Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **59**, 1-14; **http://www.math.u-szeged.hu/ejqtde/**

OSCILLATION OF SOLUTIONS TO A HIGHER-ORDER NEUTRAL PDE WITH DISTRIBUTED DEVIATING ARGUMENTS

JULIO G. DIX

Abstract. This article presents conditions for the oscillation of solutions to neutral partial differential equations. The order of these equations can be even or odd, and the deviating arguments can be distributed over an interval. We also extend our results to a nonlinear equation and to a system of equations.

1. Introduction

We study the oscillation of solutions to the neutral differential equation

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\Big(u(x,t)+\int_a^b p(t,\xi)u(x,r(t,\xi))\,d\mu(\xi)\Big)\\&=\sum_{j=1}^{m_2} a_j(t)\Delta u(x,h_j(t))-\int_c^d q(x,t,\xi)u(x,g(t,\xi))\,d\mu(\xi),\end{aligned} \tag{1.1}$$

where x is in a bounded domain Ω of $\mathbb{R}^d$, with smooth boundary $\partial\Omega$, $t\geq 0$, and the delayed arguments satisfy $r(t,\xi)\leq t$, $h_j(t)\leq t$, $g(t,\xi)\leq t$. To this equation we attach one of the following two boundary conditions:

$$u(x,t)=0 \quad x\in\partial\Omega,\ t\geq 0; \tag{1.2}$$

$$\frac{\partial u}{\partial\nu}+\gamma(x,t)u(x,t)=0 \quad x\in\partial\Omega,\ t\geq 0, \tag{1.3}$$

where ν is the unit exterior normal vector to $\partial\Omega$, and $\gamma(x,t)$ is a positive function in $C(\Omega\times[0,\infty),\mathbb{R})$. Here n, m_2 are positive integers with $n\geq 2$; $a_j(t), h_j(t)$ are in $C([0,\infty),\mathbb{R})$; $g(t,\xi)$ is in $C([0,\infty)\times[c,d],\mathbb{R})$; $p(t,\xi)$ is in $C([0,\infty)\times[a,b],\mathbb{R})$; $q(x,t,\xi)$ is in $C(\Omega\times[0,\infty)\times[a,b],\mathbb{R})$; $r(t,\xi)$ is in $C([0,\infty)\times[a,b],\mathbb{R})$; p, r have n continuous derivatives with respect to time; Δ is the Laplacian operator, $\Delta=\partial^2_{x_1^2}+\dots+\partial^2_{x_d^2}$; and the integrals are in the Stieltjes sense with μ non-decreasing. Note that these integrals can represent summations of the form $\sum_j p_j(t)u(x,r_j(t))$ and $\sum_j q_j(x,t)u(x,g_j(t))$, which we call the summation case. The study of solutions to neutral differential equations has practical importance, because they appear in population models, chemical reactions, control systems, etc. There are many publications related to the oscillation of solutions to neutral ordinary differential equations; see for example [2, 6, 11, 12, 13] and the books [1, 3, 4, 7]. There are also some publications for neutral partial differential equations, see for example [8, 9, 10, 14]. Li [8] stated that solutions to a system of type (1.1) are oscillatory for n odd. However their article has many mistakes: On page 527 "There exist

2000 *Mathematics Subject Classification.* 35B05, 35R10.
Key words and phrases. Oscillatory solution; neutral PDE; distributed deviating argument.

$M > 0$ such that $v(t) \geq M$" is not true when y decreases to zero; Lemma 2 needs the assumption that W is eventually positive; etc. Lin [9] studied a system of neutral PDEs, with n even; we will compare their hypotheses and ours in Section 3. Wang [14] stated that for a particular case of (1.1) all solutions are oscillatory. This is not true for n odd; it is easy to build an example with solution $\sin(x)e^{-t}$, which is non-oscillatory for $0 \leq x \leq \pi$. On page 570, it says "By choosing $i = 1$, we have $z'(t) > 0$", which is used later. However, by Lemma 2.1 with n odd, i can be zero and their proof fails. Luo [10] studied a system of PDEs. Their proof follows the steps in [14], including mistakes, so it fails for n odd. The main objective of this article is to present verifiable hypotheses for the oscillation of solutions to (1.1) for even and odd order, with various ranges for the coefficient $p(t)$. In Section 3, we extend our results to a nonlinear neutral equation and to an equation of the type type studied in [9]. In Section 4, we apply our results to a system of neutral partial differential equations.

2. Oscillation for the neutral PDE

By a solution, $u(x,t)$, we mean a function in $C(\Omega \times [t_-, \infty), \mathbb{R})$ that is twice continuously differentiable for $x \in \Omega$, and n times continuously differentiable for $t \geq 0$, and that satisfies (1.1) with a boundary condition (1.2) or (1.3). The value t_- is the minimum of value of functions r, h_j, g when $t \geq 0$. A solution $u(x,t)$ is called eventually positive if there exists t_0 such that $u(x,t) > 0$ for $t \geq t_0$ and all x in the interior of Ω. Eventually negative solutions are defined similarly. Solutions that are not eventually positive and not eventually negative are called oscillatory; i.e., for every $t_0 \geq 0$, there exist $t_1 \geq t_0$ and x_1 in the interior of Ω, such that $u(x_1, t_1) = 0$. The following hypotheses will be used in this article.

(H1) $0 \leq a_j(t)$ with $0 < \sum_{j=1}^{m_2} a_j(t)$ for $t \geq 0$.
There exists a continuous function $\hat{g}(t)$ such that $\hat{g}(t) \leq g(t,\xi)$ and for all t, ξ (in the summation case: $\hat{g}(t) \leq g_j(t)$ for all j).
$h_j(t), \hat{g}(t), r(t,\xi)$ approach $+\infty$ as $t \to \infty$, for all $\xi \in [a,b]$, $j \in \{1, \dots, m_2\}$;
Also,

$$0 \leq \min_{x \in \Omega} q(x,t,\xi) := Q(t,\xi), \quad 0 \leq \min_{x \in \Omega} q_j(x,t) := Q_j(t)\,.$$

