overline{U}_0$. In other words, a trajectory that starts in $\overline{U}_0$ remains in this space forever. In particular, if $u_0 \in \operatorname{span}\{\varphi_1, \dots, \varphi_k\}$, then $\Gamma(u_0) \subset \operatorname{span}\{\varphi_1, \dots, \varphi_k\}$. This means that if $\langle u_0, \varphi_j\rangle = 0$ for all but finitely many j's, then (1) is essentially a finite-dimensional problem, which can be viewed as (8).

2.2. **Well posedness and stability.** The following proposition gives some stability results of the solution of NLS boundary-value problem (1) with respect to the value of the bifurcation parameter a_0. Again, for the sake of simplicity we assume that $\gamma_2 = 0$ and $\gamma_1 > 0$. For $\gamma_1 < 0$ time intervals of the form (t_{min}, ∞) should be changed to $(-\infty, t_{max})$. The notation used in the following proposition (the ball and the sphere in $L^2(\Omega)$) we introduced in (11). Directly from the expressions (17) and (20) follow the power and the exponential rate of the convergence of $\|u(t)\|_{L^2}$ to the origin in $L^2(\Omega)$ (as the fixed point for problem (1)), respectively. More precisely, the following proposition holds.

Proposition 4. (Hopf bifurcation for Schrödinger problem) *Assume that $\gamma = \gamma_1 > 1$ and let $u_0 \in L^2(\Omega)$ be the initial function for NLS initial-boundary value problem (1), where $a_0 \in \mathbb{R}$ is the bifurcation parameter.*

(i) *For $a_0 > 0$ then the origin is exponentially stable with respect to L^2-topology for any $u_0 \neq 0$.*
(ii) *For $a_0 = 0$ the origin is power stable in $L^2(\Omega)$, with power $\alpha = 1/2$.*
(iii) *For $a_0 < 0$ let us denote $R = \sqrt{-a_0}$. Then we distinguish the following cases:*
   (a) *If $u_0 \in B_R(0)$ then the origin is exponentially unstable and the solutions are global with $\|u(t)\|_{L^2} \to R$ as $t \to \infty$.*
   (b) *If $u_0 \in L^2(\Omega) \setminus B_R(0)$ then the solution $u(t)$ is positively global and $\|u(t)\|_{L^2} \to R$ as $t \to \infty$.*
   (c) *If $u_0 \in S_R(0)$ then also $u(t) \in S_R(0)$, so the sphere $S_R(0)$ is an invariant attractor.*

The continuous dependence on the initial condition, regularity and the continuous dependence on bifurcation parameter for NLS boundary-value problem (1) when $u_0 \in L^2(\Omega)$ is given by the following proposition. Here, we point out that the same qualitative properties valid if $u_0 \in H_0^1(\Omega)$ or $H_0^1(\Omega) \cap H^2(\Omega)$. The interval of the existence of $u(t)$ we note by $I = (t_{min}, t_{max})$.

Proposition 5. *(a)* (continuous dependence on initial condition) *If $v_0 \to u_0$ in $L^2(\Omega)$ then for each fixed $t \in I$ we have that $v(t) \to u(t)$ in $L^2(\Omega)$. Moreover, the convergence is uniform on each compact interval in I.*

(b) (regularity) *If $u_0 \in L^2(\Omega)$ then $u \in C(I, L^2(\Omega))$.*
(c) (continuous dependence on bifurcation parameter) *Let $u_0 \in L^2(\Omega)$ be fixed, and let $u(t)$ and $u_{a_0}(t)$ be defined by (19) and (21) respectively. Then for any $t \in I$,*

$$\|u_{a_0}(t) - u(t)\|_{L^2} \to 0 \text{ as } a_0 \to 0,$$

where I is the interval of existence of $u(t)$, Moreover, the convergence is uniform on compact intervals J contained in I.

Proof. (a) If $v_0 = 0$ then the claim follows easily from proposition 4. Let $v_0 \neq 0$ and $u_0 \to v_0$. Denoting $\hat{u}_0 = u_0/\|u_0\|_{L^2}$ and using $|a+b|^2 \leq 2|a|^2 + 2|b|^2$ for $a, b \in \mathbb{C}$, we obtain:

$$\begin{aligned}
\|u(t) - v(t)\|_{L^2}^2 &= \sum_j |\langle \hat{u}_0, \varphi_j\rangle \rho_{u_0}(t) - \langle \hat{v}_0, \varphi_j\rangle \rho_{v_0}(t)|^2 \\
&= \sum_j |(\langle \hat{u}_0, \varphi_j\rangle - \langle \hat{v}_0, \varphi_j\rangle)\rho_{u_0}(t) + \langle \hat{v}_0, \varphi_j\rangle(\rho_{u_0}(t) - \rho_{v_0}(t))|^2 \\
&\leq 2\rho_{u_0}(t)^2 \sum_j |\langle \hat{u}_0 - \hat{v}_0, \varphi_j\rangle|^2 + 2|\rho_{u_0}(t) - \rho_{v_0}(t)|^2 \sum_j |\langle \hat{v}_0, \varphi_j\rangle|^2 \\
&= 2\rho_{u_0}(t)^2 \|\hat{u}_0 - \hat{v}_0\|_{L^2}^2 + 2|\rho_{u_0}(t) - \rho_{v_0}(t)|^2 \|\hat{v}_0\|_{L^2}^2.
\end{aligned}$$

Therefore, since $u_0 \to v_0$ in $L^2(\Omega)$ it follows that for each fixed t we have $\|u(t) - v(t)\|_{L^2} \to 0$.

(b) Assuming that $u_0 \in L^2(\Omega)$, we first write $u(t) = \rho(t)\mathcal{S}(t)$, where we note $\mathcal{S}(t) = \sum_{j=1}^{\infty} \frac{\langle u_0, \varphi_j\rangle}{\|u_0\|_{L^2}} e^{-i\lambda_j t}\varphi_j(x)$. We have $u(t) - u(s) = (\rho(t) - \rho(s))\mathcal{S}(t) + \rho(s)(\mathcal{S}(t) - \mathcal{S}(s))$, and from this

$$\|u(t) - u(s)\|_{L^2}^2 \leq 2(\rho(t) - \rho(s))^2 + 2\rho(s)^2 \sum_{j=1}^{\infty} |\langle \hat{u}_0, \varphi_j\rangle|^2 |e^{-i\lambda_j t} - e^{-i\lambda_j s}|^2. \tag{22}$$

Since $\rho(t)$ is uniformly continuous, one has that $\rho(t) \to \rho(s)$ for $t \geq s$ and it suffices to show that, for given ε the expression

$$\sum_{j=1}^{\infty} |\langle \hat{u}_0, \varphi_j\rangle|^2 |e^{-i\lambda_j t} - e^{-i\lambda_j s}|^2$$

can be made less than ε if $|t - s|$ is small enough. For any $m > 1$, the sum is less than or equal to:

$$\sum_{j=1}^{m} |e^{-i\lambda_j t} - e^{-i\lambda_j s}|^2 + 4\sum_{j=m}^{\infty} |\langle \hat{u}_0, \varphi_j\rangle|^2,$$

since $|\langle \hat{u}_0, \varphi_j\rangle| \leq 1$ and $|e^{-i\lambda_j t} - e^{-i\lambda_j s}| \leq 2$. We choose $m = m(u_0, \varepsilon)$ large enough so that the second sum is $\leq \varepsilon/8$. It is clear that the first sum can be made $\leq \varepsilon/2$ when $|t - s| \leq \delta$ for $\delta = \delta(\varepsilon, m) > 0$ small enough.

(c) The claim follows from $\|u_{a_0}(t) - u(t)\|_{L^2} = |\rho_{a_0}(t) - \rho(t)| \to 0$ uniformly in $t \in J$ as $a_0 \to 0$, since $a^{-1}[\exp(2\gamma_1 a_0 t) - 1] \to 2\gamma_1 t$ uniformly in $t \in J$ as $a_0 \to 0$. □

3. Compactness and non-rectifiability of trajectories

3.1. **Compactness.** Since we consider trajectories in infinite-dimensional spaces, it is not at all clear if they are compact sets, or even rectifiable. Using the notation given in section 1.2, the solution of NLS problem (1) can be written in the following way

$$u(t) = \rho(t) \sum_j \langle \hat{u_0}, \varphi_j \rangle e^{-i\lambda_j t} \varphi_j(x), \tag{23}$$

where u_0 is a given initial function. By $\Gamma(u_0) = \{u(t) \colon t \geq t_0\}$ we note the trajectories of solutions. The notation (23) enables us to reduce the problem of compactness of the trajectories to the problem of the compactness of the bounded sets in $\ell_2(\mathbb{C})$. In this sense, we review the following characterization of relatively compact sets in $\ell_2(\mathbb{C})$ (Weidmann [24, p. 135]):

A subset Y of $\ell_2(\mathbb{C})$ is relatively compact in $\ell_2(\mathbb{C})$ if and only if Y is bounded and for every $\varepsilon > 0$ there exists $j_0 \in \mathbb{N}$ such that for all sequences (f_j) in Y the following condition is fulfilled:

$$\sum_{j \geq j_0} |f_j|^2 \leq \varepsilon. \tag{24}$$

The following theorem establishes not only compactness of individual trajectories $\Gamma(u_0)$ of solutions of the NLS Cauchy problem (1) in $L^2(\Omega)$, but also for some bundles of trajectories. If a trajectory generated by u_0 is positively global, defined for $t \in (t_{min}, \infty)$, then by $\Gamma(u_0)$ we denote its part corresponding to $t \geq 0$. We do analogously for negatively global trajectories.

For a given nonempty base set of initial functions $A \subset L^2(\Omega)$ we can define the corresponding bundle of trajectories by

$$\Gamma(A) = \bigcup_{v_0 \in A} \Gamma(v_0).$$

The following theorem provides some sufficient conditions on A that ensure compactness of $\Gamma(A)$. To simplify, we assume that $a_0 = 0$ and $\gamma_1 > 0$ in (1).

Theorem 6. *(a) For any $u_0 \in L^2(\Omega)$ the corresponding trajectory $\Gamma(u_0)$ of (1) starting with $t_0 = 0$ is relatively compact in $L^2(\Omega)$.*

(b) Let $u_0 \in L^2(\Omega)$ be given and define

$$A(u_0) = \{v_0 \in L^2(\Omega) : |\langle v_0, \varphi_j \rangle| \leq |\langle u_0, \varphi_j \rangle|, \ \forall j\}.$$

Then we have $A(u_0) = \Gamma(A(u_0))$, and this set is compact in $L^2(\Omega)$.

