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Note on multiplicative perturbation of local C-regularized cosine functions with nondensely defined generators

Fang Li[1*] and James H. Liu[2]
[1]School of Mathematics, Yunnan Normal University
Kunming 650092, People's Republic of China
fangli860@gmail.com

[2]Department of Mathematics and Statistics, James Madison University
Harrisonburg, VA 22807, USA
liujh@jmu.edu

Abstract

In this note, we obtain a new multiplicative perturbation theorem for local C-regularized cosine function with a nondensely defined generator A. An application to an integrodifferential equation is given.

Key words : Multiplicative perturbation, local C-regularized cosine functions, second order differential equation

1 Introduction and preliminaries

Let X be a Banach space, A an operator in X. It is well known that the cosine operator function is the main propagator of the following Cauchy problem for a second order differential equation in X:

$$\begin{cases} u''(t) = Au(t), \quad t \in (-\infty, \infty) \\ u(0) = u_0, \ u'(0) = u_1, \end{cases}$$

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which controls the behaviors of the solutions of the differential equations in many cases (cf., e.g., [2, 4–10, 13, 15, 16, 19–21]); if A is the generator of a C-regularized cosine function $\{\mathrm{C}(t)\}_{t\in\mathbf{R}}$, then $u(t) = C^{-1}\mathrm{C}(t)u_0 + C^{-1}\int_0^t \mathrm{C}(s)u_1 ds$ is the unique solution of the above Cauchy problem for every pair (u_0, u_1) of initial values in $C(D(A))$(see [5, 16, 20]). So it is valuable to study deeply the properties of the cosine operator functions.

As a meaningful generalization of the classical cosine operator functions, the C-regularized cosine functions have been investigated extensively (cf., e.g., [2, 4, 5, 9, 10, 13, 15, 16, 20, 21]), where C serves as a regularizing operator which is injective.

Stimulated by these works as well as the works on integrated semigroups and C-regularized semigroups ([3, 11, 14, 17, 18]), we study further the multiplicative perturbation of local C-regularized cosine functions with nondensely defined generators, in the case where (1) the range of the regularizing operator C is not dense in a Banach space X; (2) the operator C may not commute with the perturbation operator.

Throughout this paper, all operators are linear; $\mathcal{L}(X, Y)$ denotes the space of all continuous linear operators from X to a space Y, and $\mathcal{L}(X, X)$ will be abbreviated to $\mathcal{L}(X)$; $\mathcal{L}_s(X)$ is the space of all continuous linear operators from X to X with the strong operator topology; $\mathbf{C}([0, t], \mathcal{L}_s(X))$ denotes all continuous $\mathcal{L}(X)$-valued functions, equipped with the norm $\|F\|_\infty = \sup_{r\in[0,t]} \|F(r)\|$. Moreover, we write $D(A)$, $R(A)$, $\rho(A)$, respectively, for the domain, the range and the resolvent set of an operator A. We denote by $\widetilde{A}$ the part of A in $\overline{D(A)}$, that is,

$$\widetilde{A} \subset A,\ D(\widetilde{A}) = \{x \in D(A) | Ax \in \overline{D(A)}\}.$$

We abbreviate C-regularized cosine function to C-cosine function.

Definition 1.1. *Assume $\tau > 0$. A one-parameter family $\{\mathrm{C}(t); |t| \le \tau\} \subset \mathcal{L}(X)$ is called a local C-cosine function on X if*

(i) $\mathrm{C}(0) = C$ *and* $\mathrm{C}(t + s)C + \mathrm{C}(t - s)C = 2\mathrm{C}(t)\mathrm{C}(s) \quad (\forall |s|, |t|, |s + t| \le \tau)$,

(ii) $\mathrm{C}(\cdot)x : [-\tau, \tau] \longrightarrow X$ *is continuous for every* $x \in X$.

The associated sine operator function $\mathrm{S}(\cdot)$ is defined by $\mathrm{S}(t) := \int_0^t \mathrm{C}(s)ds\ (|t| \le \tau)$.