We use the well known "averaging technique" to transform the partial differential equation into a delay differential inequality. The existence (and non-existence) of eventually positive solutions to this inequality, provides oscillation results for neutral differential equations; see for example [3, Theorem 5.1.1]. Let λ_1 be the smallest eigenvalue of the elliptic problem

$$\Delta\phi + \lambda\phi = 0 \quad \text{in } \Omega$$
$$\phi(x) = 0 \quad \text{on } \partial\Omega\,.$$

It is well know that $\lambda_1 > 0$ and that the corresponding eigenfunction ϕ_1 does not have zeros in the interior of Ω; we select $\phi_1(x) > 0$. See for example [5, Theorem 8.5.4]. Assuming that $u(x,t)$ is a solution to (1.1)-(1.2) with $u(x,t) > 0$ for $t \geq t_0$, we define the "average function"

$$v(t) = \int_\Omega u(x,t)\phi_1(x)\,dx \tag{2.1}$$

which is positive because both u and ϕ_1 are positive. Note that v is the projection of u on the first eigenspace of the Laplacian. By Green's formula,

$$\begin{aligned}\int_\Omega \Delta u(x,t)\phi_1(x)\,dx &= \int_{\partial\Omega} \phi_1\frac{\partial u}{\partial\nu} - u\frac{\partial\phi_1}{\partial\nu}\,dS + \int_{\partial\Omega} u\Delta\phi_1\,dS \\ &= -\lambda_1\int_\Omega u(x,t)\phi_1(x)\,dx < 0\,.\end{aligned} \tag{2.2}$$

We multiply each term in (1.1) by the eigenfunction ϕ_1, and integrate over Ω. Using (2.1), (2.2), (H1), and the notation

$$z(t) = v(t) + \int_a^b p(t,\xi)v(r(t,\xi))\,d\mu(\xi)\,, \tag{2.3}$$

the PDE (1.1) is transformed into the delay differential inequality

$$\begin{aligned} z^{(n)}(t) &< -\int_a^b Q(t,\xi)v(g(t,\xi))\,d\mu(\xi)\,, \\ z^{(n)}(t) &< -\sum_{j=1}^{m_1} Q_j(t)v(g_j(t)) \quad \text{(for the summation case)}\,.\end{aligned} \tag{2.4}$$

Now for the boundary condition (1.3), assuming that $u(x,t)$ is a positive solution to (1.1)-(1.3), we define the "average function"

$$v(t) = \int_\Omega u(x,t)\,dx \tag{2.5}$$

which is positive. By Green's formula,

$$\int_\Omega \Delta u(x,t)\,dx = \int_{\partial\Omega}\frac{\partial u}{\partial\nu} = -\int_\Omega \gamma(x,t)u(x,t)\,dx < 0\,. \tag{2.6}$$

Using this inequality, (2.5) and (H1), we obtain (2.4) again.

Lemma 2.1 ([7, Lemma 5.2.1])**.** *Let $z(t)$ be an n times differentiable function of constant sign, $z^{(n)}(t)$ be of constant sign and not identically zero in any interval $[t_0,\infty)$ and $z^{(n)}(t)z(t) \le 0$. Then*

(i) *There exists a time t_1 such that $z^{(0)}, z^{(1)}, \dots, z^{(n-1)}$ are of constant sign on $[t_1,\infty)$.*

(ii) *There exists an integer k in $\{1,3,5,\dots,n-1\}$ when n is even, and k in $\{0,2,4,\dots,n-1\}$ when n is odd, such that*

$$\begin{aligned} z^{(i)}(t)z(t) > 0 \quad &\text{for } i = 0,1,\dots,k, \\ (-1)^{n+i-1}z^{(i)}(t)z(t) > 0 \quad &\text{for } i = k+1,\dots,n\,.\end{aligned} \tag{2.7}$$

Remark: In our settings $z^{(n)}(t) < 0$ for $t \ge t_1$, which satisfies the "not identically zero" condition in the above lemma. However, this part was not shown in Wang [14] and Luo [10]. There it was also wrongly assumed that $z(t)$ is always increasing.

Lemma 2.2 ([7, Lemma 5.2.2]). *Assume that* $z^{(0)}, z^{(1)}, \dots, z^{(n-1)}$ *are absolutely continuous and of constant sign on the interval* (t_0, ∞)*. Moreover,* $z^{(n)}(t)z(t) \geq 0$*. Then either*

$$z^{(i)}(t)z(t) \geq 0 \quad \textit{for } i = 0, 1, \dots, n,$$

or there exists an integer k *in* $\{0, 2, \dots, n-2\}$ *when* n *is even, and* k *in* $\{1, 3, \dots, n-2\}$ *when* n *is odd, such that*

$$\begin{aligned} z^{(i)}(t)z(t) \geq 0 \quad &\textit{for } i = 0, 1, \dots, k, \\ (-1)^{n+i} z^{(i)}(t) \geq 0 \quad &\textit{for } i = k+1, \dots, n, \end{aligned} \tag{2.8}$$

Lemma 2.3 ([7, Lemma 5.2.3]). *Let* $z(t)$ *be an* n *times differentiable function of constant sign,* $z^{(n)}(t)$ *be of constant sign and not identically zero in any interval* $[t_0, \infty)$*, and* $z^{(n)}(t)z^{(n-1)}(t) \leq 0$ *for every* $t \geq t_0$*. Then for each* $0 < \lambda < 1$*,*

$$|z(\lambda t)| \geq M_1 t^{n-1} |z^{(n-1)}(t)|, \quad M_1 = \frac{\lambda^k (1-\lambda)^{n-1-k}}{2^k k!(n-1-k)!},$$

where K *is defined in Lemma 2.1 (ii).*

Our first result concerns the equation

$$\begin{aligned} &\frac{\partial^n}{\partial t^n}\Big(u(x,t) + \int_a^b p(t,\xi) u(x, r(t,\xi))\, d\mu(\xi)\Big) \\ &= \sum_{j=1}^{m_2} a_j(t) \Delta u(x, h_j(t)) - \sum_{j=1}^{m_1} q_j(x,t) u_j(x, g(t)) \end{aligned} \tag{2.9}$$

with boundary conditions (1.2) or (1.3). This equation is a particular case of (1.1), when μ is constant on $[c, d]$, except at m_1 values of ξ, where it has jumps of discontinuity.

