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POSITIVE SOLUTIONS OF THE $(n-1,1)$ CONJUGATE BOUNDARY VALUE PROBLEM

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ABSTRACT. We consider the $(n-1,1)$ conjugate boundary value problem. Some upper estimates to positive solutions for the problem are obtained. We also establish some explicit sufficient conditions for the existence and nonexistence of positive solutions of the problem.

1. INTRODUCTION

In this paper we consider the $(n-1,1)$ conjugate boundary value problem

$$u^{(n)}(t)+g(t)f(u(t))=0, \quad 0\le t\le 1, \tag{1.1}$$

$$u^{(i)}(0)=u(1)=0, \quad i=0,1,\cdots,n-2. \tag{1.2}$$

Throughout this paper, we assume that

(H1) $n\ge 3$ is a fixed integer, and $f:[0,\infty)\to[0,\infty)$ and $g:[0,1]\to[0,\infty)$ are continuous functions.

The Green's function $G:[0,1]\times[0,1]\to[0,\infty)$ for the problem (1.1)-(1.2) is given by

$$G(t,s)=\frac{1}{(n-1)!}\begin{cases} t^{n-1}(1-s)^{n-1}-(t-s)^{n-1}, & t\ge s,\\ t^{n-1}(1-s)^{n-1}, & s\ge t.\end{cases}$$

And the problem (1.1)-(1.2) is equivalent to the integral equation

$$u(t)=\int_0^1 G(t,s)g(s)f(u(s))\,ds, \quad 0\le t\le 1. \tag{1.3}$$

The $(n-1,1)$ conjugate problem and its various generalizations have been considered by many authors. For example, in 1997, Eloe and Henderson [1] considered the problem (1.1)-(1.2) and obtained some existence results for positive solutions to the problem. If $n=2$, then the problem (1.1)-(1.2) reduces to the second order problem

$$u''(t)+g(t)f(u(t))=0, \quad 0\le t\le 1,$$

$$u(0)=u(1)=0,$$

which has been extensively studied by many authors, including Graef and Yang [2].

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In 2009, Webb [4] considered a nonlocal version of the $(n-1,1)$ conjugate problem. He obtained a lower estimate for the Green's function $G(t,s)$, based on which a lower estimate to positive solutions for the problem (1.1)-(1.2) can be proved (see Lemmas 2.8 and 2.9 below). However, to our knowledge, no satisfactory upper estimates to positive solutions for the problem (1.1)-(1.2) have been obtained in the literature. We know that upper and lower estimates for positive solutions of boundary value problems have important applications. For example, once we find some *a priori* upper and lower estimates for positive solutions of a certain boundary value problem, we can use them together with the Krasnosell'skii fixed point theorem to derive a set of existence and nonexistence conditions for positive solutions of the problem (See [5] for a paper taking this approach). With this motivation, we in this paper make a further study of positive solutions to the problem (1.1)-(1.2). Our main goal is to develop some new upper estimates for positive solutions of the problem (1.1)-(1.2). Here, by a positive solution, we mean a solution $u(t)$ such that $u(t) > 0$ on $(0,1)$. Since the case $n = 2$ is a well-studied case, we in this paper assume that $n \geq 3$.

Throughout the paper, we let $X = C[0,1]$ be equipped with the supremum norm

$$\|v\| = \max_{t\in[0,1]} |v(t)|, \qquad \text{for all } v \in X.$$

Obviously X is a Banach space. Also, we define

$$F_0 = \limsup_{x\to 0^+} \frac{f(x)}{x}, \quad f_0 = \liminf_{x\to 0^+} \frac{f(x)}{x},$$

$$F_\infty = \limsup_{x\to +\infty} \frac{f(x)}{x}, \quad f_\infty = \liminf_{x\to +\infty} \frac{f(x)}{x}.$$

These constants will be used later in the statements of our existence and nonexistence theorems.

This paper is organized as follows. In Section 2, we obtain some new upper estimates to positive solutions to the $(n-1,1)$ conjugate problem, and discuss some lower estimates from the literature. In Section 3, we establish some explicit sufficient conditions for the existence and nonexistence of positive solutions to the problem.