The operator A defined by

$$\begin{aligned} D(A) &= \{x \in X; \lim_{t\to 0^+} \frac{2}{t^2}(\mathrm{C}(t)x - Cx) \text{ exists and is in } R(C)\}, \\ Ax &= C^{-1} \lim_{t\to 0^+} \frac{2}{t^2}(\mathrm{C}(t)x - Cx), \quad \forall x \in D(A), \end{aligned}$$

is called the generator of $\{\mathrm{C}(t); |t| \le \tau\}$. It is also called that A generates $\{\mathrm{C}(t);\ |t| \le \tau\}$.

Lemma 1.2. ([2]) *Let A generate a local C-cosine function $\{\mathrm{C}(t);\ |t| \le \tau\}$ on X. Then*
(i)*For $x \in D(A), t \in [-\tau, \tau], \mathrm{C}(t)x,\ \mathrm{S}(t)x \in D(A), A\mathrm{C}(t)x = \mathrm{C}(t)Ax,\ A\mathrm{S}(t)x = \mathrm{S}(t)Ax$;*
(ii)*For $x \in X,\ t \in [0, \tau],\ \int_0^t \mathrm{S}(s)xds \in D(A)$ and $A\int_0^t \mathrm{S}(s)xds = \mathrm{C}(t)x - Cx$;*
(iii)*For $x \in D(A),\ t \in [0, \tau],\quad \int_0^t \mathrm{S}(s)Axds = A\int_0^t \mathrm{S}(s)xds = \mathrm{C}(t)x - Cx$.*

2 Results and proofs

Definition 2.1. *Let $\{\mathrm{C}(t); |t| \le \tau\}$ be a local C-cosine function on X. If a closed linear operator A in X satisfies*

(1) $\mathrm{C}(t)A \subset A\mathrm{C}(t),\ |t| \le \tau,$

(2) $\mathrm{C}(t)x = Cx + A\int_0^t \int_0^s \mathrm{C}(\sigma)xd\sigma ds,\ |t| \le \tau,\ x \in X,$

then we say that A subgenerates a local C-cosine function on X, or A is a subgenerator of a local C-cosine function on X.

Remark 2.2. *The generator $\mathcal{G}$ of a local C-cosine function $\{\mathrm{C}(t); |t| \le \tau\}$ is a subgenerator of $\{\mathrm{C}(t); |t| \le \tau\}$. But for each subgenerator A, one has $A \subset \mathcal{G}$ and $\mathcal{G} = C^{-1}AC$. Moreover, if $\rho(A) \ne \emptyset$, then $C^{-1}AC = A$.*

In fact, for $x \in D(A)$, we have

$$\begin{aligned}\lim_{t\to 0^+} \frac{2(\mathrm{C}(t)x - Cx)}{t^2} &= 2\lim_{t\to 0^+} \frac{A\int_0^t \int_0^s \mathrm{C}(\sigma)xd\sigma ds}{t^2} = 2\lim_{t\to 0^+} \frac{\int_0^t \int_0^s \mathrm{C}(\sigma)Axd\sigma ds}{t^2} \\ &= CAx \in R(C),\end{aligned}$$

that is $x \in D(\mathcal{G})$ and $Ax = \mathcal{G}x$, i.e., $A \subset \mathcal{G}$.

For $x \in D(C^{-1}AC)$, then $Cx \in D(A)$ and $ACx \in R(C)$, since $A \subset \mathcal{G}$ we have $\mathcal{G}Cx = ACx \in R(C)$, then $C^{-1}ACx = C^{-1}\mathcal{G}Cx = \mathcal{G}x$, i.e., $C^{-1}AC \subset \mathcal{G}$. On the other hand, for $x \in D(\mathcal{G})$, noting

$$\lim_{n\to\infty} 2n^2 \int_0^{\frac{1}{n}} \int_0^s \mathrm{C}(\sigma)xd\sigma ds = \lim_{n\to\infty} \frac{2\int_0^{\frac{1}{n}} \int_0^s \mathrm{C}(\sigma)xd\sigma ds}{\frac{1}{n^2}} = Cx,$$