(c) If $u_0, v_0 \in L^2(\Omega)$ are given, then the bundle $\Gamma([u_0, v_0])$ generated by the line segment $[u_0, v_0] = \{(1-\lambda)u_0 + \lambda v_0 : 0 \leq \lambda \leq 1\}$ is relatively compact in $L^2(\Omega)$. More generally, if A is a finite set in $L^2(\Omega)$ and $\operatorname{conv} A$ its convex hull, then $\Gamma(\operatorname{conv} A)$ is relatively compact.

Proof. (a) Let $u_0 \neq 0$ be a fixed initial function (for $u_0 = 0$ the claim is trivial, since $\Gamma(0) = \{0\}$). The trajectory $\Gamma(u_0)$ is identified with the corresponding trajectory in $\ell_2(\mathbb{C})$, that is

$$u(t) = \sum_j z_j(t)\varphi_j \ \mapsto \ (z_j(t))_j \in \ell_2(\mathbb{C}).$$

where $u(t)$ is generated by u_0 and

$$z_j(t) = \rho(t)\langle \hat{u_0}, \varphi_j\rangle e^{-i\lambda_j t}.$$

Now, we show that the set $\{(z_j(t))_j : t \geq 0)\}$ is relatively compact in $\ell_2(\mathbb{C})$. Using (18) and (17), for all $t \geq 0$ we have that

$$\begin{aligned} \sum_{j\geq j_0} |z_j(t)|^2 &= \rho^2(t)\sum_{j\geq j_0} r_j(t)^2 = \frac{1}{1+2\gamma_1\rho_0^2 t}\sum_{j\geq j_0} |\langle u_0, \varphi_j\rangle|^2 \\ &\leq \sum_{j\geq j_0} |\langle u_0, \varphi_j\rangle|^2 \leq \varepsilon, \end{aligned}$$

that is, condition (24) is fulfilled provided j_0 is large enough. The case of $a_0 \neq 0$ is treated similarly.

(b) Let us first prove that $A(u_0) = \Gamma(A(u_0))$. The inclusion $\subseteq$ is clear. To prove the converse inclusion, let $v(t) \in \Gamma(A(u_0))$, where $v(t)$ is a trajectory generated by $v_0 \in A(u_0)$. Then

$$\begin{aligned} |\langle v(t), \varphi_j\rangle| &= |\rho(t)\langle \hat{v}_0, \varphi_j\rangle| = \frac{\rho(t)}{\|v_0\|_{L^2}}|\langle v_0, \varphi_j\rangle| \\ &\leq |\langle v_0, \varphi_j\rangle| \leq |\langle u_0, \varphi_j\rangle|, \end{aligned}$$

where we used the monotonicity of $\rho(t)$ for $\gamma_1 > 0$, so that for $t \geq 0$ we have $\rho(t) \leq \rho(0) = \|u_0\|_{L^2}$. Hence $v(t) \in A(u_0)$.

The compactness of $A(u_0)$ is proved using a slight change in the proof of (a). Denoting by $A'(u_0)$ the set in $\ell_2(\mathbb{C})$ corresponding to $A(u_0)$ in $L^2(\Omega)$, then

$$A'(u_0) = \prod_j \overline{B_{r_j}(0)}, \tag{25}$$

where $r_j = |\langle u_0, \varphi_j\rangle|$ and $B_{r_j}(0)$ is the open disk of radius r_j in $\mathbb{C} = \mathbb{R}^2$ imbedded into the j-th component of $\ell_2(\mathbb{C})$. To prove (25), it suffices to note that if $v = \sum_j z_j\varphi_j$, then $z = (z_j)_j \in A(u_0)$ if and only if $z_j \in \overline{B_{r_j}(0)}$ for all j.

(c) Let $r_j = |\langle u_0, \varphi_j\rangle|$ and $s_j = |\langle v_0, \varphi_j\rangle|$. Defining $w = \sum_j \max\{r_j, s_j\}\varphi_j$ it is clear that $w \in L^2(\Omega)$ since

$$\sum_j \max\{r_j, s_j\}^2 \leq \sum_j (r_j^2 + s_j^2) = \|u_0\|_{L^2}^2 + \|v_0\|_{L^2}^2 < \infty.$$

Let us show that $[u_0, v_0] \subset A(w)$. Indeed, taking $v \in [u_0, v_0]$, for any j we have

$$|\langle v, \varphi_j\rangle| = |\langle (1-\lambda)u_0 + \lambda v_0, \varphi_j\rangle| \leq (1-\lambda)r_j + \lambda s_j \leq \max\{r_j, s_j\} = |\langle w, \varphi_j\rangle|.$$

Since $\Gamma([u_0, v_0]) \subset \Gamma(A(w))$ and due to the result given by (b), one concludes that the bundle $\Gamma([u_0, v_0])$ is compact in $L^2(\Omega)$. □

Remark 7. Similar results like those presented in Theorem 6 valid if $u_0 \in H_0^1(\Omega)$ or $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. More precisely, in then the trajectory $\Gamma(u_0)$ is relatively compact in $H_0^1(\Omega)$ and $H_0^1(\Omega) \cap H^2(\Omega)$, respectively.

Indeed, the subspace $H_0^1(\Omega)$ of $L^2(\Omega)$ is isometrically isomorphic to the subspace ℓ_2' of $\ell_2(\mathbb{C})$, consisting of all sequences $z = (z_j)$ such that $\|z\|_{\ell_2'}^2 = \sum_j \lambda_j |z_j|^2 < \infty$. It is easy to see that the analogous characterization of compact sets as for ℓ_2 in (a) holds also in ℓ_2' with respect to the new norm. Similarly for the subspace ℓ_2'' corresponding to $H_0^1(\Omega) \cap H^2(\Omega)$, with the norm $\|z\|_{\ell_2''}^2 = \sum_j \lambda_j^2 |z_j|^2$.

3.2. **Rectifiability.** Assuming that $t_{max}(u_0) = \infty$ and $\|u(t)\|_{L^2} \to 0$ as $t \to \infty$, we define the length of trajectory $\Gamma(u_0)$ over the interval $[0, t)$ by

$$l(u_0, t) = \int_0^t \|\dot{z}(s)\|_{\ell_2}\, ds,$$

where $z(t) = (z_j(t))_j \in \ell_2(\mathbb{C})$ is isometrically assigned to $u(t) \in L^2(\Omega)$, generated by $u_0 \in L^2(\Omega)$, see (1), (2) and (3). We consider rectifiability and nonrecitifiability of $\Gamma(u_0)$ only for $t \in [0, \infty)$ or $t \in (-\infty, 0]$.

Moreover, we are interested in the asymptotic behaviour of $l(u_0, t)$ as $t \to \infty$. For that purpose, we introduce here some notation. We say that:

- $f(t) \sim g(t)$ as $t \to \infty$ if $\lim_{t\to\infty} \dfrac{f(t)}{g(t)} = 1$.
- $f(t) \simeq g(t)$ as $t \to \infty$ if $\lim_{t\to\infty} \dfrac{f(t)}{g(t)} < \infty$.

The following theorem states the dependence of the (non)-rectifiability of trajectories $\Gamma(u_0)$ near the origin of the NLS initial-boundary value problem (1) due to the value of the bifurcation parameter a_0.

Theorem 8. *Let $\Gamma(u_0)$ be the trajectory of the solution of the NLS initial-boundary value problem* (1) *and let be $\gamma_1 > 0$ and $u_0 \neq 0$. Each trajectory $\Gamma(u_0)$ of (1) near the origin is nonrectifiable for $a_0 = 0$ and rectifiable for $a_0 > 0$. Moreover, for $a_0 = 0$ and $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ we have the following asymptotic result:*

$$l(u_0, t) \sim \frac{\|u_0\|_{H_0^1 \cap H^2}}{\|u_0\|_{L^2}} (2\gamma_1^{-1} t)^{1/2} \quad \textit{as } t \to \infty. \tag{26}$$

In particular, $l(u_0, t) \simeq t^{1/2}$ as $t \to \infty$.

If $a_0 < 0$, then for $u_0 \in \mathcal{B}(R)$ the trajectory is rectifiable for $t \in (-\infty, 0]$, and nonrectifiable for $t \in [0, \infty)$. If $u_0 \notin \mathcal{B}(R)$, the trajectories are nonrectifiable. In both of these nonrectifiable cases we have $l(u_0, t) \simeq t$ as $t \to \infty$.

Proof. Assume that $a_0 = 0$ and $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$. For the sake of simplicity we assume that $\gamma_2 = 0$, so that $\dot{\theta}_j = -\lambda_j$, see (18) (the case of $\gamma_2 \neq 0$ can be treated with slight modifications of the proof). Using $z_j = r_j \exp(i\theta_j)$, $|\dot{z}_j|^2 = \dot{r}_j^2 + r_j^2 \dot{\theta}_j^2$, ant expressions (18) and (13), we obtain the following estimate for $l(u_0, t)$:

$$\begin{aligned} l(u_0,t) &= \int_0^t \left(\sum_{j=1}^{\infty} |\dot{z}_j|^2 \right)^{1/2} ds = \int_0^t \left(\sum_j \dot{r}_j^2 + \sum_j r_j^2 \dot{\theta}_j^2 \right)^{1/2} ds \\ &= \int_0^t \left(\gamma_1^2 (2\gamma_1 s + \rho_0^{-2})^{-3} + (2\gamma_1 s + \rho_0^{-2})^{-1} \frac{\|u_0\|^2_{H_0^1 \cap H^2}}{\rho_0^2} \right)^{1/2} ds \qquad (27) \\ &\geq \frac{\|u_0\|_{H_0^1 \cap H^2}}{\rho_0} \int_0^t (2\gamma_1 s + \rho_0^{-2})^{-1/2} ds, \end{aligned}$$

where $\rho_0 = \|u_0\|_{L^2(\Omega)}$.