Theorem 2.4. *Assume* (H1)*,* $0 \leq p(t, \xi)$ *and*

$$\int_0^\infty \sum_{j=1}^{m_1} Q_j(t)\, dt = \infty\,. \tag{2.10}$$

Then every solution of (2.9) *is oscillatory or its "average"* $v(t)$ *converges to zero, as* $t \to \infty$*.*

Proof. Assuming that $u(x,t)$ is an eventually positive solution of (2.9), we show that the "average" function approaches zero. By (H1) there exists a time t_0 such that $u(x,t)$, $u(x, r(t,\xi))$, $u(x, h_j(t))$, and $u(x, g_i(t))$ are positive for all $t \geq t_0$ and all j, ξ. Then we define z by (2.3), so that $z(t) > 0$, and (2.4) and (2.7) hold. For the value k defined in (2.7), the function $z^{(k)}(t)$ is positive and decreasing. Therefore, $L := \lim_{t\to\infty} z^{(k)}(t)$ exists as a finite number. Note that

$$0 \leq z^{(k)}(t) - L = \frac{(-1)^{n-k}}{(n-1-k)!} \int_t^\infty (s-t)^{n-1-k} z^{(n)}(s)\, ds\,. \tag{2.11}$$

Note that the left-hand side is a finite number for each t; therefore, the integral on the right-hand side is convergent. From (2.4), it follows that for every $j \in$

$\{1, \ldots, m_1\}$,

$$0 \leq \int_0^\infty t^{n-1-k} Q_j(t) v(g_j(t))\, dt < \infty\,.$$

Using (2.10) and the limit comparison test,

$$\limsup_{t\to\infty} \frac{t^{n-1-k} Q_j(t) v(g_j(t))}{Q_j(t)} = 0$$

for at least one index j. Since $0 \leq k \leq n-1$, for this index, $\lim_{t\to\infty} v(g_j(t)) = 0$. Since g_j is continuous and approaches ∞ as $t \to \infty$, we have $\lim_{t\to\infty} v(t) = 0$. For an eventually negative solution u, we note that $-u$ is also a solution and it is eventually positive. This completes the proof. □

In the next theorem, we relax the conditions on Q_j, but restrict the values of $p(t, \xi)$.

Theorem 2.5. *Assume:* (H1) *holds; there exist a positive constant $\hat{p}$ such that $0 \leq \int_a^b p(t, \xi)\, d\mu(\xi) \leq \hat{p} < 1$ for $t > 0$; $\hat{g}(t)$ is differentiable and strictly increasing; and there exist positive constants α, γ such that $\gamma t^\alpha \leq \hat{g}(t)$ for t sufficiently large, and β with $0 \leq \beta < \alpha(n-1)$ such that*

$$\int_0^\infty t^\beta \sum_{j=1}^{m_1} Q_j(t)\, dt = \infty\,. \tag{2.12}$$

Then every solution of (2.9) *is oscillatory or its "average" $v(t)$ converges to zero as $t \to \infty$.*

Proof. Assuming that $u(x, t)$ is an eventually positive solution, we show that the "average" function approaches zero. Define z by (2.3), so that $z(t)$ is positive, and (2.4) and (2.7) hold.

Case 1: $z(t)$ is decreasing. In this case $k = 0$ in (2.7); thus $L := \lim_{t\to\infty} z(t)$ exists as a finite number. The same process as in the proof of Theorem 2.4 shows that $\lim_{t\to\infty} v(t) = 0$.

Case 2: $z(t)$ is increasing. This happens when n is even, because $k \geq 1$ in (2.7), and sometimes when n is odd. Note that $r(t, \xi) \leq t$ and $z(r) \leq z(t)$. Also note that $v(r) \leq z(r)$, so that by (2.3),

$$\begin{aligned} (1-\hat{p}) z(t) &\leq \Big(1 - \int_a^b p(t, \xi)\, d\mu(\xi)\Big) z(t) \\ &\leq z(t) - \int_a^b p(t, \xi) z(r)\, d\mu(\xi) \\ &\leq z(t) - \int_a^b p(t, \xi) v(r)\, d\mu(\xi) = v(t)\,. \end{aligned} \tag{2.13}$$

From (2.4), using that $\hat{g}(t) \leq g(t, \xi)$, we have

$$z^{(n)}(t) < -z(\hat{g}(t))(1-\hat{p}) \sum_{j=1}^{m_1} Q_j(t)\,, \tag{2.14}$$

Then for $\beta \geq 0$, we define

$$w(t) = \frac{z^{(n-1)}(t)}{z(\frac{1}{2}\hat{g}(t))} t^\beta$$

and differentiate with respect to t,

$$w'(t) = \frac{z^{(n)}(t)}{z(\frac{1}{2}\hat{g})} t^\beta - \frac{z^{(n-1)}(t) z'(\frac{1}{2}\hat{g}) \frac{1}{2}\hat{g}'(t)}{(z(\frac{1}{2}\hat{g}))^2} t^\beta + \frac{z^{(n)}(t)}{z(\frac{1}{2}\hat{g})} \beta t^{\beta-1} .$$

To estimate the first term in the right-hand side, we use (2.14) and the fact that $z(\hat{g})/z(\frac{1}{2}\hat{g}) \geq 1$ because z is increasing. To estimate the second term, we use Lemma 2.3. Since $0 \leq k \leq n-1$, we can make M_1 independent of k, hence independent of the function z. By setting $\lambda = 1/2$ and using z' instead of z and $\hat{g}(t)$ instead of t, we have constants M and t_2 such that

$$z'(\frac{1}{2}\hat{g}(t)) \geq M(\hat{g}(t))^{n-2} z^{(n-1)}(\hat{g}(t)), \quad \text{for } t \geq t_2 . \tag{2.15}$$

To estimate the third term, we multiply and divide by t^β. Then

$$w'(t) \leq -t^\beta (1-\hat{p}) \sum_{j=1}^{m_1} Q_j(t) - \big[\frac{M\hat{g}^{n-2}\hat{g}'}{2t^\beta} w^2 - \frac{\beta}{t} w\big] .$$

By completing the square in the brackets,

$$w'(t) \leq -t^\beta (1-\hat{p}) \sum_{j=1}^{m_1} Q_j(t) + \frac{\beta^2 t^{\beta-2}}{2M\hat{g}^{n-2}(t)\hat{g}'(t)} .$$