and

$$\lim_{n\to\infty} A(2n^2\int_0^{\frac{1}{n}}\int_0^s \mathrm{C}(\sigma)xd\sigma ds) = \lim_{n\to\infty} 2n^2(\mathrm{C}(\frac{1}{n})x - Cx) = C\mathcal{G}x,$$

the closedness of A ensures $Cx \in D(A)$ and $ACx = C\mathcal{G}x$, therefore, we have $\mathcal{G} \subset C^{-1}AC$.

¿From Proposition 1.4 in [12], we can obtain $C^{-1}AC = A$ if $\rho(A) \neq \emptyset$.

Theorem 2.3. *Let nondensely defined operator A generate a local C-cosine function $\{\mathrm{C}(t); |t| \le \tau\}$ on X, $\mathrm{S}(t) = \int_0^t \mathrm{C}(s)ds$, and $B \in \mathcal{L}(\overline{D(A)})$. Then*

(1) *there exists an operator family $\{\mathrm{E}(t); |t| \le \tau\} \subset \mathcal{L}(X)$ such that*

$$\mathrm{E}(t)x = Cx + A(I+B)\int_0^t\int_0^s \mathrm{E}(\sigma)xd\sigma ds,\ \ |t| \le \tau,\ x \in \overline{D(A)},$$

provided that

(H1)

$$\left\| A\int_0^t \mathrm{S}(t-s)C^{-1}B\Phi(s)ds \right\| \le M\int_0^t \sup_{0\le s\le \sigma} \|\Phi(s)\| d\sigma,\ \ t \in [0,\,\tau],$$

where $\Phi \in \mathbf{C}([0,\tau],\,X)$, and $M > 0$ is a constant.

(2) *$(I+B)\widetilde{A}$ generates a local C_1-cosine function on $\overline{D(A)}$ provided that*

(H1')

$$\left\| \int_0^t \Phi(s)C^{-1}BA\mathrm{S}(t-s)xds \right\| \le M\|x\|\int_0^t \sup_{0\le s\le \sigma} \|\Phi(s)\| d\sigma,\ \ t \in [0,\,\tau],$$

where $x \in D(A)$, $\Phi \in \mathbf{C}([0,\tau],\,\mathcal{L}_s(X))$, and $M > 0$ is a constant,

(H2) *there exists an injective operator $C_1 \in \mathcal{L}(\overline{D(A)})$ such that $R(B) \subset R(C_1) \subset C(\overline{D(A)})$, $C_1(I+B)\widetilde{A} \subset (I+B)\widetilde{A}C_1$, and $C^{-1}C_1(D(\widetilde{A}))$ is a dense subspace in $D(A)$,*

(H3) $\rho((I+B)\widetilde{A}) \neq \emptyset$.

(3) *$\widetilde{A}(I+B)$ subgenerates a C_1-cosine function on $\overline{D(A)}$ provided that $C_1B = BC_1$, and* (H1'), (H2), *and* (H3) *hold.*

Proof. First, we prove the conclusion (2).

Define the operator functions $\{\overline{\mathrm{C}}_n(t)\}_{t\in[\,0,\,\tau]}$ as follows:

$$\begin{cases} \overline{\mathrm{C}}_0(t)x = \mathrm{C}(t)x, \\ \overline{\mathrm{C}}_n(t)x = \displaystyle\int_0^t \overline{\mathrm{C}}_{n-1}(s)C^{-1}BA\mathrm{S}(t-s)xds, \qquad x \in D(A),\, t \in [0,\,\tau], n = 1, 2, \cdots. \end{cases}$$

By induction, we obtain:

(i) $\overline{\mathrm{C}}_n(t) \in \mathbf{C}([\,0,\tau],\, \mathcal{L}_s(\overline{D(A)}))$;