On the other hand, using inequality $(a^2 + b^2)^{1/2} \leq |a| + |b|$, see (27), we obtain the estimate from below for $l(u_0, t)$ which reads

$$\begin{aligned} l(u_0,t) &\leq \gamma_1 \int_0^t (2\gamma_1 s + \rho_0^{-2})^{-3/2} ds + \frac{\|u_0\|_{H_0^1 \cap H^2}}{\rho_0} \int_0^t (2\gamma_1 s + \rho_0^{-2})^{-1/2} ds \\ &\leq \rho_0 + \frac{\|u_0\|_{H_0^1 \cap H^2}}{\rho_0} \int_0^t (2\gamma_1 s + \rho_0^{-2})^{-1/2} ds, \end{aligned}$$

The claim in (26) follows by direct computation. The case when $a_0 \neq 0$ is treated similarly, using (21). □

Remark 9. If $u_0 \in L^2(\Omega) \setminus (H_0^1(\Omega) \cap H^2(\Omega))$ then $l(u_0, t) = \infty$ for each $t > 0$. Indeed, in this case we have that $\|u_0\|_{H_0^1 \cap H^2} = \sum_j \lambda_j^2 |\langle u_0, \varphi_j \rangle|^2 = \infty$, and the claim follows from (27).

Remark 10. Let A be a bounded subset of $H_0^1(\Omega) \cap H^2(\Omega)$ such that $0 \notin \overline{A}$, with closure taken in this space. Let us consider the bundle of trajectories $\Gamma(A)$, and let us define lower and upper length of bundle $\Gamma(A)$ in time interval $[0, t)$ by:

$$\underline{l}(A,t) = \inf\{l(u_0,t) : u_0 \in A\}, \ \ \overline{l}(A,t) = \sup\{l(u_0,t) : u_0 \in A\}.$$

Assuming that $a_0 = 0$, $\gamma_1 > 0$, $u_0 \neq 0$, it can be shown by reconsidering the proof of Theorem 8 that

$$\underline{l}(A,t) \simeq t^{1/2}, \ \ \overline{l}(A,t) \simeq t^{1/2} \ \text{ as } t \to \infty.$$

4. Minkowski sequence associated to a trajectory

In this section we consider NLS boundary-initial value problem (1) for $a_0 = 0$. Let $\Gamma(u_0)$ be a trajectory generated by $u_0 \neq 0$, corresponding to $t \geq 0$ if the solution $u(t)$ is positively global, and to $t \leq 0$ if it is negatively global. Denote by $\Gamma_j(u_0)$ the orthogonal projection of $\Gamma(u_0) \subset L^2(\Omega)$ onto the φ_j-component, where φ_j are defined in section 1.2. Here, we recall that the solution of (1) can be written in form $u(t,x) = \sum_{j=1}^{\infty} z_j(t)\varphi_j(x)$. Now, $\Gamma_j(u_0)$ can be viewed as a curve in the complex plane defined by $z_j(t) = \langle u(t), \varphi_j \rangle = r_j(t)\exp(i\theta_j(t))$.

Figure 2 shows an example of $\Gamma_j(u_0)$, $j = 1,2,3,4$, where $z_j(t) = (t+1)^{-1/2} c_j e^{-i\lambda_j t}$ with $c_j = \langle u_0, \varphi_j \rangle$. For some chosen eigenvalues λ_j and Fourier coefficients c_j we plotted $z_j(t)$ on the time interval $[0,\infty)$. The trajectory $t \mapsto z(t)$ of the considered system in $\ell_2(\mathbb{C})$ is equal to a sequence $t \mapsto (z_j(t))_{j\in\mathbb{N}}$. In this way Figure 2 should indicate the projection of the considered trajectory into $\mathbb{C}^4$.

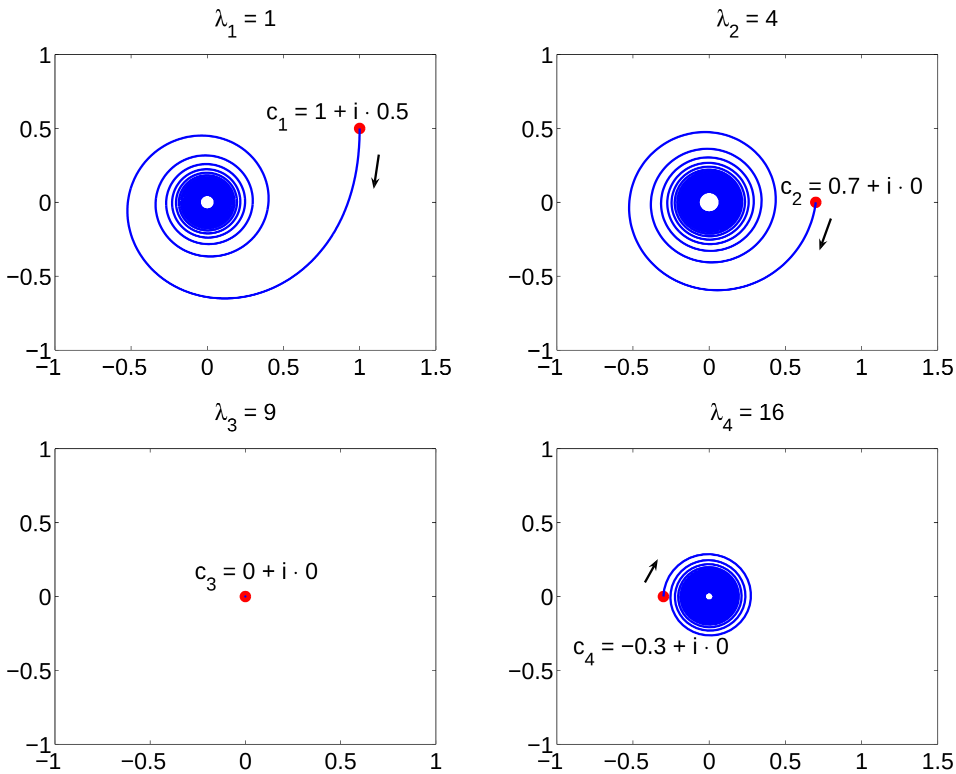

Figure 2. Projections $\Gamma_j(u_0)$ of $\Gamma(u_0)$, $j = 1,2,3,4$.

The following result describes some properties of the sequence of d-dimensional Minkowski contents of curves $\Gamma_j(u_0)$ for $d = 4/3$, $j \in \mathbb{N}$. Recall that if a set $A \subset \mathbb{R}^k$ is such that $d = \dim_B A > 0$, then $\mathcal{M}^s(A) = \infty$ for $0 \leq s < d$ and $\mathcal{M}^s(A) = 0$ for $s > d$.

Hence, since $\dim_B \Gamma_j(u_0) = 4/3$ for all j, see Theorem 11(a) below, it has sense to consider $\mathcal{M}^d(\Gamma_j(u_0))$ for $d = 4/3$ only.

The sequence $(\mathcal{M}^d(\Gamma_j(u_0)))_j$ will be called the *Minkowski sequence* associated to the trajectory $\Gamma(u_0)$. More generally, for any trajectory $\Gamma(u_0)$ in $L^2(\Omega)$ the corresponding Minkowski sequence is $(\mathcal{M}^{d_j}(\Gamma_j(u_0)))_j$, where $d_j = \dim_B \Gamma_j(u_0)$, provided the box dimension exists for each j. The following result deals with properties of the Minkowski sequence associated to the trajectory of the NLS initial-boundary value problem (1) at the point of the bifurcation. As the cruical step in the proof of Theorem 11 we use the result of Žubrinić and Županović [26, Theorem 6]. The simpler equivalent formulation can be found in [17, Theorem 3]. Because of the completeness, we briefly recall that result here.

Let Γ be a planar spiral defined in polar coordinates by $r = f(\theta)$, where $f(\theta)$ is decreasing to zero as $\theta \to \infty$, such that $f'(\theta)/(\theta^{-\alpha})' \to p$ as $\theta \to \infty$, $\alpha \in (0, 1)$, $p > 0$, and $|f''(\theta)| \le C\theta^{-\alpha}$. Then $\dim_B \Gamma = d$, where we defined $d = 2/(1 + \alpha)$, and

$$\mathcal{M}^d(\Gamma) = p^d \pi (\pi\alpha)^{-2\alpha/(1+\alpha)} \frac{1+\alpha}{1-\alpha}. \tag{28}$$

The asymptotic behaviour of the Minkowski sequence $(\mathcal{M}^d(\Gamma_j(u_0)))_j$ for $j \to \infty$ we obtain using the following well known asymptotic result for the eigenvalues of $-\Delta$, due to H. Weyl. More precisely,

$$\lambda_j \simeq j^{2/N}, \; j \to \infty, \tag{29}$$

that is, there exist positive constants a and b such that for all j, $a \le \lambda_j / j^{2/N} \le b$, see e.g. Mikhailov [19, Section IV.1.5] or Davies [9, Theorem 6.3.1].

Theorem 11. *Assume that $a_0 = 0$ in (1), and $\gamma_1 \neq 0$.*

(a) For any $u_0 \in L^2(\Omega)$ and j such that $\langle u_0, \varphi_j \rangle \neq 0$ we have $\dim_B \Gamma_j(u_0) = 4/3$. Moreover,

$$\mathcal{M}^{4/3}(\Gamma_j(u_0)) = 3\pi^{1/3} \left(\frac{\lambda_j |\langle u_0, \varphi_j \rangle|^2}{|\gamma_1| \, \|u_0\|_{L^2}^2} \right)^{2/3}. \tag{30}$$

In particular,

$$\mathcal{M}^{4/3}(\Gamma_j(u_0)) \le 3\pi^{1/3} \left(\frac{\lambda_j}{|\gamma_1|} \right)^{2/3}, \tag{31}$$

and equality is achieved if and only if $u_0 \in span\{\varphi_j\}$, $u_0 \neq 0$. Furthermore, we have the following asymptotic behaviour

$$\max_{u_0 \in L_2(\Omega)} \mathcal{M}^{4/3}(\Gamma_j(u_0)) \simeq j^{4/(3N)} \quad \text{as } j \to \infty. \tag{32}$$

(b) For any $u_0 \in H_0^1(\Omega)$, $u_0 \neq 0$, we have the following identity

$$\sum_{j=1}^{\infty}[\mathcal{M}^{4/3}(\Gamma_j(u_0))]^{3/2} = \frac{\sqrt{27\pi}}{|\gamma_1|}\left(\frac{\|u_0\|_{H_0^1}}{\|u_0\|_{L^2}}\right)^2. \tag{33}$$

In particular,

$$\sum_{j=1}^{\infty}[\mathcal{M}^{4/3}(\Gamma_j(u_0))]^{3/2} \geq \frac{\sqrt{27\pi}}{|\gamma_1|}\lambda_1, \tag{34}$$

and equality is achieved if and only if $u_0 \in \operatorname{span}\{\varphi_1\}$, $u_0 \neq 0$.