Integrating from t_1 to s,

$$w(s) \leq w(t_1) - \int_{t_1}^{s} t^\beta (1-\hat{p}) \sum_{j=1}^{m_1} Q_j(t) + \int_{t_1}^{s} \frac{\beta^2 t^{\beta-2}}{2M\hat{g}^{n-2}(t)\hat{g}'(t)} \, dt .$$

Note that the left-hand side remains positive while the right-hand side approaches $-\infty$ as $x \to \infty$. By (2.12) the first integral approaches ∞ while the second integral converges as explained below. This contradiction indicates that there are no eventually positive solutions under assumption (2.12). To study the convergence of the second integral, we use the limit comparison test and L'Hôpital's Rule, so that $\int^\infty t^{\beta-2} \frac{1}{\hat{g}^{n-2}\hat{g}'}$ and $\int^\infty t^{\beta-2} \frac{t}{\hat{g}^{n-1}}$ both converge or both diverge. Now, we use the comparison test,

$$0 < \frac{t^{\beta-1}}{\hat{g}^{n-1}} \leq \frac{t^{\beta-1}}{\gamma^{n-1} t^{\alpha(n-1)}} = \frac{1}{\gamma^{n-1}} t^{\beta-1-\alpha(n-1)} .$$

By the p-test, the integral converges if $\beta - 1 - \alpha(n-1) < -1$; i.e., $\beta < \alpha(n-1)$ which is assumed in this theorem. For an eventually negative solution u, we note that $-u$ is also a solution and it is eventually positive. This completes the proof. □

Remark. Instead of t^β, Wang [14] and Luo [10] used a positive nondecreasing function. They also used a function $H(t,s)\rho(s)$. However, their hypotheses are not easy to verify, and do not seem to cover a much wider range of coefficients for (1.1). An increasing function $\phi(t)$ played the role of t^β in [9], for n even. In the next theorem, we restrict n to be even, so we can study the case when $\sum Q$ is replaced by $\int Q$. Also we obtain results stronger than in Theorem 2.5.

Theorem 2.6. *Assume:* (H1) *holds; n is even; there exist a positive constant $\hat{p}$ such that $0 \le \int_a^b p(t,\xi)\,d\mu(\xi) \le \hat{p} < 1$ for $t > 0$; $\hat{g}(t)$ is differentiable and strictly increasing; there exist positive constants α, γ such that $\gamma t^\alpha \le \hat{g}(t)$ for t sufficiently large; and there exists β with $0 \le \beta < \alpha(n-1)$ such that*

$$\int_0^\infty t^\beta \int_c^d Q(t,\xi)\,d\mu(\xi) = \infty\,. \tag{2.16}$$

Then every solution of (1.1) *is oscillatory.*

Proof. Assuming that $u(x,t)$ is an eventually positive solution, we find a contradiction. Define z by (2.3), so that $z(t)$ is positive, and (2.4) and (2.7) hold. Because n is even, $k \ge 1$ in (2.7); therefore $z(t)$ is positive and increasing. The rest of the proof is as in the proof of case 2 in Theorem 2.5, except for using $\int_c^d Qt,\xi)$ instead of $\sum_{j=1}^{m_1} Q_j(t)$. □

Note that α in Theorems 2.5 and 2.6 can not exceed 1, because $\hat{g}(t) \le t$. Also note that when $\alpha = 1$, the exponent β can be close to $n-1$, which seems to be the optimal exponent, even for special cases of (1.1); see [3, Theorem 5.2.6] Next, we allow the coefficient p_1 to be negative in the equation

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\Big(u(x,t) + p_1(t)u(x,r_1(t))\Big)\\ &= \sum_{j=1}^{m_2} a_j(t)\Delta u(x,h_j(t)) - \sum_{j=1}^{m_1} q_j(x,t)u(x,g_j(t))\,,\end{aligned} \tag{2.17}$$

with boundary conditions (1.2) or (1.3).

Theorem 2.7. *Assume* (H1) *holds; there exists a constant $\hat{p}$ such that $\hat{p} < p_1(t) \le 0$; and*

$$\int_0^\infty \sum_{j=1}^{m_1} Q_j(t)\,dt = \infty\,.$$

Then every solution of (2.17) *is oscillatory, or its "average" $v(t)$ converges to zero, or $v(t)$ approaches infinity at least at the rate of t^{n-2} (as $t \to \infty$).*

Proof. Assuming that $u(x,t)$ is an eventually positive solution, we show that the "average" function approaches zero, or the "average" is bounded below a constant times t^{n-2} for t large. Define z by (2.3). Then (2.4) holds and $z^{(n)}(t) < 0$, for $t \ge t_0$. As in Lemma 2.1(i), there exists a $t_1 \ge t_0$ such that $z^{(0)}(t), \dots, z^{(n-1)}(t)$ are of constant sign on $[t_1, \infty)$.

Case 1: $z^{(i)}(t) > 0$ for some $i \in \{0, \dots, n-1\}$. Then the conditions of Lemma 2.1 are satisfied with $z^{(i)}(t)$ instead of $z(t)$. We proceed as in the proof of Theorem 2.4 to show that $\lim_{t\to\infty} v(t) = 0$.

Case 2: $z^{(i)}(t) < 0$ for all $i \in \{0, \dots, n-1\}$. By a repeated integration of $z^{(n)}$, we obtain a negative constant M, such that $z(t) < Mt^{n-2}$ for t sufficiently large.

From (2.3) and $\hat{p} < p_1(t)$, we obtain $z(t) = v(t) + p_1(t)v(r_1(t)) > \hat{p}v(r_1(t))$. Since $r_1(t) \le t$, we have

$$\frac{v(r_1(t))}{(r_1(t))^{n-2}} \ge \frac{v(r_1(t))}{t^{n-2}} > \frac{M}{\hat{p}} > 0\,.$$

Recall that $\lim_{t\to\infty} r_1(t) = \infty$. Therefore, $v(t) \ge Mt^{n-2}/\hat{p}$ for all t sufficiently large. This completes the proof. □

In the next theorem, we impose restrictions on n, r_1 and p_1, so that we obtain results stronger than those in Theorem 2.7.