2. Upper and Lower Estimates for Positive Solutions

Throughout the paper we define the constants

$$p = \frac{(n-1)^{n-1}}{(n-1)^{n-1} + (n-2)^{n-2}}, \quad q = \frac{n-2}{n-1}.$$

We also define the functions $w_1 : [0,1] \to [0,+\infty)$ and $w_2 : [0,1] \to [0,\infty)$ by

$$w_1(t) = \begin{cases} (n-1)^{n-1}(n-2)^{2-n}(t^{n-2} - t^{n-1}), & \text{if} \quad t \geq p, \\ t^{n-1}, & \text{if} \quad t \leq p, \end{cases}$$

and

$$w_2(t) = \begin{cases} (n-1)^{n-1}(n-2)^{2-n}(t^{n-2} - t^{n-1}), & \text{if} \quad t \leq q, \\ 1, & \text{if} \quad t \geq q. \end{cases}$$

The functions $w_1(t)$ and $w_2(t)$ will be used to estimate positive solutions of the problem (1.1)-(1.2). It is easy to see that $\|w_2\| = 1$ and that both $w_1(t)$ and $w_2(t)$ are continuous functions. An equivalent definition for the function $w_1(t)$ is

$$w_1(t) = \min\{t^{n-1}, (n-1)^{n-1}(n-2)^{2-n}(t^{n-2} - t^{n-1})\}, \quad 0 \leq t \leq 1.$$

The function $w_1(t)$ first appeared in [4]. It can be shown that

$$w_1(t) \leq w_2(t), \quad 0 \leq t \leq 1. \tag{2.1}$$

The verification of (2.1) is straightforward and is therefore left to the reader.

Lemma 2.1. *If $u \in C^n[0,1]$ satisfies (1.2), and*

$$u^{(n)}(t) \leq 0 \quad \textit{for} \quad 0 \leq t \leq 1, \tag{2.2}$$

then $u(t) \geq 0$ for $0 \leq t \leq 1$.

Proof. It is well known that the Green's function $G(t,s)$ has the property

$$G(t,s) > 0 \quad \text{for} \quad (t,s) \in (0,1) \times (0,1).$$

If $u^{(n)}(t) \leq 0$ on $[0,1]$, then

$$u(t) = \int_0^1 G(t,s)(-u^{(n)}(s))ds \geq 0, \quad 0 \leq t \leq 1.$$

The proof of the lemma is complete. □

The next lemma was proved by Eloe and Henderson in [1].

Lemma 2.2. *If $u \in C^n[0,1]$ satisfies (1.2) and (2.2), and $u(t_0) > 0$ for some $t_0 \in (0,1)$, then*

$$u(t) > 0 \quad \textit{for} \quad 0 < t < 1, \tag{2.3}$$

and there exists $c \in (0,1)$ such that $u'(c) = 0$,

$$u'(t) > 0 \quad \textit{for} \quad 0 < t < c, \quad \textit{and} \quad u'(t) < 0 \quad \textit{for} \quad c < t < 1. \tag{2.4}$$

In other words, c is the unique zero of u' in $(0,1)$.

The next lemma provides a new upper estimate for functions satisfying (1.2), (2.2), and (2.3).

Lemma 2.3. *Suppose that* $u \in C^n[0,1]$ *satisfies* (1.2), (2.2), *and* (2.3). *Let* c *be the unique zero of* u' *in* $(0,1)$, *then*

$$u(t) \le \beta(t)\|u\| \quad \text{for} \quad 0 \le t \le 1, \tag{2.5}$$

where

$$\beta(t) = \frac{t^{n-2}}{c^{n-1}}((n-1)c-(n-2)t), \quad 0 \le t \le 1.$$

Proof. We see from Lemma 2.2 that $u(c) = \|u\|$. The inequality in (2.5) is trivial for $t=0$, $t=c$, and $t=1$. We need only to show that the inequality holds for $0<t<c$ and $c<t<1$. Let $x \in (0,c)\cup(c,1)$ be a fixed number. Define

$$h(t) = u(t) - \beta(t)u(c) - (u(x)-\beta(x)u(c))\frac{t^{n-2}(t-c)^2}{x^{n-2}(x-c)^2}, \quad 0 \le t \le 1.$$

It is easy to verify the following facts:

$$h(0) = h'(0) = \cdots = h^{(n-3)}(0) = 0,$$

$$h(c) = h'(c) = 0, \quad h(x) = 0.$$

$$h^{(n)}(t) = u^{(n)}(t) - (u(x)-\beta(x)u(c))\frac{n!}{x^{n-2}(x-c)^2}, \quad 0 \le t \le 1.$$

We take two cases to continue the proof.