(c) For $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_0 \neq 0$, besides (33) we have the following identity:

$$\sum_{j=1}^{\infty}\lambda_j[\mathcal{M}^{4/3}(\Gamma_j(u_0))]^{3/2} = \frac{\sqrt{27\pi}}{|\gamma_1|}\left(\frac{\|u_0\|_{H_0^1\cap H^2}}{\|u_0\|_{L^2}}\right)^2, \tag{35}$$

and

$$\mathcal{M}^{4/3}(\Gamma_j(u_0)) = o(j^{-4/(3N)}) \quad \text{as } j \to \infty. \tag{36}$$

In particular,

$$\sum_{j=1}^{\infty}\lambda_j[\mathcal{M}^{4/3}(\Gamma_j(u_0))]^{3/2} \geq \frac{\sqrt{27\pi}}{|\gamma_1|}\lambda_1^2, \tag{37}$$

and equality is achieved if and only if $u_0 \in \operatorname{span}\{\varphi_1\}$, $u_0 \neq 0$.

Proof. (a) We consider the case $\gamma_1 > 0$ (for $\gamma_1 < 0$ the proof is analogous). After eliminating variable t from the system (18) one obtains

$$\begin{aligned} r_j &= \frac{|\langle u_0, \varphi_j\rangle|}{\rho_0}\left(2\gamma_1\frac{\arg\langle u_0, \varphi_j\rangle - \theta_j}{\lambda_j} + \rho_0^{-2}\right)^{-1/2} \\ &= m(-\theta_j + \theta_0)^{-1/2}, \end{aligned} \tag{38}$$

where $\theta_j \to -\infty$, $m = \dfrac{|\langle u_0, \varphi_j\rangle|}{\rho_0}\sqrt{\dfrac{\lambda_j}{2\gamma_1}}$, and θ_0 is a constant. The expression (38) enables to represent the spiral $\Gamma_j(u_0) \subset \mathbb{C}$ in polar coordinates (r_j, θ_j) in a form $r_j = f(\theta_j)$, where $f(\theta_j) = m(\theta_0 - \theta_j)^{-1/2}$. Now, the expression (30) follows directly from the result of Žubrinić and Županović [26, Theorem 6], which we briefly recalled at the beginning of this Section. We use the mentioned result in our situation by taking $\alpha = 1/2$. Direct calculation gives

$$\lim_{\theta_j \to \infty}\frac{f'(\theta_j)}{\theta_j^{-1/2}} = m,$$

and the expression (30) follows directly from (28). Furthermore, inequality (31) follows from (30) since $|\langle u_0, \varphi_j\rangle| \leq \|u_0\|_{L^2}$.

The last claim follows from (31) and the Weyl asymptotic result (29) for the eigenvalues of $-\Delta$ subject to zero boundary data.

(b) Note that if $\langle u_0, \varphi_j\rangle = 0$ then $\Gamma_j = \{0\}$, hence $\mathcal{M}^{4/3}(\Gamma_j(u_0)) = 0$. ¿From this and using (30), we obtain

$$\sum_j [\mathcal{M}^{4/3}(\Gamma_j(u_0))]^{3/2} = \frac{\sqrt{27\pi}}{\gamma_1}\rho_0^{-2}\sum_j \lambda_j|\langle u_0, \varphi_j\rangle|^2. \tag{39}$$

Due to (13) this proves (33). Inequality (34) follows from the Poincaré inequality $\lambda_1\|u_0\|^2_{L^2} \le \|u_0\|^2_{H^1_0}$, and it is optimal since the constant λ_1 is optimal.

(c) To prove (36), note that by (35) we have $\lambda_j[\mathcal{M}^{4/3}(\Gamma_j(u_0))]^{3/2} \to 0$ as $j \to \infty$. Hence, $\mathcal{M}^{4/3}(\Gamma_j(u_0)) = o(\lambda_j^{-2/3}) = o(j^{-4/(3N)})$, where we exploited again the Weyl asymptotic result (29).

The last claim follows from inequality $\|u_0\|_{H^1_0\cap H^2} \ge \lambda_1\|u_0\|_{L^2}$, which is an immediate consequence of (13). □

Remark 12. ¿From (30) we see that $\mathcal{M}^{4/3}(\Gamma_j(u_0)) \to \infty$ as $\gamma_1 \to 0$ in (1), provided $\langle u_0, \varphi_j\rangle \ne 0$. Another interesting consequence is that for any $\alpha \ne 0$,

$$\mathcal{M}^{4/3}(\Gamma_j(\alpha u_0)) = \mathcal{M}^{4/3}(\Gamma_j(v_0)).$$

Hence, the mapping $v_0 \mapsto \mathcal{M}^{4/3}(\Gamma_j(u_0))$ is constant along rays through the origin in $L^2(\Omega)$. Since $\Gamma_j(-u_0) = -\Gamma_j(u_0)$ for each j, this mapping can be viewed as an even function defined on the unit sphere S^1 in $L^2(\Omega)$.

Remark 13. For $a_0 \ne 0$ all curves $\Gamma_j(u_0)$ are rectifiable near the origin, so that $\dim_B \Gamma_j(u_0) = 1$. Moreover, $\mathcal{M}^1(\Gamma_j(u_0))$ is equal to the length of the curve up to a multiplicative constant independent of j and u_0, see Federer [12, 3.2.39. Theorem], and its length depends on the choice of $t_0 = t_{0j} > t_{min}(u_0)$. If $d = \dim_B \Gamma_j(u_0) > 1$ (like in Theorem 11), then $\mathcal{M}^d(\Gamma_j(u_0))$ does not depend on the choice t_{0j} due to excision property of d-dimensional Minkowski content, see [25, Lemma 5.6(b)].

Remark 14. To see how the Minkowski sequence depends on the domain, let us denote by $\Omega_\varepsilon = \varepsilon\Omega$ the domain obtained from Ω by scaling using the factor $\varepsilon > 0$. To any $u_0 \in L^2(\Omega)$ we assign $u_{0\varepsilon} \in L^2(\Omega_\varepsilon)$ with $u_{0\varepsilon}(x) = u_0(x/\varepsilon)$, and similarly for $\varphi_{j\varepsilon}$. It is easy to see that $\lambda_{j\varepsilon} = \varepsilon^{-2}\lambda_j$, and from this using (30) we obtain:

$$\mathcal{M}^{4/3}(\Gamma_j(u_{0\varepsilon})) = \varepsilon^{-4/3}\mathcal{M}^{4/3}(\Gamma_j(u_0)).$$

5. Box dimension of the trajectory

For calculation of the box dimension of trajectories of NLS initial-boundary value problem (1) we use the well known fact that the box dimension is invariant with the respect to the bilipshitz mappings. As the references for this important results we refer to Falconer

[11, Corollary 2.4(b)] and Tricot [23, see p. 121]. The main result of this section is given by theorem 17 and says that at the bifurcation value $a_0 = 0$ the box dimension of trajectories of (1) viewed near the origin in $L^2(\Omega)$ (or near the origin of $\ell_2(\mathbb{C})$ for (3)) has a jump. More precisely, for $u_0 \neq 0$ the mapping $a_0 \mapsto \dim_B \Gamma_{a_0}(u_0)$ is discontinuous at $a_0 = 0$, where $\Gamma_{a_0}(u_0)$ is the trajectory of (1) corresponding to u_0 and a_0.

5.1. **Multiple spirals.** For a given trajectory $\Gamma(u_0)$ in $L^2(\Omega)$ we define its box dimension via finite-dimensional approximations:

$$\dim_B \Gamma(u_0) = \lim_{k\to\infty} \dim_B \Pi_k(\Gamma(u_0)), \tag{40}$$

where Π_k is the orthogonal projection of the Lebesgue space onto $\operatorname{span}\{\varphi_1, \dots, \varphi_k\}$, or equivalently, from $\ell_2(\mathbb{C})$ onto $\mathbb{C}^k$, corresponding to the first k components of $\ell_2(\mathbb{C})$. The above limit exists due to the monotonicity property of box dimension, see Falconer [11, p. 37].

By the *multiple spiral* (or n-spiral) Γ_{2N} we mean a curve in $\mathbb{C}^N = \mathbb{R}^{2N}$ defined by

$$\Gamma_{2N} = \{(t^{-\alpha_1} e^{i\lambda_1 t}, \dots, t^{-\alpha_N} e^{i\lambda_N t}) \in \mathbb{C}^N : t \geq t_0\}, \tag{41}$$

where $t_0 > 0$. The following result will be fundamental for the computation of box dimension of trajectories of problem (1). Its consequence is that box dimension of multiply oscillating trajectories in $\mathbb{C}^N$ is always less then 2. The claim extends the formula of box dimension of planar spirals due to Tricot [23, p. 121] to oscillating curves in $\mathbb{C}^N$. Since the result seems to be interesting for itself, we state it in a slightly more general form.

Theorem 15. *Let $\alpha_k > 0$ and $\lambda_k \neq 0$ be given numbers, $k = 1, \dots, N$. Then the corresponding multiple spiral Γ_{2N} has box dimension*

$$\dim_B \Gamma_{2N} = \max\left\{1, \frac{2}{1 + \min_k \alpha_k}\right\}. \tag{42}$$

The curve Γ_{2N} is Minkowski nondegenerate if and only if $\min_k \alpha_k \neq 1$. It is rectifiable if and only if $\min_k \alpha_k > 1$.