(ii) $\|\overline{\mathrm{C}}_n(t)\| \le \dfrac{M^n t^n}{n!} \sup\limits_{s\in[0,\,\tau]} \|\mathrm{C}(s)\|, \quad t \in [0,\, \tau],\ \forall\, n \ge 0.$

It follows that the series $\displaystyle\sum_{n=0}^{\infty} \frac{M^n t^n}{n!}$ converges uniformly on $[\,0,\, \tau]$ and consequently,

$$\overline{\mathrm{C}}(t)x := \sum_{n=0}^{\infty} \overline{\mathrm{C}}_n(t)x \in \mathbf{C}([\,0,\, \tau],\, \overline{D(A)}),\ \forall x \in \overline{D(A)},$$

and satisfies

$$\overline{\mathrm{C}}(t)x = \mathrm{C}(t)x + \int_0^t \overline{\mathrm{C}}(s)C^{-1}BA\mathrm{S}(t-s)xds, \quad x \in D(A),\ t \in [\,0,\, \tau]. \tag{2.1}$$

Using (H1') and Gronwall's inequality, we can see the uniqueness of solution of (2.1).

Put

$$\widehat{\mathrm{C}}(t) := \overline{\mathrm{C}}(t)C^{-1}C_1,\ t \in [0,\, \tau].$$

It follows from (2.1) and $C^{-1}C_1 \in \mathcal{L}(\overline{D(A)})$ that for $x \in \overline{D(A)}$,

$$\widehat{\mathrm{C}}(t)x \in \mathbf{C}([\,0,\, \tau],\, \overline{D(A)}),$$

and satisfies

$$\widehat{\mathrm{C}}(t)x = \mathrm{C}(t)C^{-1}C_1x + \int_0^t \widehat{\mathrm{C}}(s)C_1^{-1}BA\mathrm{S}(t-s)C^{-1}C_1xds, \quad x \in D(A),\ t \in [\,0,\, \tau]. \tag{2.2}$$

Note that $D(\widetilde{A}) \subset D(C_1^{-1}B\widetilde{A}C_1)$ and $C^{-1}C_1$ maps $D(\widetilde{A})$ into $D(\widetilde{A})$. So, for $x \in D(\widetilde{A})$, by (2.2), we have

$$\begin{aligned} &\int_0^t \int_0^s \widehat{\mathrm{C}}(\sigma)C_1^{-1}B\widetilde{A}C_1xd\sigma ds \\ = &\int_0^t \int_0^s \mathrm{C}(\sigma)C^{-1}B\widetilde{A}C_1xd\sigma ds \\ &+ \int_0^t \int_0^s \widehat{\mathrm{C}}(\sigma)C_1^{-1}B\big[\mathrm{C}(s-\sigma)C^{-1}B\widetilde{A}C_1x - B\widetilde{A}C_1x\big]d\sigma ds. \end{aligned} \tag{2.3}$$

Therefore, for $x \in D(\widetilde{A})$, we have

$$
\begin{aligned}
& \int_0^t \int_0^s \widehat{\mathrm{C}}(\sigma)(I+B)\widetilde{A}x d\sigma ds \\
= \quad & \int_0^t \int_0^s \mathrm{C}(\sigma)C^{-1}(I+B)\widetilde{A}C_1 x d\sigma ds \\
& + \int_0^t \int_0^s \widehat{\mathrm{C}}(\sigma)C_1^{-1}B\big[\mathrm{C}(s-\sigma)C^{-1}(I+B)\widetilde{A}C_1x - (I+B)\widetilde{A}C_1x\big] d\sigma ds \\
= \quad & \mathrm{C}(t)C^{-1}C_1x - C_1x + \int_0^t \int_0^s \mathrm{C}(\sigma)C^{-1}B\widetilde{A}C_1 x d\sigma ds \\
& + \int_0^t \int_0^s \widehat{\mathrm{C}}(\sigma)C_1^{-1}B\mathrm{C}(s-\sigma)C^{-1}\widetilde{A}C_1 x d\sigma ds - \int_0^t \int_0^s \widehat{\mathrm{C}}(\sigma)C_1^{-1}B\widetilde{A}C_1 x d\sigma ds \\
& + \int_0^t \int_0^s \widehat{\mathrm{C}}(\sigma)C_1^{-1}B[\mathrm{C}(s-\sigma)C^{-1}B\widetilde{A}C_1x - B\widetilde{A}C_1x] d\sigma ds \\
\overset{(2.3)}{=} \quad & \widehat{\mathrm{C}}(t)x - C_1x. \qquad (2.4)
\end{aligned}
$$