Case I: $0<x<c$. In this case, because $h(0)=h(x)=h(c)=0$, there exist $t_1 \in (0,x)$ and $s_1 \in (x,c)$ such that $h'(t_1)=h'(s_1)=0$.

Because $h'(0) = h'(t_1) = h'(s_1) = h'(c) = 0$, there exist $t_2 \in (0,t_1)$, $s_2 \in (t_1,s_1)$, and $r_2 \in (s_1,c)$ such that $h''(t_2)=h''(s_2)=h''(r_2)=0$.

Because $h''(0) = h''(t_2) = h''(s_2) = h''(r_2) = 0$, there exist $t_3 \in (0,t_2)$, $s_3 \in (t_2,s_2)$, and $r_3 \in (s_2,r_2)$ such that $h'''(t_3)=h'''(s_3)=h'''(r_3)=0$.

If we continue this procedure, we can show that for each $i = 2,3,\cdots,n-2$, there exist t_i, s_i, and r_i such that $0<t_i<s_i<r_i<c$ and $h^{(i)}(t_i)=h^{(i)}(s_i)=h^{(i)}(r_i)=0$.

In particular, we have $h^{(n-2)}(t_{n-2}) = h^{(n-2)}(s_{n-2}) = h^{(n-2)}(r_{n-2}) = 0$. This implies that there exist $t_{n-1} \in (t_{n-2},s_{n-2})$ and $s_{n-1} \in (s_{n-1},r_{n-2})$ such that $h^{(n-1)}(t_{n-1}) = h^{(n-1)}(s_{n-1}) = 0$. Because $h^{(n-1)}(t_{n-1}) = h^{(n-1)}(s_{n-1}) = 0$, there exists $t_n \in (t_{n-1},s_{n-1})$ such that $h^{(n)}(t_n)=0$, which implies that

$$0 = u^{(n)}(t_n) - (u(x)-\beta(x)u(c))\frac{n!}{x^{n-2}(x-c)^2}.$$

The above equation implies that

$$u(x)-\beta(x)u(c) = \frac{u^{(n)}(t_n)}{n!}x^{n-2}(x-c)^2 \le 0.$$

In summary, if $0 < x < c$ then $u(x) \leq \beta(x)u(c)$. Hence, the inequality in (2.5) holds for $0 < t < c$.

Case II: $c < x < 1$. In this case, because $h(0) = h(c) = h(x) = 0$, there exist $t_1 \in (0, c)$ and $s_1 \in (c, x)$ such that $h'(t_1) = h'(s_1) = 0$.

Because $h'(0) = h'(t_1) = h'(c) = h'(s_1) = 0$, there exist $t_2 \in (0, t_1)$, $s_2 \in (t_1, c)$, and $r_2 \in (c, s_1)$ such that $h''(t_2) = h''(s_2) = h''(r_2) = 0$. If we continue from here and follow the same lines as in Case I, we can show that $u(x) \leq \beta(x)u(c)$ for $c < x < 1$. The proof in Case II is now complete.

In summary, we have $u(t) \leq \beta(t)u(c)$ for $t \in (0, c) \cup (c, 1)$. The proof of the lemma is complete. □

As a by-product of Lemma 2.3, we have

Lemma 2.4. *Suppose that* $u \in C^n[0,1]$ *satisfies* (1.2), (2.2), *and* (2.3). *Let* c *be the unique zero of* u' *in* $(0,1)$. *Then* $q \leq c \leq 1$.

Proof. By Lemma 2.3 we have

$$0 \leq u(t) \leq \beta(t)u(c), \quad 0 \leq t \leq 1.$$

Substituting $t = 1$ into the above inequality gives

$$0 \leq \frac{u(c)}{c^{n-1}}((n-1)c - (n-2)).$$

This implies that $(n-1)c - (n-2) \geq 0$. Hence $c \geq (n-2)/(n-1)$. The proof is complete. □

Lemma 2.4 is interesting in its own right. The following lower estimate was given in [1].

Lemma 2.5. *Suppose that* $u \in C^n[0,1]$ *satisfies* (1.2), (2.2), *and* (2.3). *Let* c *be the unique zero of* u' *in* $(0,1)$. *Then*

$$u(t) \geq u(c)\min\{(t/c)^{n-1}, (1-t)/(1-c)\}, \quad 0 \leq t \leq 1. \tag{2.6}$$

Combining Lemmas 2.4 and 2.5, we get

Lemma 2.6. *If* $u \in C^n[0,1]$ *satisfies* (1.2), (2.2), *and* (2.3), *then*

$$u(t) \geq \|u\|\gamma(t), \quad 0 \leq t \leq 1, \tag{2.7}$$

where $\gamma(t) = \min\{t^{n-1}, (n-1)(1-t)\}$, $0 \leq t \leq 1$.