Proof. (a) We assume without loss of generality that α_1 is minimal among all α_i. Let Γ_2 be the curve in $\mathbb{C}$ defined by $t \mapsto t^{-\alpha_1} e^{i\lambda_1 t}$. Let us introduce the mapping $F : \Gamma_2 \to \Gamma_{2N}$ in the following way. First, we view the curves Γ_2 and Γ_{2N} as subsets of $\mathbb{R}^2$ and $\mathbb{R}^{2N}$ respectively, and define (writing $z_k = x_k + iy_k$):

$$F(x_1, y_1) = (x_1, y_1, f_2(x_1, y_1), g_2(x_1, y_1), \dots, f_n(x_1, y_1), g_n(x_1, y_1)), \tag{43}$$

where

$$\begin{aligned} f_k(x_1, y_1) &= \left(\lambda_1^{-1} \arctan \frac{y_1}{x_1}\right)^{-\alpha_k} \cos\left[\lambda_k \lambda_1^{-1} \arctan \frac{y_1}{x_1}\right], \\ g_k(x_1, y_1) &= \left(\lambda_1^{-1} \arctan \frac{y_1}{x_1}\right)^{-\alpha_k} \sin\left[\lambda_k \lambda_1^{-1} \arctan \frac{y_1}{x_1}\right], \end{aligned}$$

for $2 \le k \le N$. It is easy to check that F maps Γ_2 bijectively onto Γ_{2N}. Let us estimate the expression

$$\begin{aligned}\frac{\partial f_k}{\partial x_1} &= C_1\Big(\arctan\frac{y_1}{x_1}\Big)^{-\alpha_k-1}\frac{y_1}{x_1^2+y_1^2}\cos\Big[\lambda_k\lambda_1^{-1}\arctan\frac{y_1}{x_1}\Big]\\ &\quad+C_2\Big(\arctan\frac{y_1}{x_1}\Big)^{-\alpha_k}\sin\Big[\lambda_k\lambda_1^{-1}\arctan\frac{y_1}{x_1}\Big]\frac{y_1}{x_1^2+y_1^2}\end{aligned}$$

along Γ_2. Here C_1 and C_2 are real constants. It follows:

$$\left|\frac{\partial f_k}{\partial x_1}\right| \le C(t^{\alpha_1-\alpha_k-1}+t^{\alpha_1-\alpha_k}) = O(t^{\alpha_1-\alpha_k}), \quad t\to\infty. \tag{44}$$

Since $\alpha_1 \le \alpha_k$ the function $\frac{\partial f_k}{\partial x_1}$ is bounded by a constant along Γ_2. Similarly for $\frac{\partial f_k}{\partial y_1}$, $\frac{\partial g_k}{\partial x_1}$ and $\frac{\partial g_k}{\partial y_1}$. This proves that the mapping F is Lipschitzian. It is easy to see that the inverse $F^{-1}:\Gamma_4\to\Gamma_2$ is the projection of Γ_4 onto Γ_2, which is also Lipschitzian. Hence, F is a bilipschitz function, so that we may use Falconer [11, Corollary 2.4(b)] with Tricot [23, see p. 121] to obtain that:

$$\dim_B\Gamma_{2N} = \dim_B F(\Gamma_2) = \dim_B\Gamma_2 = \max\Big\{1,\frac{2}{1+\alpha_1}\Big\}.$$

Minkowski nondegeneracy of Γ_{2N} for $\alpha_1\in(0,1)$ follows from [27, Theorem 1], since Γ_2 is Minkowski nondegenerate, and moreover Minkowski measurable, see [26, Corollary 2], and Γ_{2N} is lipeomorphic (i.e. bilipshitz equivalent) to Γ_2. If $\alpha_1\ge 1$, then $\dim_B\Gamma_{2N} = \dim_B\Gamma_2 = 1$. For $\alpha_1 = 1$ we have $\mathcal{M}^1(\Gamma_{2N}) = \infty$ since nonrectifiability of Γ_2 implies that Γ_{2n} is also not rectifiable. $\square$

Remark 16. The condition on α_i to be positive is essential. Indeed, if $N = 2$ and $\alpha_1 = \alpha_2 = 0$, then we obtain the curve on the 2-torus, and assuming that λ_1/λ_2 is rational we obtain that the curve is periodic, hence its box dimension is 1. Therefore, formula (42) does not hold in this case. A more general formulation of the first part of the theorem is as follows: if $\alpha_j\ge 0$ for all j and the set $J=\{j:\alpha_j>0,\ \lambda_j\ne 0\}$ is nonempty, then

$$\dim_B\Gamma_{2N} = \max\Big\{1,\frac{2}{1+\min\{\alpha_k: k\in J\}}\Big\},$$

and analogously for the second part.

Now, using the result given by theorem 15 we are able to show that at the bifurcation value $a_0 = 0$ the box dimension of trajectories of (1) viewed near the origin in $L^2(\Omega)$ (or near the origin of $\ell_2(\mathbb{C})$ for (3)) has a jump.

Theorem 17. *Assume that $u_0\ne 0$ and $\gamma\in\mathbb{R}\setminus\{0\}$. Let $\Gamma(u_0)$ be the trajectory of (1) viewed in a bounded neighbourhood of the origin. If $a_0 = 0$ then $\dim_B\Gamma(u_0) = 4/3$. If $a_0\ne 0$ then $\dim_B\Gamma(u_0) = 1$.*

Proof. We consider the case of $\gamma = \gamma_1 + i\gamma_2$ with $\gamma_2 = 0$ and $\gamma_1 > 0$ (for $\gamma_1 < 0$ the proof is analogous). The case when $\gamma_1 \neq 0$ and $\gamma_2 \neq 0$, see (18), can be treated with slight modifications.

The projection of solution $u(t)$ defined by (19), viewed in $\ell_2(\mathbb{C})$, onto its first n components, defines the following curve in $\mathbb{C}^N$:

$$\Gamma'_{2N} \ldots \; t \mapsto u_N(t) = \frac{1}{\sqrt{2\gamma_1 t + \|u_0\|_{L^2}^{-2}}}(c_1 e^{-i\lambda_1 t}, \ldots, c_N e^{-i\lambda_N t})$$

where $c_j = \frac{\langle u_0, \varphi_j \rangle}{\|u_0\|_{L^2}}$, $t \geq t_0$ with t_0 large enough. We assume for simplicity that $c_j \neq 0$ for all j. It is natural to define the curve

$$\Gamma_{2N} \ldots \; t \mapsto v_N(t) = t^{-1/2}(e^{-i\lambda_1 t}, \ldots, e^{-i\lambda_N t}).$$

The mapping $G : \Gamma'_{2N} \to \Gamma_{2N}$ defined by $G(u_N(t)) = v_N(t)$ for all $t \geq t_0 > 0$, is clearly bilipschitzian, hence $\dim_B \Gamma'_{2N} = \dim_B \Gamma_{2N}$. The claim follows from Theorem 15. The case when $c_j \neq 0$ for at least one j, is treated with minor modifications of the above proof.

If $a_0 \neq 0$, then all components $z_j(t)$ are rectifiable, hence any finite-dimensional projection $\pi_k(\Gamma_0)$ is rectifiable, where $\pi_k : \ell_2(\mathbb{C}) \to \mathbb{C}^k$ it the natural projection onto the first k components of $\ell_2(\mathbb{C})$. This implies that $\dim_B \pi_k(\Gamma_0) = 1$ for each k, and the claim follows. ☐

As we saw before, assuming that $a_0 = 0$ and $\gamma_1 \neq 0$, the natural projection of any trajectory $\Gamma(u_0)$ into j-th component, that is, into the eigenspace spanned by φ_j, can be viewed as a curve in the Gauss plane either of box dimension 4/3 if $\langle u_0, \varphi_j \rangle \neq 0$ (in this case the projection is a spiral), or zero if $\langle u_0, \varphi_j \rangle = 0$ (in this case the projection is simply a point).

6. Applications

In this section we introduce the notion of spiral chirp. The aim is to study the box dimension of spiral chirps and associated trajectories of NLS problems with different nonlinearities. To achieve this goal, we shall use the results on box dimensions of trajectories obtained in previous sections. Finally, in Subsection 6.3 we show that a class of planar polynomial dynamical systems can be interpreted in terms of a NLS problem of nonlocal type.

6.1. **Box dimension of spiral chirps.** Results on box dimension of trajectories of NLS initial-boundary value problem (1) obtained in the previous section can be used in order to calculate the box dimension of some other curves, for example spiral chirps. More precisely, in the preceding subsection we have studied box dimension of projections of Γ_0 into subspaces of the form span $\{\varphi_1, \ldots, \varphi_N\}$, that is, on $\mathbb{R}^{2N}$. Now we would like to consider projections onto $\mathbb{R}^{2N-1}$. The basic model in $\mathbb{R}^3$ is the curve that we call *spiral chirp*:

$$\Gamma = \Big\{(t^{-\alpha}\cos t, t^{-\alpha}\sin t, t^{-\alpha}\cos t) \in \mathbb{R}^3 \colon t \ge t_0\Big\}, \tag{45}$$

with $\alpha \in (0,1)$. A more general model could be (α,β)-spiral chirp:

$$\Gamma_3 = \Big\{(t^{-\alpha}\cos(\lambda_1 t), t^{-\alpha}\sin(\lambda_1 t), t^{-\beta}\cos(\lambda_2 t)) \in \mathbb{R}^3 \colon t \ge t_0\Big\}, \tag{46}$$

for fixed positive α, β, t_0, and $\lambda_j \neq 0$, $j = 1, 2$. Of course, box dimension of Γ_3 will remain the same if we replace the last component with $t^{-\beta}\sin\lambda_2 t$.

Theorem 18. *Let α and β be fixed positive numbers such that either*

$$\alpha \le \beta, \text{ or } \beta \le \alpha < 1, \text{ or } 1 < \beta \le \alpha.$$

Furthermore, assume that λ_1, λ_2 are nonzero real numbers, and $t_0 > 0$. Then for the spiral chirp Γ_3 defined by expression (46) we have:

$$\dim_B \Gamma_3 = \max\Big\{1, \frac{2}{1+\min\{\alpha,\beta\}}\Big\}. \tag{47}$$

The curve Γ_3 is rectifiable if and only if $\min\{\alpha,\beta\} > 1$.

Proof. Due to the simplicity we assume that $\lambda_1 = \lambda_2 = 1$.

Firstly, we consider the case $\alpha \le \beta$.