Theorem 2.8. *Assume* (H1) *holds, n is odd, $-1 \le p_1(t) \le 0$, $r_1(t)$ is strictly increasing, and*

$$\int_0^\infty \sum_{j=1}^{m_1} Q_j(t)\,dt = \infty\,.$$

Then every solution of (2.17) *is oscillatory, or its "average" $v(t)$ converges to zero.*

Proof. Assuming that $u(x,t)$ is an eventually positive solution, we show that the "average" function approaches zero. Define z by (2.3). Then (2.4) holds and $z^n(t) < 0$, for $t \ge t_0$. As in Lemma 2.1(i), there exists a $t_1 \ge t_0$ such that $z^{(0)}(t), \dots, z^{(n-1)}(t)$ are of constant sign on $[t_1,\infty)$.

Claim: $z(t) > 0$ for $t \ge t_1$. The proof of this claim is a generalization of the proof in [3, Lemma 5.14]. On the contrary assume that $z(t) < 0$ for $t \ge t_1$. Since n is odd and $z^{(n)}(t) < 0$, by Lemma 2.2, $z^{(1)}(t) < 0$. Thus $z(t)$ is negative and decreasing. For $t > t_1$, we have

$$0 > z(t_1) > z(t) = v(t) + p_1(t)v(r_1(t)) \ge v(t) - v(r_1(t))\,.$$

For t, $r_1^{-1}(t)$, $r_1^{-1}(r_1^{-1}(t))$, …, the above inequality yields $z(t_1) > v(t) - v(r_1(t))$, $z(t_1) > v(r_1^{-1}(t)) - v(t)$, $z(t_1) > v(r_1^{-1}(r_1^{-1}(t))) - v(r_1^{-1}(t))$, …. Adding k of these inequalities, we have

$$kz(t_1) > v(r_1^{-1}(r_1^{-1}\dots(t))) - v(r_1(t)) \ge -v(r_1(t))$$

For a fixed value of t, the left-hand side approaches $-\infty$ as $k \to \infty$, while the right-hand side is a finite number. This contradiction proves the claim. Once we know that $z(t)$ is positive, we proceed as in Theorem 2.4 to show that $\lim_{t\to\infty} v(t) = 0$. □

3. Oscillation for related neutral PDEs

In this section, we study a nonlinear PDE, and then an equation more general than (1.1). We consider the neutral differential equation

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\Big(u(x,t) + \int_a^b p(t,\xi)u(x,r(t,\xi))\,d\mu(\xi)\Big)\\&= \sum_{j=1}^{m_2} a_j(t)\Delta u(x,h_j(t)) - q(x,t)F\big(u(x,g_1(t)),\dots,u(x,g_{m_1}(t))\big)\,,\end{aligned} \tag{3.1}$$

where $F \in C(\mathbb{R}^{m_1} \to \mathbb{R})$ satisfies the following two properties: If $u_j > 0$ for all $j \in \{1,\dots,m_1\}$, then $F(u_1,\dots,u_{m_1}) \ge u_j$ for all j; if $u_j < 0$ for all $j \in \{1,\dots,m_1\}$, then $F(u_1,\dots,u_{m_1}) \le u_j$ for all j. The conditions in (H1) are also assumed with the part corresponding to q replaced by

$$0 \le \min_{x\in\Omega} q(x,t) := Q(t)\,.$$

For defining the "average" $v(t)$, we consider the eventually positive and eventually negative solutions separately. When $u(x,t)>0$, define $v(t)$ by (2.1), and $z(t)$ by (2.3). Then instead of 2.4 we obtain

$$z^{(n)}(t) < -Q(t)v(g_j(t)) \quad \text{for all } j\in\{1,\dots,m_1\}\,. \tag{3.2}$$

When $u(x,t)<0$, for the boundary condition (1.2), define $v(t) = -\int_\Omega \phi(x)u(x,t)\,dx$ so that $v(t)>0$. Then multiply each term in (3.1) by $-\phi(x)$ and integrate over Ω. Using the assumptions on F, we obtain (3.2) again. For the boundary condition (1.3), define $v(t) = -\int_\Omega u(x,t)\,dx$ so that $v(t)>0$. Then multiply each term in (3.1) by -1 and integrate over Ω. Using the assumptions on F, we obtain (3.2) again. Once the differential inequality (3.2) is established for non-oscillatory solutions, the results in Theorems 2.4, 2.5, 2.6, 2.7, 2.8 follow with a minor change in notation: Use $\int_0^\infty Q(t)=\infty$ instead of $\int_0^\infty \sum Q_j(t)=\infty$. The second equation to be considered in this section is the neutral equation

$$\begin{aligned}\frac{\partial}{\partial t}\Big(b(t)\frac{\partial^{n-1}}{\partial t^{n-1}}\Big(u(x,t)+\int_a^b p(t,\xi)u(x,r(t,\xi))\,d\mu(\xi)\Big)\Big)\\ =\sum_{j=1}^m a_j(t)\Delta u(x,h_{ij}(t)) - \sum_{j=1}^{m_1} q_j(x,t)u(x,g_j(t)),\end{aligned} \tag{3.3}$$

where $n\ge 3$, and $b(t)$ is a positive function in $C^1([0,\infty),\mathbb{R})$ such that

$$\int_0^\infty \frac{1}{b(t)}\,dt=\infty. \tag{3.4}$$

A system of this form was studied in [9] when n is odd, with $\int p$ replaced by $\sum c_j$, where $0\le \sum c_j<1$. Assuming that $u(x,t)>0$ for $t\ge t_0$, we define $z(t)$ by (2.3). From (3.3) and each one of the two boundary conditions, (1.2) and (1.3), we have

$$\big(b(t)z^{(n-1)}(t)\big)' < -\sum_{j=1}^{m_1} Q_j(t)v(g_j(t)) \quad \text{for } t\ge t_0\,. \tag{3.5}$$

Therefore, $b(t)z^{(n-1)}(t)$ is a decreasing function; hence, eventually positive or eventually negative.