Proof. Let c be the unique zero of u' in $(0,1)$. By Lemma 2.4, we have $(n-2)/(n-1) \leq c \leq 1$. For $0 \leq t \leq c$ we have

$$u(t) \geq \|u\|(t/c)^{n-1} \geq \|u\|t^{n-1} \geq \|u\|\gamma(t).$$

For $c \le t \le 1$ we have

$$u(t) \ge \|u\| \frac{1-t}{1-c} \ge \|u\| \frac{1-t}{1-(n-2)/(n-1)} = \|u\|(n-1)(1-t) \ge \|u\|\gamma(t).$$

The proof is complete. □

Both Lemmas 2.5 and 2.6 provide a lower estimate to functions satisfying (1.2), (2.2), and (2.3). The difference is that the lower estimate in Lemma 2.5 depends on the unique zero c of u' in $(0,1)$, while the lower estimate in Lemma 2.6 does not. If we know where c is, then (2.6) is a better estimate that (2.7). However, if we don't know where c is, then we have to do with (2.7).

Lemma 2.7. *If $u \in C^n[0,1]$ satisfies* (1.2), (2.2), *and* (2.3), *then*

$$u(t) \le \|u\| w_2(t), \quad 0 \le t \le 1. \tag{2.8}$$

Proof. It is obvious that $u(t) \le \|u\|w_2(t)$ for $q \le t \le 1$. We need only to show that $u(t) \le \|u\|w_2(t)$ for $0 \le t \le q$. Let c be the unique zero of u' in $(0,1)$. We have $q \le c$ and $u(c) = \|u\|$.

If we define

$$h(t) = w_2(t)u(c) - u(t), \quad 0 \le t \le q,$$

then we have $h(0) = h'(0) = h''(0) = \cdots = h^{(n-3)}(0) = 0$, and

$$h^{(n)}(t) = -u^{(n)}(t) \ge 0, \quad 0 \le t \le q. \tag{2.9}$$

We also note that

$$h(q) = w_2(q)u(c) - u(q) = u(c) - u(q) \ge 0,$$

$$h'(q) = w_2'(q)u(c) - u'(q) = -u'(q) \le 0.$$

To prove the lemma, it suffices to show that $h(t) \ge 0$ for $0 \le t \le q$. Assume the contrary that $h(t_0) < 0$ for some $t_0 \in (0,q)$. Because $h(0) = 0 > h(t_0)$ and $h(t_0) < 0 \le h(q)$, there exist $t_1 \in (0,t_0)$ and $s_1 \in (t_0,q)$ such that $h'(t_1) < 0$ and $h'(s_1) > 0$.

Because $h'(0) = 0 > h'(t_1)$, $h'(t_1) < 0 < h'(s_1)$, and $h'(s_1) > 0 \ge h'(q)$, there exist $t_2 \in (0,t_1)$, $s_2 \in (t_1,s_1)$, and $r_2 \in (s_1,q)$ such that $h''(t_2) < 0$, $h''(s_2) > 0$, and $h''(r_2) < 0$.

Because $h''(0) = 0 > h''(t_2)$, $h''(t_2) < 0 < h''(s_2)$, and $h''(s_2) > 0 > h''(r_2)$, there exist $t_3 \in (0,t_2)$, $s_3 \in (t_2,s_2)$, and $r_3 \in (s_2,r_2)$ such that $h'''(t_3) < 0$, $h'''(s_3) > 0$, and $h'''(r_3) < 0$.