Note that we have natural projections $\pi_{43} : \Gamma_4 \to \Gamma_3$ and $\pi_{32} : \Gamma_3 \to \Gamma_2$, where $\Gamma_2 \subset \mathbb{R}^2$ is the spiral defined by $t \mapsto z_1(t) = t^{-\alpha}e^{it}$, and $\Gamma_4 \subset \mathbb{R}^4$ is 2-spiral defined by $t \mapsto (z_1(t), z_2(t))$, with $z_2(t) = t^{-\beta}e^{it}$. Hence, since the projections are Lipschitzian, using known box dimensions of Γ_2 (see Tricot [23, see p. 121]) and Γ_4 (see Theorem 15), it follows

$$\dim_B \Gamma_2 = \dim_B \pi_{32}(\Gamma_3) \le \dim_B \Gamma_3 = \dim_B \pi_{43}(\Gamma_4) \le \dim_B \Gamma_4.$$

Now, the desired result follows directly since $\dim_B \Gamma_2 = \dim_B \Gamma_4 = d$, where we used the notation $d = \max\{1, 2/(1+\alpha)\}$.

Next, we consider the case when $\beta \le \alpha < 1$.

Since the natural projection $\pi_{43} : \Gamma_4 \to \Gamma_3$ is Lipschitzian, it follows that $\dim_B \Gamma_3 \le 2/(1+\beta)$. To prove the opposite inequality, it suffices to construct a surjective Lipschitz mapping $F : \Gamma_3 \to \Gamma_2$, where Γ_2 is described by $r = \varphi^{-\beta}$, since then

$$\frac{2}{1+\beta} = \dim_B \Gamma_2 = \dim_B F(\Gamma_3) \le \dim_B \Gamma_3.$$

In this way let us construct an auxiliary spiral Γ_2' defined by

$$\Gamma_2' = \{(t^{-a}\cos t^b, t^{-a}\sin t^b) \in \mathbb{R}^2 : t \ge t_0 > 0\},$$

with positive parameters a and b to be chosen later. It is easy to see that the mapping $F = (F_1, F_2)$ where

$$\begin{aligned} F_1(x,y) &= \Big(\arctan\frac{y}{x}\Big)^{-a}\cos\Big(\arctan\frac{y}{x}\Big)^{b}, \\ F_2(x,y) &= \Big(\arctan\frac{y}{x}\Big)^{-a}\sin\Big(\arctan\frac{y}{x}\Big)^{b} \end{aligned}$$

maps Γ_3 onto Γ_2'. Now, it remains to to find a and b such that F is Lipschitzian on Γ_3. First, since $\frac{\partial}{\partial x}(\arctan(y/x)) = O(t^\alpha)$, and similarly $\frac{\partial}{\partial y}(\arctan(y/x)) = O(t^\alpha)$, when $t \to \infty$ it follows that

$$\frac{\partial F_1}{\partial x} = O(t^{\alpha-a-1}) + O(t^{\alpha-a+b-1}),$$

and similarly for the remaining first order derivatives of F_1 and F_2. Finally, in order to have the boundness of $|\frac{\partial F_i}{\partial x}|$ and $|\frac{\partial F_i}{\partial y}|$, it suffices to impose that $\alpha-a-1 \le 0$ and $\alpha-a+b-1 \le 0$.

The curve Γ_2' is defined by $r = \varphi^{-a/b}$, and since $a/b \le 1$, we have $\dim_B \Gamma_2' = 2/(1+(a/b))$, see Tricot [23, p. 121]. In order to have $\dim_B \Gamma_2' = 2/(1+\beta)$ we choose a and b so that $\beta = a/b$. To have $b - a \le 1 - \alpha$ we impose even $b - a = 1 - \alpha$. ¿From $a = \beta b$ we obtain $b - a = b(1-\beta) = 1 - \alpha$, hence $b = (1-\alpha)/(1-\beta)$, and from this $a = \beta(1-\alpha)/(1-\beta)$. This choice for a and b satisfies all the requirements.
Finally, the proof for the case $1 < \beta \le \alpha$ goes similarly to the previous considerations.

At the end of the proof, let us consider the rectifiability in the case when $\min\{\alpha,\beta\} > 1$. The spiral Γ_2 is rectifiable, as well as the graph of the chirp $z_1(t) = t^\beta \cos t^{-1}$ for t near zero, corresponding to the third component of Γ_3. In this case the curve Γ_3 is rectifiable as well. Indeed, since $z_1(t)$ has local extrema for t_k such that $t_k^{-1/\beta} = \pm\frac{\pi}{2} + 2k\pi$, then $t_k \simeq k^{-\beta}$ as $k \to \infty$, hence,

$$l(\Gamma_3) \le l(\Gamma_2) + 2\sum_k |z_1(t_k)| \le l(\Gamma_2) + c\sum_k k^{-\beta} < \infty, \tag{48}$$

since $\beta > 1$. This implies that $\dim_B \Gamma_3 = 1$. The first inequality in (48) can be justified by isometrically transforming Γ_3 onto the chirp defined in rectangular coordinates, defined on the interval of length of the spiral Γ_2.

On the other side, if $\min\{\alpha,\beta\} < 1$ then from (47) we see that $\dim_B \Gamma_3 > 1$, hence, Γ_3 is not rectifiable. In the case when $\min\{\alpha,\beta\} = 1$, let us consider the following two cases:
(i) If $\alpha = 1$, the spiral Γ_2 defined by $z(t) = t^{-1}e^{it}$ is nonrectifiable. Since $\mathcal{M}^1(\Gamma_3) \ge \mathcal{M}^1(\Gamma_2) = \infty$, the same holds for Γ_3.
(ii) If $\beta = 1$, then the chirp Γ_2' is not rectifiable. Using a suitable parametrization of Γ_3 and the fact that $F(\Gamma_3) = \Gamma_2'$ with a Lipschitzian map F, since Γ_2' is nonrectifiable then also Γ_3 is nonrectifiable. □

Finally, using theorems 15 and 18 we can easily derive the following interesting consequence about box dimension of multiple spiral chirps.

Corollary 19. *Let* $\Gamma_{2N} \subset \mathbb{R}^{2N}$*,* $N \geq 1$*, be a multiple spiral defined by (41), with* $0 < \alpha_k < 1$ *and* $\lambda_k \neq 0$*,* $k = 1, \dots, N$*. Let* $e_1, \dots, e_{2N}$ *be the canonical orthonormal base in* $\mathbb{R}^{2N}$*. Then for the multiple spiral chirp* Γ_{2N-1} *obtained by projecting* Γ_{2N} *into* $(2N-1)$*-dimensional subspace of* $\mathbb{R}^{2N}$ *spanned by any* $2N-1$ *vectors from the base, we have*

$$\dim_B \Gamma_{2N-1} = \dim_B \Gamma_{2N} = \frac{2}{1 + \min_k \alpha_k}.$$

6.2. **Hopf bifurcation for other types of nonlinearities.** Up to now we studied the initial-boundary value problem for Schrödinger equation

$$u_t = i\Delta u - \gamma u V(\|u\|_{L^2(\Omega)}), \tag{49}$$

where $\Omega \subset \mathbb{R}^N$ and $\gamma = \gamma_1 + i\gamma_2 \in \mathbb{C}$. More precisely, we considered the nonlinearity $V(\rho) = \rho^2 + a_0$, where a_0 was a real bifurcation parameter. In this section we consider some polynomial types of nonlinearity given by the expression

$$V(\rho) = \rho^{2l} + a_{l-1}\rho^{2(l-1)} + \cdots + a_1\rho^2 + a_0, \tag{50}$$

where a_i, $i = 0, \dots, l-1$ are prescribed real parameters. The NLS equation (49) corresponds to the *standard generic generalized Hopf bifurcation*, see Takens [22], also called the *standard Hopf-Takens bifurcation*. The following result extends Theorem 17.

Theorem 20. *Let* $\Gamma(u_0)$ *be a part of trajectory of* (49) *near the origin with* $V(\rho)$ *defined by (50), such that* $u_0 \neq 0$*,* $\gamma_2 = 0$ *and* $\gamma_1 \neq 0$*. Assume that* $k := \min\{j : a_j \neq 0\} \geq 1$*. Then*

$$\dim_B \Gamma(u_0) = \frac{4k}{2k+1}. \tag{51}$$

Proof. The proof uses the idea of the proof of Theorem 17, together with the proof of [26, Theorem 9(b)]. Here $\alpha = 1/(2k)$, so that the box dimension is obtained from Theorem 15. □

Up to now we studied trajectories with spiral components $z_j(t)$ converging to zero with equal rate $\alpha > 0$, that is, $|z_j(t)| \simeq t^{-\alpha}$ as $t \to \infty$, for all j, see for example expressions (45) or (46). Next, we are interested in dynamical systems yielding solutions in which spiral components converge to zero with different rates.

A. As the simplest model we consider a polynomial system in in $\mathbb{C}^2$, defined by

$$\dot{z}_j = -i\lambda_j z_j - \frac{1}{4m_j} z_j(|z_1|^{2m_1} + |z_2|^{2m_2} + a_0), \ \ j = 1, 2, \tag{52}$$

where m_j are fixed positive integers and a_0 is the real bifurcation parameter. It is easy to see that for $a_0 < 0$ an attracting invariant set $\mathcal{S}(a_0)$ is born, defined by $|z_1|^{2m_1} + |z_2|^{2m_2} = -a_0$, diffeomorphic to the unit 3-sphere in $\mathbb{C}^2$. Each trajectory starting in the point $(z_{10}, z_{20}) \in \mathcal{S}(a_0)$ is contained in the torus $|z_{10}|S^1 \times |z_{20}|S^1$ (provided that both components z_{j0} are nonzero). The invariant set $\mathcal{S}(a_0)$ is clearly equal to the union of these trajectories.

Assume that $a_0 = 0$ and let $z(0) = (z_{01}, z_{02}) \in \mathbb{C}^2$ be a given initial value, $z(0) \neq 0$. It can be shown that the unique solution $z(t) = (z_j(t))$ of system (52) is given by

$$z_j(t) = |z_{0j}|(t+1)^{-1/(2m_j)} \exp(-i\lambda_j t + i \arg z_{0j}), \quad j = 1, 2.$$

Here $z_j(t)$ is a spiral converging to zero with rate $\alpha_j = 1/(2m_j)$. Assume that both z_{01} and z_{02} are different from zero. According to Theorem 15 the box dimension of the corresponding trajectory $\Gamma_4 = \{(z_1(t), z_2(t)) \in \mathbb{C}^2 : t \geq t_0 > 0\}$ is equal to

$$\dim_B \Gamma_4 = \frac{2}{1 + 0.5 \max\{m_1, m_2\}^{-1}}. \tag{53}$$

System (52) is a special case of the following Schrödinger equation:

$$u_t = i\Delta u - F(u), \tag{54}$$

with initial and boundary values as in (1), and

$$F(u) = \left(|\langle u, \varphi_1\rangle|^{2m_1} + |\langle u, \varphi_2\rangle|^{2m_2} + a_0\right) \sum_{j=1}^{2} \frac{1}{4m_j} \langle u, \varphi_j\rangle\, \varphi_j(x),$$

with scalar products in $L^2(\Omega)$. System (52) is obtained from (54) by considering solutions of the form $u(t,x) = z_1(t)\varphi_1(x) + z_2(t)\varphi_2(x)$, where φ_1 and φ_2 are defined in the subsection 1.2.