Claim: $b(t)z^{(n-1)}(t)$ and $z^{(n-1)}(t)$ are eventually positive, when $z(t)>0$. We proceed as in [9]: If there is a time for which $b(t)z^{(n-1)}(t)\le 0$, then because $b(t)z^{(n-1)}(t)$ is decreasing, there is t_1 such that for all $t\ge t_1$,

$$z^{(n-1)}(t) \le \frac{1}{b(t)}b(t_1)z^{(n-1)}(t_1) < 0\,.$$

Integrating the above inequality, we obtain

$$z^{(n-2)}(s) \le z^{(n-2)}(t_1) + b(t_1)z^{(n-1)}(t_1)\int_{t_1}^s \frac{1}{b(t)}\,dt\,.$$

By (3.4), the right-hand side approaches $-\infty$ as $s \to \infty$; thus $\lim_{t\to\infty} z^{(n-2)}(t) = -\infty$. Integrating $z^{(n-2)}(t)$, we show that $\lim_{t\to\infty} z^{(n-3)}(t) = -\infty$. Repeating the integration, we show that $z^{(n-4)}(t), \dots, z^{(0)}(t)$ approach $-\infty$ as $t \to \infty$. This contradicts $z(t) > 0$, and proves that $b(t)z^{(n-1)}(t)$ is eventually positive, and so is $z^{(n-1)}(t)$. The main result for (3.3) reads as follows.

Theorem 3.1. *Assume:* (H1) *holds; there exist a positive constant $\hat{p}$ such that $0 \le \int_a^b p(t,\xi)\, d\mu(\xi) \le \hat{p} < 1$ for $t > 0$; b is bounded; and*

$$\int_0^\infty \sum_{j=1}^{m_1} Q_j(t)\, dt = \infty\,.$$

Then every solution of (3.3) *is oscillatory, or its "average" $v(t)$ converges to zero, or $v(t)$ approaches infinity at least at the rate of t^{n-3} (as $t \to \infty$).*

Proof. Assuming that $u(x,t)$ is an eventually positive solution of (3.3), we show that the "average" function approaches zero, or the "average" is bounded below by a constant times t^{n-3}. By (H1) there exists a time t_0 such that $u(x,t)$, $u(x,r(t,\xi))$, $u(x,h_j(t))$, and $u(x,g_i(t))$ are positive for all $t \ge t_0$ and all j,ξ. Then we define z by (2.3), so that $z(t) > 0$. Then by the claim after (3.5), $z^{(n-1)}(t) > 0$. As in Lemma 2.1(i), there exists a $t_1 \ge t_0$ such that $z^{(0)}(t), \dots, z^{(n-1)}(t)$ are of constant sign on $[t_1,\infty)$. By Lemma 2.3 with $n-1$ instead of n, we have only two possible cases.

Case 1: $z^{(i)}(t) < 0$ for some $i \in \{1,\dots,n-2\}$. By Lemma 2.1, $z^{(n-2)}(t)$ is negative and increasing; so $L := \lim_{t\to\infty} z^{(n-2)}(t)$ exists as finite number. Then $L - z^{(n-2)}(t) = \int_t^\infty z^{(n-1)}(s)\, ds$. Therefore, $\lim_{t\to\infty} z^{(n-1)}(t) = 0$ and because $b(t)$ is bounded, $\lim_{t\to\infty} b(t)z^{(n-1)}(t) = 0$. Then using (3.2)

$$0 - b(t)z^{(n-1)}(t) = \int_t^\infty \big(b(s)z^{(n-1)}(s)\big)' < -\int_t^\infty \sum_{j=1}^{m_1} Q_j(t)v(g(_j(t))\,.$$

Therefore, $0 \le \int_0^\infty Q_j(t)v(g_j(t)) < +\infty$ for each $j \in \{0,\dots,m_1\}$. We proceed as in the proof of Theorem 2.4 to show that $\lim_{t\to\infty} v(t) = 0$.

Case 2: $z^{(i)}(t) > 0$ for all $i \in \{0,\dots,n-1\}$. Note that $z(t)$ is positive and increasing. By repeated integration of $z^{(n-1)}$, we obtain a positive constant M such that $z(t) \ge Mt^{n-3}$ for t sufficiently large. Since $z(t)$ is increasing and $v(r(t)) \le z(r(t)) \le z(t)$, by (2.13), $(1-\hat{p})z(t) \le v(t)$. Therefore, $(1-\hat{p})Mt^{n-3} < v(t)$, which completes the proof. □

Now, we allow the coefficient p_1 to be negative in the equation

$$\begin{aligned}&\frac{\partial}{\partial t}\Big(b(t)\frac{\partial^{n-1}}{\partial t^{n-1}}\Big(u(x,t) + p_1(t)u(x,r_1(t))\Big)\Big)\\ &= \sum_{j=1}^{m} a_j(t)\Delta u(x,h_{ij}(t)) - \sum_{j=1}^{m_1} q_j(x,t)u(x,g_j(t)),\end{aligned} \tag{3.6}$$

Theorem 3.2. *Assume* (H1) *holds;* $b(t)$ *is bounded; there is constant* $\bar{p}$ *such that* $\bar{p} < p_1(t) \leq 0$*; and*

$$\int_0^\infty \sum_{j=1}^{m_1} Q_j(t)\,dt = \infty\,.$$

Then every solution of (3.6) *is oscillatory, or its "average"* $v(t)$ *converges to zero, or* $v(t)$ *approaches infinity at least at the rate of* t^{n-3} *(as* $t \to \infty$*).*

Proof. Assuming that $u(x,t)$ is an eventually positive solution of (3.6), we show that the "average" function approaches zero, or the "average" is bounded below by a constant times t^{n-3}. By (H1) there exists a time t_0 such that $u(x,t)$, $u(x,r(t,\xi))$, $u(x,h_j(t))$, and $u(x,g_i(t))$ are positive for all $t \geq t_0$ and all j,ξ. Then we define z by (2.3). By (3.5) $b(t)z^{(n-1)}(t)$ is decreasing; thus $b(t)z^{(n-1)}(t)$ is eventually positive or eventually negative. Using that $b(t) > 0$, we consider four possible cases.