Now we consider the integral equation

$$
v(t)x = C_1x + \int_0^t \int_0^s v(\sigma)(I+B)\widetilde{A}x d\sigma ds, \quad x \in D(\widetilde{A}),\ t \in [0,\ \tau], \qquad (2.5)
$$

where $v(t) \in \mathbf{C}([0,\tau], \mathcal{L}_s(\overline{D(A)}))$. Let $\widetilde{v}(t)$ satisfy the equation (2.5). Then from (2.5), we obtain, for $x \in D(\widetilde{A})$,

$$
\begin{aligned}
& \int_0^t \widetilde{v}(s)\mathrm{S}(t-s)C^{-1}C_1 x ds - C_1 \int_0^t \mathrm{S}(s)C^{-1}C_1 x ds \\
= \quad & \int_0^t \int_0^s \widetilde{v}(\sigma)(I+B)\widetilde{A} \int_0^{s-\sigma} \mathrm{S}(r)C^{-1}C_1 x dr d\sigma ds \\
= \quad & \int_0^t \widetilde{v}(s)\mathrm{S}(t-s)C^{-1}C_1 x ds - \int_0^t \int_0^s \widetilde{v}(\sigma)C_1 x d\sigma ds \\
& + \int_0^t \int_0^s \widetilde{v}(\sigma)B\widetilde{A} \int_0^{s-\sigma} \mathrm{S}(r)C^{-1}C_1 x dr d\sigma ds.
\end{aligned}
$$

Hence,

$$
\int_0^t \int_0^s \widetilde{v}(\sigma)C_1 x d\sigma ds = C_1 \int_0^t \mathrm{S}(s)C^{-1}C_1 x ds + \int_0^t \int_0^s \widetilde{v}(\sigma)B\widetilde{A} \int_0^{s-\sigma} \mathrm{S}(r)C^{-1}C_1 x dr d\sigma ds,
$$

that is,

$$
(\widetilde{v}(t)C)C^{-1}C_1 x = C_1\mathrm{C}(t)C^{-1}C_1 x + \int_0^t (\widetilde{v}(s)C)C^{-1}B\widetilde{A}\mathrm{S}(t-s)C^{-1}C_1 x ds.
$$

Note that $C^{-1}C_1(D(\widetilde{A})) \subset D(\widetilde{A})$ is dense in $\overline{D(A)}$, and the solution $\bar{w}(t)$ of the equation

$$
w(t)y = C_1\mathrm{C}(t)y + \int_0^t w(s)C^{-1}B\widetilde{A}\mathrm{S}(t-s)y ds, \quad y \in C^{-1}C_1(D(\widetilde{A})),\ t \in [0,\ \tau]
$$

in $\mathbf{C}([0,\tau],\mathcal{L}_s(\overline{D(A)}))$ is unique, we can see the solution of (2.5) is also unique.