B. Various extensions of (52) and (54) are possible, like for example a cyclic polynomial system in $\mathbb{R}^{2k}$, which we view as a bifurcation problem in $\mathbb{C}^k$:

$$\begin{aligned}
\dot z_1 &= -i\lambda_1 z_1 - \frac{1}{4m_1} z_1(|z_1|^{2m_1} + |z_2|^{2m_2} + a_0) \\
\dot z_2 &= -i\lambda_2 z_2 - \frac{1}{4m_2} z_2(|z_2|^{2m_2} + |z_3|^{2m_3} + a_0) \\
&\vdots \\
\dot z_k &= -i\lambda_k z_k - \frac{1}{4m_k} z_k(|z_k|^{2m_k} + |z_1|^{2m_1} + a_0).
\end{aligned} \tag{55}$$

For $a_0 < 0$ an invariant set $\mathcal{S}(a_0)$ is born in $\mathbb{C}^k$ defined by $|z_1|^{2m_1} + |z_2|^{2m_2} = |z_2|^{2m_2} + |z_3|^{2m_3} = \cdots = |z_k|^{2m_k} + |z_1|^{2m_1} = -a_0$, which is a subset the surface defined by $\sum_j |z_j|^{2m_j} = -\frac{k}{2}a_0$, diffeomorphic to the $(2k-1)$-sphere in $\mathbb{C}^k$.

Here system (55) again has the form of the Schrödinger equation (54), but with

$$F(u) = \sum_{j=1}^{k} \frac{1}{4m_j} \langle u, \varphi_j\rangle \left(|\langle u, \varphi_j\rangle|^{2m_j} + |\langle u, \varphi_{j+1}\rangle|^{2m_{j+1}} + a_0\right) \varphi_j(x),$$

and the summation of indices in $j+1$ is taken modulo k. Here we consider solutions of the form $u(t,x) = \sum_{j=1}^{k} z_j(t)\varphi_j(x)$.

For $a_0 = 0$ and any prescribed initial value $z(0) = (z_{0j}) \in \mathbb{C}^k$ it can be shown that the unique solution is the curve $\Gamma_{2k} = \{(z_1(t), \dots, z_k(t)) \in \mathbb{C}^k : t \geq 0\}$ of (55), defined by

$$z_j(t) = |z_{0j}|(t+1)^{-1/(2m_j)} \exp(-i\lambda_j t + i \arg z_{0j}),\ j = 1, \dots, k. \tag{56}$$

Using Theorem 15 with $\alpha_j = 1/(2m_j)$ we obtain the following result.

Corollary 21. *Let m_j be positive integers and $a_0 = 0$. For any trajectory Γ_{2k} of (55) such that $z(0) = (z_{01}, \dots, z_{0k}) \neq 0$ we have*

$$\dim_B \Gamma_{2k} = \frac{2}{1 + 0.5 \max\{m_j : z_{0j} \neq 0\}^{-1}}.$$

C. Now we would like to construct a NLS Cauchy problem possessing trajectories of box dimension equal to two. Let us consider the Schrödinger equation of the form (54) such that

$$F(u) = \sum_{k=1}^{\infty} (|\langle u, \varphi_{2k-1}\rangle|^{2m_{2k-1}} + |\langle u, \varphi_{2k}\rangle|^{2m_{2k}} + a_0) \sum_{j=-1}^{0} \frac{1}{4m_{2k+j}} \langle u, \varphi_{2k+j}\rangle \varphi_j(x), \tag{57}$$

where m_j is a prescribed sequence of positive integers. First, if $u \in L^2(\Omega)$ then also $F(u) \in L^2(\Omega)$. Indeed, the expression in the round brackets is bounded by a constant $M = M(u, a_0, m_1, m_2, \dots)$ independent of j, since $\langle u, \varphi_j\rangle$ converges to zero. Hence,

$$\|F(u)\|_{L^2}^2 \leq M^2 \left\| \sum_{k=1}^{\infty} \sum_{j=-1}^{0} \frac{1}{4m_{2k+j}} \langle u, \varphi_{2k+j}\rangle \varphi_j \right\|_{L^2}^2 \leq \frac{M^2}{16} \|u\|_{L^2}^2.$$

It can be shown that if $u(t) = \sum_j z_j(t)\varphi_j$ is a solution of (54) with initial value $u(0) = u_0 \in L^2(\Omega)$, then $z_j(t)$ is uniquely determined, and

$$z_j(t) = |\langle u_0, \varphi_j\rangle|(t+1)^{-1/(2m_j)} \exp(i\lambda_j t + i \arg\langle u_0, \varphi_j\rangle),\ \ j \geq 1.$$

Denote by $\Gamma(u_0)$ the trajectory in $L^2(\Omega)$ corresponding to $t \geq 0$. Using the definition of $\dim_B \Gamma(u_0)$ in (40) and Theorem 15 we obtain the following result.

Corollary 22. *Let $a_0 = 0$. Let $u(t)$ be the solution of (54) with $F(u)$ defined by (57), and $a_0 = 0$, satisfying initial condition $u(0) = u_0 \in L^2(\Omega)$. Then*

$$\dim_B \Gamma(u_0) = \frac{2}{1 + 0.5 \sup\{m_j : \langle u_0, \varphi_j\rangle \neq 0\}^{-1}}. \tag{58}$$

In particular, if $m_j \to \infty$ as $j \to \infty$, and $\langle u_0, \varphi_j\rangle \neq 0$ for infinitely many j's, then $\dim_B \Gamma(u_0) = 2$.

6.3. **Box dimension of trajectories of some nonintegrable Schrödinger problems.** Many dynamical systems important for applications, like for example Liénard systems, weakly damped systems etc., are not explicitly solvable. However, in spite of their nonintegrability it is possible to get the information about the box dimension of corresponding trajectories, see [30] for some examples. Here, using the connection between the nonlinear system of ODE's and the Schrödinger equation pointed out already in Section 1.2, we would like to study the box dimension of trajectories of some nonintegrable Schrödinger evolution problems. For this purpose let us consider the following polynomial planar system:

$$\begin{aligned} \dot{x} &= f(x,y) = \quad \lambda_1 y + p(x,y) \\ \dot{y} &= g(x,y) = -\lambda_1 x + q(x,y), \end{aligned} \tag{59}$$

where we define $p(x,y) = f(x,y) - \lambda_1 y$ and $q(x,y) = g(x,y) + \lambda_1 x$. Here λ_1 is the first eigenvalue of the $-\Delta|_{H_0^1 \cap H^2}$ corresponding to an open and bounded domain $\Omega \subset \mathbb{R}^N$. An initial point in the phase space is prescribed by $x(0) = x_{10}$ and $y(0) = y_{10}$. Naturally, we say that the system (59) is *equivalent* to (54) if there is a bijection between the corresponding solution sets. In this sense it is easy to see that the problem (59) is equivalent to Schrödinger equation (54) with $F\colon \mathbb{C}\varphi_1 \to \mathbb{C}\varphi_1$ defined by

$$[F(u)](\xi) = \Big(p(\langle \operatorname{Re} u, \varphi_1\rangle, \langle \operatorname{Im} u, \varphi_1\rangle) + iq(\langle \operatorname{Re} u, \varphi_1\rangle, \langle \operatorname{Im} u, \varphi_1\rangle)\Big)\varphi_1(\xi), \quad \xi \in \Omega, \tag{60}$$

and with initial function $u(0) = z_{01}\varphi_1$, where $z_{01} = x_{10} + iy_{10} \in \mathbb{C}$. Note that the nonlinear term $F(u)$ is of nonlocal type, i.e. F it is not defined in pointwise manner, but via scalar products, that is, by means of two integrals. The solution of (54) in this case has the form $u(t,x) = z(t)\varphi_1(x)$, where $z(t) = x(t) + iy(t)$, and the components satisfy (59). Note that (54) is of nonlocal type, and in general not explicitly solvable. Moreover, it is interesting that the study of *any* system of ODE's can be considered as the study of a Schrödinger problem.

Proposition 23. *Any Cauchy problem for the system $\dot{x} = f(x)$, where $x \in \mathbb{R}^n$ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a Lipschitz function, can be naturally interpreted as a Schrödinger problem of the form* (54)*, with $F(u)$ written explicitly, see* (62) *below.*

Proof. We can assume without loss of generality that n is even, $n = 2k$ (if $n = 2k-1$, we can add a new trivial equation $\dot{x}_{2k} = 0$, $x_{2k}(0) = 0$). Now the system can be written in the form

$$\begin{aligned} \dot{x}_j &= f_j(x,y) \\ \dot{y}_j &= g_j(x,y) \end{aligned} \qquad j = 1, \dots, k \tag{61}$$

where $x, y \in \mathbb{R}^k$. Defining $z_j = x_j + iy_j$ and $p_j(x,y) = f_j(x,y) - \lambda_j y_j$, $q_j(x,y) = g_j(x,y) + \lambda_j x_j$, $j = 1, \dots, k$, where λ_j are the first k eigenvalues of $-\Delta$ (counting the multiplicity) with zero boundary data on a given bounded open domain Ω in $\mathbb{R}^N$, then we have $\dot{z}_j = -i\lambda_j z_j + (p_j + iq_j)$. Multiplying with eigenfunctions $\varphi_j = \varphi_j(\xi)$ normalized in $L^2(\Omega)$, and defining $u(t,\xi)=\sum_j z_j(t)\varphi_j(\xi)$ we easily obtain (54), where the operator $F\colon X_k\to X_k$, $X_k=\operatorname{span}_{\mathbb{C}}\{\varphi_1,\dots,\varphi_k\}\subseteq L^2(\Omega)$, is defined by

$$[F(u)](\xi)=\left[F\left(\sum_{j=1}^k z_j\varphi_j\right)\right](\xi)=\sum_{j=1}^k\big(p_j(x,y)+iq_j(x,y)\big)\,\varphi_j(\xi),\ \ \xi\in\Omega, \tag{62}$$

with $x_j=\operatorname{Re}z_j=\langle\operatorname{Re}u,\varphi_j\rangle$, $y_j=\operatorname{Im}z_j=\langle\operatorname{Im}u,\varphi_j\rangle$. If we prescribe the initial value $(x,y)(0)=(x_0,y_0)\in\mathbb{R}^{2k}$ for the system (61), then the corresponding initial function $u(0)=\sum_{j=1}^k(x_{0j}+iy_{0j})\varphi_j\in X_k$ of the Schrödinger problem (54) generates the trajectory in X_k. It can be identified with the trajectory in $\mathbb{R}^{2k}$ corresponding to (61) using the natural isomorphism between X_k and $\mathbb{R}^{2k}$. □