Case 1.1: $z^{(n-1)}(t) > 0$ and $z^{(i)}(t) < 0$ for some $i \in \{0,\dots,n-2\}$. By Lemma 2.1, $z^{(n-2)}(t)$ negative and increasing. So $L := \lim_{t\to\infty} z^{(n-2)}(t)$ exists as finite number. Then by (3.5), $L - z^{(n-2)}(t) = \int_t^\infty z^{(n-1)}(s)\,ds$. Therefore, $\lim_{t\to\infty} z^{(n-1)}(t) = 0$ and because $b(t)$ is bounded, $\lim_{t\to\infty} b(t)z^{(n-1)}(t) = 0$. Then

$$0 - b(t)z^{(n-1)}(t) = \int_t^\infty \big(b(s)z^{(n-1)}(s)\big)' < -\int_t^\infty \sum_{j=1}^{m_1} Q_j(t)v(g(_j(t))\,.$$

Therefore, $0 \leq \int_0^\infty Q_j(t)v(g_j(t)) < +\infty$ for each $j \in \{0,\dots,m_1\}$. We proceed as in the proof of Theorem 2.4 to show that $\lim_{t\to\infty} v(t) = 0$.

Case 1.2: $z^{(n-1)}(t) > 0$ and $z^{(i)}(t) > 0$ for all $i \in \{0,\dots,n-2\}$. By repeated integration of $z^{(n-1)}$, we obtain a positive constant M such that $z(t) \geq Mt^{n-3}$ for t sufficiently large. Since $p_1(t) \leq 0$, $z(t) < v(t)$. Therefore, $Mt^{n-3} < v(t)$.

Case 2.1: $z^{(n-1)}(t) < 0$ and $z^{(i)}(t) > 0$ for some $i \in \{0,\dots,n-2\}$. By Lemma 2.1, $z^{(n-2)}(t)$ is positive and decreasing, As in case 1.1.

$$0 - b(t)z^{(n-1)}(t) = \int_t^\infty \big(b(s)z^{(n-1)}(s)\big)' < -\int_t^\infty \sum_{j=1}^{m_1} Q_j(t)v(g(_j(t))\,.$$

Note that the left-hand side is positive, while the right-hand side is negative. This contradiction indicates that this case does not happen.

Case 2.2: $z^{(n-1)}(t) < 0$ and $z^{(i)}(t) < 0$ for all $i \in \{0,\dots,n-2\}$. By repeated integration of $z^{(n-1)}$, we obtain a negative constant M such that $z(t) \leq Mt^{n-3}$ for t sufficiently large. Since $\bar{p} \leq p_1(t) \leq 0$, $z(t) > p_1(t)v(r(t)) \geq \bar{p}v(r(t))$. Using that $r(t) \leq t$, we have $M(r(t))^{n-3} \geq Mt^{n-3} > \bar{p}v(r(t))$. Recall that $\bar{p}$ and M are negative, and that $\lim_{t\to\infty} r(t) = \infty$. Then $v(t) \geq M_1 t^{n-3}$ for t large, where M_1 is a positive constant. This completes the proof. □

4. Oscillation for a system of neutral PDEs

In this section, we study the system

$$\begin{aligned}&\frac{\partial^n}{\partial t^n}\Big(u_i(x,t)+\int_a^b p_i(t)u_i(x,r_i(t))\Big)\\&=\sum_{j=1}^{m_2} a_{ij}(t)\Delta u_i(x,h_{ij}(t))-\sum_{j=1}^{m}\int_c^d q_{ij}(x,t,\xi)u_j(x,g_j(t,\xi))\,d\mu(\xi),\end{aligned} \tag{4.1}$$

for $x\in\Omega$, $t\ge 0$, $i=1,\dots,m$. To the above system we attach one the following two boundary conditions

$$u_i(x,t)=0 \quad x\in\partial\Omega,\ t\ge 0,\ i=1,\dots,m \tag{4.2}$$

$$\frac{\partial u_i}{\partial\nu}+\gamma_i(x,t)u_i(x,t)=0 \quad x\in\partial\Omega,\ t\ge 0,\ i=1,\dots,m\,. \tag{4.3}$$

A system of this type was studied by [9], when n is even and the delays are constants. Their results correspond to Theorem 2.6 with $\beta=0$, but their hypotheses need additional assumptions on p, or q, or f to guarantee that $[\lambda(t)V^{(n-1)}(t)]'<0$ on page 110. A solution $(u_1,u_2,\dots,u_m)$ of (4.1) is said to be oscillatory, if at least one component is oscillatory. For eventually positive or eventually negative components, we define $\delta_i=\operatorname{sign}(u_i)$; thus, $|u_i|=\delta_i u_i$. Our next step is to transform the coupled system of PDEs (4.1) into an uncoupled system of differential inequalities. When the boundary condition (4.2) is satisfied, we multiply each equation in (4.1) by the first eigenvalue of the Laplacian and by δ_i. Then integrate over Ω and add over $i=1,\dots,m$. Let

$$v_i(t)=\int_\Omega \phi_1(x)\delta_i u_i(x,t)\,dx\,, \tag{4.4}$$

which is eventually positive when u_i is non-oscillatory. Let

$$z_i(t)=v_i(t)+\int_a^b p_i(t)v_i(r_i(t))\,. \tag{4.5}$$

By Green's formula and (4.2), the first summation in the right-hand side of (4.1) leads to a negative quantity. Therefore,

$$\sum_{i=1}^m z_i^{(n)}(t)<-\sum_{i,j}^m\int_\Omega\int_a^b q_{ij}(x,t,\xi)\delta_i u_j(x,g_j(t,\xi))\,d\mu(\xi)\,.$$

Let $\bar q_{ij}(t,\xi)=\min_{x\in\Omega}q_{ij}(x,t,\xi)$ and $\hat q_{ij}(t,\xi)=\max_{x\in\Omega}q_{ij}(x,t,\xi)$. Note that

$$\int_\Omega q_{jj}(x,t,\xi)\delta_j u_j(x,g_j)\ge\bar q_{jj}(t,\xi)\int_\Omega\delta_j u_j(x,g_j)=\bar q_{jj}(t,\xi)v_j(g_j(t,\xi))\,,$$

where v_j is defined by (4.4). Also note that $\delta_i u_j\le\delta_j u_j$ and

$$\int_\Omega q_{ij}(x,t,\xi)\delta_i u_j(x,g_j)\le\hat q_{ij}(t,\xi)\int_\Omega\delta_j u_j(x,g_j)=\hat q_{ij}(t,\xi)v_j(g_j(t,\xi))\,.$$