If we continue this procedure, then finally we can show that there exist t_{n-2}, s_{n-2}, and r_{n-2} such that $0 < t_{n-2} < s_{n-2} < r_{n-2} < q$ and

$$h^{(n-2)}(t_{n-2}) < 0, \quad h^{(n-2)}(s_{n-2}) > 0, \quad \text{and} \quad h^{(n-2)}(r_{n-2}) < 0.$$

Because $h^{(n-2)}(t_{n-2}) < 0 < h^{(n-2)}(s_{n-2})$ and $h^{(n-2)}(s_{n-2}) > 0 > h^{(n-2)}(r_{n-2})$, there exist $t_{n-1} \in (t_{n-2}, s_{n-2})$ and $s_{n-1} \in (s_{n-2}, r_{n-2})$ such that

$$h^{(n-1)}(t_{n-1}) > 0, \quad h^{(n-1)}(s_{n-1}) < 0.$$

Therefore, there exists $t_n \in (t_{n-1}, s_{n-1})$ such that $h^{(n)}(t_n) < 0$, which contradicts (2.9). The proof is complete. □

Both Lemmas 2.3 and 2.7 provide an upper estimate to functions satisfying (1.2), (2.2), and (2.3). The difference is that the upper estimate in Lemma 2.3 depends on the unique zero c of u' in $(0, 1)$, while the upper estimate in Lemma 2.7 does not.

The next lemma was proved by Webb in [4].

Lemma 2.8. *For $t, s \in [0, 1]$ we have*

$$w_1(t)\Phi_0(s) \leq G(t, s) \leq \Phi_0(s),$$

where

$$\Phi_0(s) = \frac{(\tau(s))^{n-2} s(1-s)^{n-1}}{(n-1)!}, \quad \tau(s) = \frac{s}{1-(1-s)^{(n-1)/(n-2)}}.$$

Using Lemma 2.8, we can easily establish the following lower estimate.

Lemma 2.9. *If $u \in C^n[0, 1]$ satisfies (1.2), (2.2), and (2.3), then*

$$u(t) \geq w_1(t)\|u\|, \quad 0 \leq t \leq 1. \tag{2.10}$$

Proof. On one hand, for $0 \leq t \leq 1$ we have

$$u(t) = \int_0^1 G(t, s)(-u^{(n)}(s))ds \leq \int_0^1 \Phi_0(s)(-u^{(n)}(s))ds.$$

This means that

$$\|u\| \leq \int_0^1 \Phi_0(s)(-u^{(n)}(s))ds.$$

On the other hand, for $0 \leq t \leq 1$, we have

$$\begin{aligned} u(t) &= \int_0^1 G(t, s)(-u^{(n)}(s))ds \\ &\geq w_1(t) \int_0^1 \Phi_0(s)(-u^{(n)}(s))ds \\ &\geq w_1(t)\|u\|. \end{aligned}$$

The proof is complete. □

The estimates (2.7) and (2.10) are of the same type, that is, both are independent of c. The two estimates are similar but (2.10) is a little better than (2.7). Both estimates are listed here because they are obtained by different methods, and both methods are useful in finding estimates.

Theorem 2.10. *If $u \in C^n[0,1]$ satisfies* (1.2), (2.2), *and* (2.3), *then*

$$w_2(t)\|u\| \geq u(t) \geq w_1(t)\|u\|, \quad 0 \leq t \leq 1, \tag{2.11}$$

and

$$u(t) \leq u(p)w_2(t)/w_1(p), \quad 0 \leq t \leq 1. \tag{2.12}$$

In particular, if $u \in C^n[0,1]$ is a positive solution to the boundary value problem (1.1)–(1.2), *then $u(t)$ satisfies* (2.11) *and* (2.12).

Proof. Suppose $u \in C^n[0,1]$ satisfies (1.2), (2.2), and (2.3). By Lemmas 2.7 and 2.9, the inequalities in (2.11) hold. Note that $u(p) \geq w_1(p)\|u\|$. For $0 \leq t \leq 1$ we have

$$u(t) \leq w_2(t)\|u\| = w_2(t)w_1(p)\|u\|/w_1(p) \leq w_2(t)u(p)/w_1(p).$$

Thus we proved (2.12).

If u is a positive solution to the problem (1.1)–(1.2), then $u(t)$ satisfies (1.2), (2.2), and (2.3). By the first half of the theorem, u also satisfies (2.11) and (2.12). The proof of the theorem is now complete. □

We have shown that, for functions u satisfying (1.2), (2.2), and (2.3), there are several upper and lower estimates of different types — the lower estimates (2.6), (2.7), (2.10), the upper estimates (2.5), (2.8), (2.12), and the "natural" upper estimate

$$u(t) \leq \|u\|, \quad 0 \leq t \leq 1.$$

These upper and lower estimates can be used in different situations. In the next section, we will show how to use the upper estimate (2.10) and the lower estimate (2.8) to establish some explicit existence and nonexistence conditions for positive solution of the problem (1.1)-(1.2).