Remark 24. Note that the operator $F\colon X_k\to X_k$ in the proof of Proposition 23 is *nonlocal*, since (62) contains integral terms $x_j=\langle\operatorname{Re}u,\varphi_j\rangle=\int_\Omega(\operatorname{Re}u)(\xi)\varphi_j(\xi)\,d\xi$ on the right-hand side, $j=1,\dots,k$, and similarly for y_j. The operator F can be naturally extended to $\tilde F\colon L^2(\Omega)\to L^2(\Omega)$ by defining $\tilde F(u)=\tilde F\left(\sum_{j=1}^\infty z_j\varphi_j\right)=\sum_{j=1}^k\big(p_j(x,y)+iq_j(x,y)\big)\varphi_j$, where $u=\sum_{j=1}^\infty z_j\varphi_j$ is the Fourier expansion of $u\in L^2(\Omega)$ with respect to the orthonormal basis (φ_j) of eigenfunctons of $-\Delta|_{H_0^1\cap H^2}$. The claim in the proposition holds (with the same proof) for nonautonomous systems of ODE's as well. In this case the nonlinearity in (54) takes the form $F(t,u)$.

Remark 25. Proposition 23 enables us to reformulate the second part of the 16th Hilbert problem in terms of the Schrödinger equation (54) as follows. Let $F(u)$ be defined by (60), where $p(x,y)$ and $q(x,y)$ are polynomials in 2 real variables, such that the maximum of their degrees d is at least 2. Assuming that the value of $d\geq 2$ is fixed, find an upper bound for the number of limit cycles in the class of Schrödinger equations (54) in terms of d. For $d=2$ (that is, for the class of quadratical planar systems of ODE's) it is even not known if the bound can be finite.

As an example, consider the following nonlocal Schrödinger evolution problem on a given bounded domain Ω in $\mathbb{R}^N$:

$$u_t=i\Delta u+\sum_{j=k}^m a_{2j}\left(\int_\Omega(\operatorname{Re}u)\varphi_1\,dx\right)^{2j}+\sum_{j=k}^m a_{2j+1}\left(\int_\Omega(\operatorname{Re}u)\varphi_1\,dx\right)^{2j+1}, \tag{63}$$

where $1\leq k\leq m$.

Theorem 26. *Let $\Gamma(u_0)$ be the trajectory in $L^2(\Omega)$ of the Schrödinger evolution problem (63) viewed near the zero function in $L^2(\Omega)$, and generated by initial condition $u(0)=z_0\varphi_1$, with $z_0=x_0+iy_0\neq 0$, and k is a positive integer. Assume that the coefficients $a_2,a_4\dots,a_{2m}$ and $a_{2k+1},a_{2k+3},\dots,a_{2m+1}$ are real. Assume that $a_{2k+1}\neq 0$, i.e. a_{2k+1} is the first nonzero* *coefficient corresponding to an odd exponent of the integral term in* (63)*. Then*

$$\dim_B \Gamma(u_0) = 2 - \frac{2}{2k+1}. \tag{64}$$

Proof. Here we have $p(x,y) = \sum_{j=1}^{m} a_{2j}x^{2k+1} + \sum_{j=k}^{m} a_{2j+1}x^{2j+1}$ and $q(x,y) = 0$. Let us write $u(t,\xi) = (x(t) + iy(t))\varphi_1(\xi)$. Then (63) with the indicated initial condition is equivalent to the following Liénard system in the plane, see Proposition 23:

$$\begin{aligned} \dot{x} = f(x,y) = &\quad \lambda_1 y + \sum_{j=1}^{m} a_{2j}x^{2k+1} + \sum_{j=k}^{m} a_{2j+1}x^{2j+1} \\ \dot{y} = g(x,y) = &-\lambda_1 x. \end{aligned} \tag{65}$$

The claim follows immediately from [28, Theorem 6]. □

Remark 27. Note that under the condition of $a_{2k+1} \neq 0$ the Liénard problem (65) is not explicitly solvable for $(x_0, y_0) \neq (0,0)$. Therefore, the corresponding Schrödinger evolution problem (63) is also not explicitly solvable for $u(0) \neq 0$.

References

[1] H. Brezis, Analyse Fonctionnelle, Masson, 1983.

[2] T. Cazenave, Semilinear Schrödinger Equations, Courant Lecture Notes in Mathematics 10, 2004.

[3] T. Cazenave, A. Haraux, F.B. Weissler, *Une équation des ondes complètement intégrable avec non-linéarité homogène de degré trois*, C. Rend. Acad. Sci. Paris, 313 (1991), 237–241.

[4] T. Cazenave, A. Haraux, F.B. Weissler, *Detailed asymptotics for a convex Hamiltonian system with two degrees of freedom*, J. Dynam. Diff. Eq., 5 (1993), 155-187.

[5] T. Cazenave, A. Haraux, F.B. Weissler, *A class of nonlinear completely integrable abstract wave equations*, J. Dynam. Diff. Eq., 5 (1993), 129–154.

[6] J. Chen, B. Guo, *Strong instability of standing waves for a nonlocal Schrödinger equation*, Phys. D 227 (2007), no 2, 142–148.

[7] C. Chicone, Ordinary Differential Equations with Applications, 2nd Edition, Springer Verlag, 2006.

[8] M. Christ, *Power series solution of a nonlinear Schrödinger equation*, Mathematical Aspects of Nolinear Dispersive Equations, eds J. Bourgain, C.E. Kenig and S. Klainerman, Princeton University Press, Princeton and Oxford, 2007, 131–155

[9] E.D. Davies, Spectral Theory and Differential Operators, Cambridge University Press, 1995.

[10] N. Elezović, V. Županović, D. Žubrinić. *Box dimension of trajectories of some discrete dynamical systems*, Chaos, Solitons & Fractals 34 (2007) 244–252.

[11] K. Falconer, Fractal Geometry, Chichester: Wiley, 1990.

[12] H. Federer, Geometric Measure Theory, Springer-Verlag, 1969.

[13] J. Guckenheimer, P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer Verlag, 1983.

[14] C.Q. He, M.L. Lapidus, *Generalized Minkowski content, spectrum of fractal drums, fractal strings and the Riemann zeta-function*, Mem. Amer. Math. Soc. 127 (1997), no. 608.

[15] D. Henry, *Geometric theory of semilinear parabolic equations*, Lecture Notes in Mathematics, no 840, Springer-Verlag, Berlin, 1981.

[16] L. Horvat-Dmitrović, *Box dimension and bifurcation of one-dimensional discrete dynamical systems*, Preprint (2010)
[17] L. Korkut, D. Vlah, D. Žubrinić, V. Županović, *Generalized Fresnel integrals and fractal properties of related spirals, Appl. Mathematics and Computation*, 206 (2008), 236–244.
[18] M. Lapidus, M. van Frankenhuysen, Fractal geometry and number theory. Complex dimensions of fractal strings and zeros of zeta functions, Birkhäuser Boston, Inc., Boston, MA, 2000.
[19] V.P. Mikhailov, Partial Differential Equations (in Russian), Nauka, Moscow 1976.
[20] M. Pašić, D. Žubrinić, V. Županović, *Oscillatory and phase dimensions of solutions of some second-order differential equations*, Bulletin des Sciences Mathématiques, Vol 133(8) 2009, 859–874.
[21] L. Perko, Differential Equations and Dynamical Systems, Texts in Applied Mathematics, Springer–Verlag, 1991.
[22] F. Takens, *Unfoldings of certain singularities of vector fields: Generalized Hopf bifurcations*, J. Differential Equations, 14 (1973) 476–493.
[23] C. Tricot, Curves and Fractal Dimension, Springer–Verlag, 1995.
[24] J. Weidmann, Linear Operators in Hilbert Spaces, Springer, 1980.
[25] D. Žubrinić, *Analysis of Minkowski contents of fractal sets and applications*, Real Anal. Exchange, Vol 31(2), 2005/2006, 315–354.
[26] D. Žubrinić, V. Županović, *Fractal analysis of spiral trajectories of some planar vector fields*, Bulletin des Sciences Mathématiques, 129/6 (2005), 457–485.
[27] D. Žubrinić, V. Županović, *Fractal analysis of spiral trajectories of some vector fields in* $\mathbb{R}^3$, C. R. Acad. Sci. Paris, Série I 342, (2006), 959–963.
[28] D. Žubrinić, V. Županović, *Poincaré map in fractal analysis of spiral trajectories of planar vector fields*, Bull. Belg. Math. Soc. Simon Stevin, Vol. 15 (2008), 947-960.
[29] V. Županović, D. Žubrinić, Fractal dimensions in dynamics, Encyclopedia of Mathematical Physics, eds. J.-P. Françoise, G.L. Naber and Tsou S.T. Oxford: Elsevier, (2006), vol 2, 394–402.
[30] V. Županović, D. Žubrinić, *Recent results on fractal analysis of trajectories of some dynamical systems*, *FUNCTIONAL ANALYSIS IX* - Proceedings of the Postgraduate School and Conference held at the Inter-University Centre, Dubrovnik, Croatia, 15-23 June, 2005. (eds. G. Muic, J. Hoffmann-Jorgensen), Aarhus, Denmark: University of Aarhus, Department of Mathematical Sciences, (2007), 126–140.

(Received April 1, 2010)

University of Zagreb, Faculty of Electrical Engineering and Computing, Unska 3, 10000 Zagreb, Croatia