By the uniqueness of solution of (2.5), we can obtain that

$$\widehat{\mathrm{C}}(-t)x=\widehat{\mathrm{C}}(t)x,\ \widehat{\mathrm{C}}(t)C_1x=C_1\widehat{\mathrm{C}}(t)x,\ \text{for each } x\in\overline{D(A)},\ t\in[0,\ \tau].$$

Moreover, for $t,h,t\pm h\in[0,\ \tau]$, we have

$$\begin{aligned}
&\widehat{\mathrm{C}}(t+h)C_1x+\widehat{\mathrm{C}}(t-h)C_1x\\
=&\int_0^{t+h}\int_0^s\widehat{\mathrm{C}}(\sigma)(I+B)AC_1xd\sigma ds+\int_0^{t-h}\int_0^s\widehat{\mathrm{C}}(\sigma)(I+B)AC_1xd\sigma ds+2C_1^2x\\
=&\int_0^h\int_0^s\widehat{\mathrm{C}}(t+\sigma)(I+B)AC_1xd\sigma ds+\int_0^t\int_0^{t-s}\widehat{\mathrm{C}}(\sigma)(I+B)AC_1xd\sigma ds\\
&+\int_0^h\int_0^t\widehat{\mathrm{C}}(t-\sigma)(I+B)AC_1xd\sigma ds+\int_0^h\int_0^s\widehat{\mathrm{C}}(t-\sigma)(I+B)AC_1xd\sigma ds\\
&+\int_0^t\int_0^{t-s}\widehat{\mathrm{C}}(\sigma)(I+B)AC_1xd\sigma ds-\int_0^h\int_0^t\widehat{\mathrm{C}}(t-\sigma)(I+B)AC_1xd\sigma ds\\
&+2C_1^2x\\
=&\int_0^h\int_0^s\left[\widehat{\mathrm{C}}(t+\sigma)C_1+\widehat{\mathrm{C}}(t-\sigma)C_1\right](I+B)Axd\sigma ds+2\int_0^t(t-s)\widehat{\mathrm{C}}(s)(I+B)AC_1xds\\
&+2C_1^2x,
\end{aligned}$$

and for all $x\in D(\widetilde{A})$, $t,h\in[0,\ \tau]$, we have

$$\begin{aligned}
2\widehat{\mathrm{C}}(t)\widehat{\mathrm{C}}(h)x &= 2\widehat{\mathrm{C}}(t)\Big[\int_0^h\int_0^s\widehat{\mathrm{C}}(\sigma)(I+B)\widetilde{A}xd\sigma ds+C_1x\Big]\\
&= \int_0^h\int_0^s 2\widehat{\mathrm{C}}(t)\widehat{\mathrm{C}}(\sigma)(I+B)Axd\sigma ds+2\int_0^t(t-s)\widehat{\mathrm{C}}(s)(I+B)\widetilde{A}C_1xds\\
&\quad +2C_1^2x.
\end{aligned}$$

Therefore, for $x\in D(\widetilde{A})$, $t\in[0,\ \tau]$,

$$\begin{aligned}
&[\widehat{\mathrm{C}}(t+h)C_1x+\widehat{\mathrm{C}}(t-h)C_1x]-2\widehat{\mathrm{C}}(t)\widehat{\mathrm{C}}(h)x\\
=&\int_0^t\int_0^s\Big\{[\widehat{\mathrm{C}}(\sigma+h)C_1+\widehat{\mathrm{C}}(\sigma-h)C_1]-2\widehat{\mathrm{C}}(\sigma)\widehat{\mathrm{C}}(h)\Big\}(I+B)\widetilde{A}xd\sigma ds.
\end{aligned}$$

It follows from the uniqueness of solution of (2.5) and the denseness of $\widetilde{A}$ in $\overline{D(A)}$ that

$$2\widehat{\mathrm{C}}(t)\widehat{\mathrm{C}}(h)=\widehat{\mathrm{C}}(t+h)C_1+\widehat{\mathrm{C}}(t-h)C_1$$

on $\overline{D(A)}$, for $t,h,t\pm h\in[0,\ \tau]$. Therefore, $\{\widehat{\mathrm{C}}(t)\}_{t\in[-\tau,\ \tau]}$ is a local C_1-cosine operator function on $\overline{D(A)}$.

Next, we show that the subgenerator of $\{\widehat{\mathrm{C}}(t)\}_{t\in[-\tau,\,\tau]}$ is operator $(I+B)\widetilde{A}$.