The three inequalities above imply

$$\sum_{i=1}^m z_i^{(n)}(t)<-\sum_{j=1}^m\int_c^d\big[\bar q_{jj}(t,\xi)-\sum_{i\ne j}^m\hat q_{ij}(t,\xi)\big]v_j(g_j(t,\xi))\,d\mu(\xi)\,. \tag{4.6}$$

With the notation in (H2) below, provided that $z_i(t) \ge 0$, we obtain the uncoupled system of inequalities

$$\begin{aligned} z_i^{(n)}(t) &< -\int_c^d Q_i(t,\xi) v_i(g_i(t,\xi))\, d\mu(\xi), \\ z_i^{(n)}(t) &< -\sum_{k=1}^{m_1} Q_{ik}(t) v_i(g_{ik}(t)) \quad \text{(for the summation case)} \end{aligned} \tag{4.7}$$

where $i = 1,\dots,m$. The hypotheses in (H1) are modified as follows

(H2) $0 \le a_{ij}(t)$ with $0 < \sum_{j=1}^{m_2} a_{ij}(t)$ for $t \ge 0$, $1 \le i \le m$.
There exist continuous functions $\hat g_i(t)$ such that $\hat g_i(t) \le g_i(t,\xi)$ and for all t,ξ,i (in the summation case: $\hat g_i(t) \le g_{ik}(t)$ for all k).
$h_{ij}(t), \hat g_i(t), r_i(t,\xi)$ approach $+\infty$ as $t \to \infty$, for all $\xi \in [a,b]$, $j \in \{1,\dots m_2\}$, $i \in \{1,\dots m\}$;
let $\bar q_{ij}(t,\xi) = \min_{x\in\Omega} q_{ij}(x,t,\xi)$ and $\hat q_{ij}(t,\xi) = \max_{x\in\Omega} q_{ij}(x,t,\xi)$; assume that

$$0 \le \bar q_{jj}(t,\xi) - \sum_{i\ne j}^{m} \hat q_{ij}(t,\xi) := Q_j(t,\xi)\,. \tag{4.8}$$

When p is non-negative, each component z_i is positive, then the inequality (4.7) holds. So that the proofs of Theorems 2.4, 2.5 and 2.6 apply to each component. However, when $p \le 0$, the inequality (4.7) may not hold. So we cannot state analogs for Theorems 2.7 and 2.8. We state only the analog to Theorem 2.4. The other theorems require similar changes in notation. Consider the system

$$\begin{aligned} &\frac{\partial^n}{\partial t^n}\Big(u_i(x,t) + \int_a^b p_i(t) u_i(x, r_i(t))\Big) \\ &= \sum_{j=1}^{m_2} a_{ij}(t) \Delta u_i(x, h_{ij}(t)) - \sum_{j=1}^{m}\sum_{k=1}^{m_1} q_{ijk}(x,t) u_j(x, g_{jk}(t))\,. \end{aligned} \tag{4.9}$$

Theorem 4.1. *Assume* (H2) *holds;* $0 \le p_i(t,\xi)$ *for all* i,t,ξ*; and*

$$\int_0^\infty \sum_{k=1}^{m_1} Q_{ik}(t)\, dt = \infty\,.$$

Then each component of each solution of (4.9) *is oscillatory, or its "average"* $v_i(t)$ *converges to zero, as* $t \to \infty$.

Concluding Remarks. We studied oscillation only for a few range intervals of the coefficient $p(t)$, but there are many intervals to be considered. The case when p changes sign is also an open question. Another open question is oscillation for nonlinearities more general than those in Section 3.

References

[1] D. D. Bainov, D. P. Mishev; *Oscillation theory for neutral differential equations with delay*, Adam Hilger New York, 1990.

[2] Julio G. Dix, Dillip Kumar Ghose, Radhanath Rath; Oscillation of a higher order neutral differential equation with a sub-linear delay term and positive and negative coefficients. Mathematica Bohoemica, Vol. 134 (2009), No. 4, pp 411-425.

[3] L. H. Erbe, Qungkai Kong, B. G. Zhang; *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, Inc., New York, 1995.
[4] I. Gyori, G. Laddas; *Oscillation theory of delay differential equations*, Claendon Press, Oxford, 1991.
[5] Jost, Jürgen; *Partial differential equations.* Graduate Texts in Mathematics, 214. Springer-Verlag, New York, 2002.
[6] B. Karpuz, R. N. Rath and L. N. Padhy; On Oscillation and asymptotic behaviour of a higher order neutral differential equation with positive and negative coefficients, *Electronic J. of Diff. Eqs.*, vol 2008 (2008), No. 113, pp. 1-15.
[7] G. S. Ladde, V. Laxmikantam and B. G. Zhang: *Oscillation Theory of Differential Equations with Deviating Arguments*, Marcel Dekker Inc., New York, 1987.
[8] Wei Nian Li; L. Debnath; Oscillation of higher-order neutral partial functional differential equations., *Appl. Math. Lett.* 16 (2003), no. 4, 525–530.
[9] Wen-Xian Lin; Oscillation for systems of higher-order neutral type delay partial differential equations, *Appl. Math. Comput.* 156 (2004), no. 1, 107–114.
[10] Lipping Luo, Zhenghui Gao, Liu Yang; Oscillation criteria for boundary-value problems of systems of high-order partial functional differential equations. Preprint.
[11] Seshadev Padhi, Julio G. Dix; Oscillation of functional differential equations of n-th order with distributed deviating arguments, *Journal of contemporary Mathematics*, Vol. 1 no. 1 (2008), 11-24.
[12] R. N. Rath, N. Misra, P. P. Mishra and L. N. Padhy; Non-oscillatory behaviour of higher order functional differential equations of neutral type, *Electron. J. Diff. Eqns.*, Vol. 2007 (2007), No. 163, pp. 1-14.
[13] Y. Sahiner and A. Zafer; Bounded oscillation of non-linear neutral differential equations of arbitrary order, *Czech. Math. J.*, 51 (126)(2001), 185–195.
[14] Peiguang Wang; Yonghong Wu; Lou Caccetta; Oscillation criteria for boundary value problems of high-order partial functional differential equations, *J. Comput. Appl. Math.*, 206 (2007), no. 1, 567–577.

(Received July 15, 2010)

Department of Mathematics, Texas State University, 601 University Dr., San Marcos, TX 78666, USA
E-mail address: jd01@txstate.edu