By the equality (2.4), (H3), the uniqueness of solution of (2.5), we obtain on $\overline{D(A)}$

$$(\lambda-(I+B)\widetilde{A})^{-1}\widehat{\mathrm{C}}(t)=\widehat{\mathrm{C}}(t)(\lambda-(I+B)\widetilde{A})^{-1},\ t\in[0,\,\tau],\,\lambda\in\rho((I+B)\widetilde{A}),$$

therefore,

$$(I+B)\widetilde{A}\widehat{\mathrm{C}}(t)x=\widehat{\mathrm{C}}(t)(I+B)\widetilde{A}x,\,x\in D(\widetilde{A}),\,t\in[0,\,\tau], \tag{2.6}$$

that is,

$$\widehat{\mathrm{C}}(t)(I+B)\widetilde{A}\subset(I+B)\widetilde{A}\widehat{\mathrm{C}}(t),\,t\in[0,\,\tau].$$

Moreover, since $\rho((I+B)\widetilde{A})\neq\emptyset$, $(I+B)\widetilde{A}$ is a closed operator. It follows from (2.4) and the closedness of $(I+B)\widetilde{A}$ that $\int_0^t\int_0^s\widehat{\mathrm{C}}(\sigma)xd\sigma ds\in D(\widetilde{A})$ and

$$\widehat{\mathrm{C}}(t)x=C_1x+(I+B)\widetilde{A}\int_0^t\int_0^s\widehat{\mathrm{C}}(\sigma)xd\sigma ds, \tag{2.7}$$

for each $x\in\overline{D(A)},\ t\in[0,\,\tau]$. Therefore, $(I+B)\widetilde{A}$ is a subgenerator of $\{\widehat{\mathrm{C}}(t)\}_{t\in[-\tau,\,\tau]}$. By (H3) and remark 2.2, we can see that $(I+B)\widetilde{A}$ is the generator of $\{\widehat{\mathrm{C}}(t)\}_{t\in[-\tau,\,\tau]}$. This completes the proof of statement (2).

By a combination of similar arguments as above and those given in the proof of [11, Theorem 2.1], we can obtain the conclusion (1).

Next, we prove the conclusion (3).

In view of statement (1) just proved, we can see that $(I+B)\widetilde{A}$ subgenerates $\{\widehat{\mathrm{C}}(t)\}_{t\in[-\tau,\,\tau]}$ on $\overline{D(A)}$. Set

$$Q(t)x=C_1x+\widetilde{A}\int_0^t\int_0^s\widehat{\mathrm{C}}(\sigma)(I+B)xd\sigma ds,\ |t|\leq\tau,\,x\in\overline{D(A)}.$$

Obviously, by (2.7) and the fact that the graph norms of $\widetilde{A}$ and $(I+B)\widetilde{A}$ are equivalent, we can see that $\{Q(t)\}_{t\in[-\tau,\,\tau]}$ is a strongly continuous operator family of bounded linear operators on $\overline{D(A)}$. Moreover, by (2.6) and (2.7), for $|t|\leq\tau$, $x\in\overline{D(A)}$, we obtain

$$\begin{aligned}
Q(t)\widetilde{A}(I+B)x &= C_1\widetilde{A}(I+B)x+\widetilde{A}\int_0^t\int_0^s\widehat{\mathrm{C}}(\sigma)(I+B)\widetilde{A}(I+B)xd\sigma ds,\\
&= \widetilde{A}(I+B)C_1x+\widetilde{A}(I+B)\widetilde{A}\int_0^t\int_0^s\widehat{\mathrm{C}}(\sigma)(I+B)xd\sigma ds\\
&= \widetilde{A}(I+B)Q(t)x
\end{aligned}$$

and

$$
\begin{aligned}
(I+B)\int_0^t\int_0^s Q(\sigma)x\,d\sigma ds &= \int_0^t\int_0^s (I+B)C_1x + \int_0^t\int_0^s [\widehat{\mathrm{C}}(\sigma)(I+B)x - C_1(I+B)x]d\sigma ds \\
&= \int_0^t\int_0^s \widehat{\mathrm{C}}(\sigma)(I+B)x\,d\sigma ds.
\end{aligned}
$$

It follows that for any $|t| \le \tau$, $x \in \overline{D(A)}$, $(I+B)\int_0^t\int_0^s Q(\sigma)x\,d\sigma ds \in D(\widetilde{A})$ and

$$
\widetilde{A}(I+B)\int_0^t\int_0^s Q(\sigma)x\,d\sigma ds = \widetilde{A}\int_0^t\int_0^s \widehat{\mathrm{C}}(\sigma)(I+B)x\,d\sigma ds = Q(x)x - C_1x.
$$

According to Definition 2.1, we see that $\widetilde{A}(I+B)$ subgenerates a local C_1-cosine function $\{Q(t)\}_{t\in[-\tau,\,\tau]}$ on $\overline{D(A)}$. □

Example 2.4. Let Ω be a domain in R^n and write

$$
\begin{aligned}
C_0(\Omega) :=\{f \in C(\Omega) :\ & \text{for each } \varepsilon > 0 \text{ there is a compact } \Omega_\varepsilon \subset \Omega \\
& \text{such that } |f(s)| < \varepsilon \ \text{ for all } \ s \in \Omega \setminus \Omega_\varepsilon\}.
\end{aligned}
$$

Given $q \in C(\Omega)$ with $q(\eta) \ge 0$, $b \in C_0(\Omega)$ with

$$
be^{\tau q}, \quad qbe^{\tau q} \in C_0(\Omega), \tag{2.8}
$$

and $K \in L^1(\Omega)$, we consider the following Cauchy problem

$$
\begin{cases}
\dfrac{\partial^2 u(t,\eta)}{\partial t^2} = q(\eta)\left(u(t,\eta) + b(\eta)\displaystyle\int_\Omega K(\sigma)u(t,\sigma)d\sigma\right), \\
u(0,\eta) = f_1(\eta), \quad u'(0,\eta) = f_2(\eta), \quad \eta \in \Omega,\ 0 \le t \le \tau,
\end{cases} \tag{2.9}
$$

where $f_1, f_2 \in C_0(\Omega)$.

Set $Af =: qf$ with $D(A) =: \{f \in C_0(\Omega); qf \in C_0(\Omega)\}$.

When q is bounded, A generates a classical cosine function $\mathrm{C}(t)$ on $C_0(\Omega)$ (i.e., I-cosine function on $[0,\infty)$), with

$$
\|\mathrm{C}(t)\| \le e^{\left(\sup\limits_{\eta\in\Omega} q(\eta)\right)t}, \qquad t \ge 0
$$

(for the exponential growth bound of a cosine function (which is closely related to a strongly continuous semigroup in some cases), as well as its relation with the spectral bound of the generator, we refer to, e.g., [1, 16]). Nevertheless, when q is unbounded, A

does not generate a global C-cosine function $\mathrm{C}(t)$ on $C_0(\Omega)$ for any C. On the other hand, A generates a local C-cosine function $\mathrm{C}(t)$ on $C_0(\Omega)$:

$$\mathrm{C}(t)f = \left\{ \frac{1}{2} \left[e^{t\sqrt{q}} + e^{-t\sqrt{q}} \right] e^{-\tau\sqrt{q}} f \right\}_{t\in[-\tau,\ \tau]},$$

with $Cf = e^{-\tau\sqrt{q}} f$. Set

$$(Bf)(\eta) = b(\eta) \int_\Omega K(\sigma) f(\sigma) d\sigma, \quad f \in C_0(\Omega).$$

¿From (2.8), we see the hypothesis (H1) in Theorem 2.3 holds. This means, by Theorem 2.3 (1) and [20, Theorem 2.4], that the Cauchy problem (2.9) has a unique solution in $C^2([0,\tau]; C_0(\Omega))$ for every couple of initial values in a large subset of $C_0(\Omega)$.

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