3. Nonexistence and Existence Results

We begin by fixing some notations. First, we define

$$P = \{v \in X \mid v(t) \geq 0 \ \text{ on } \ [0,1]\}.$$

Clearly, P is a positive cone of the Banach space X. Define the operator $T : P \to X$ and its associated linear operator $L : X \to X$ by

$$Tu(t) = \int_0^1 G(t,s)g(s)f(u(s))ds, \quad 0 \le t \le 1, \qquad \text{for all } u \in P,$$

$$Lu(t) = \int_0^1 G(t,s)g(s)u(s)ds, \quad 0 \le t \le 1, \qquad \text{for all } u \in X.$$

It is well known that $T : P \to X$ and $L : X \to X$ are completely continuous operators. It is easy to see that $T(P) \subset P$ and $L(P) \subset P$. Now the integral equation (1.3) is equivalent to the equality

$$Tu = u, \ \ u \in P.$$

In order to solve the problem (1.1)–(1.2), we only need to find a fixed point u of T in P such that $u \neq 0$. We also define the constants

$$A = \int_0^1 G(p,s)g(s)w_1(s)\,ds \quad \text{and} \quad B = \int_0^1 G(p,s)g(s)w_2(s)\,ds.$$

Now we give some explicit sufficient conditions for the nonexistence of positive solutions.

Theorem 3.1. *If* $(B/w_1(p))f(x) < x$ *for all* $x \in (0, +\infty)$*, then the problem* (1.1)–(1.2) *has no positive solutions.*

Proof. Assume the contrary that $u(t)$ is a positive solution of the problem (1.1)–(1.2). Then $u \in P$, $u(t) > 0$ for $0 < t < 1$, and

$$\begin{aligned} u(p) &= \int_0^1 G(p,s)g(s)f(u(s))\,ds \\ &< w_1(p)B^{-1}\int_0^1 G(p,s)g(s)u(s)\,ds \\ &\le w_1(p)B^{-1}(u(p)/w_1(p))\int_0^1 G(p,s)g(s)w_2(s)ds \\ &= u(p), \end{aligned}$$

which is a contradiction. □

Theorem 3.2. *If* $Af(x) > x$ *for all* $x \in (0, +\infty)$*, then the problem* (1.1)–(1.2) *has no positive solutions.*

The proof of Theorem 3.2 is similar to that of Theorem 3.1 and is therefore omitted. The next theorem is from [4].

Theorem 3.3. *Let $r(L)$ be the radius of the spectrum of L. Then $r(L) > 0$ and L has a positive eigenfunction $\phi \in P \setminus \{0\}$ corresponding to the principal eigenvalue $r(L)$ of L. Let $\mu_1 = 1/r(L)$. If either*

$$0 \leq F_0 < \mu_1 \quad \text{and} \quad \mu_1 < f_\infty \leq \infty$$

or

$$0 \leq F_\infty < \mu_1 \quad \text{and} \quad \mu_1 < f_0 \leq \infty,$$

then the problem (1.1)-(1.2) *has at least one positive solution.*

Here comes the natural question — how can we find the value of μ_1? In general, there is no explicit formula for finding μ_1, and some kind of approximation has to be made. One approach is to use a numerical method to find an approximation for μ_1. Another approach is to develop some theoretic upper and lower bounds for μ_1. Both approaches are interesting. The following upper and lower bounds for μ_1 were given in [4].

Theorem 3.4. *Let $r(L)$ be the radius of the spectrum of L. Let $\mu_1 = 1/r(L)$. Then*

$$m < \mu_1 < M,$$

where

$$m := \left(\sup_{0 \leq t \leq 1} \int_0^1 G(t,s)g(s)ds \right)^{-1},$$

$$M := \inf_{0 \leq a < b \leq 1} \left(\inf_{a \leq t \leq b} \int_a^b G(t,s)g(s)ds \right)^{-1}.$$

With the newly found upper estimate (2.8) from Section 2, we can now improve the lower bound m for μ_1. First, we introduce some notations. For each $n \geq 1$, we let $\theta_n = T^n w_2$ and $\sigma_n = T^n w_1$. In other words, we define

$$\theta_1(t) = \int_0^1 G(t,s)g(s)w_2(s)ds, \quad 0 \leq t \leq 1,$$

$$\theta_2(t) = \int_0^1 G(t,s)g(s)\theta_1(s)ds, \quad 0 \leq t \leq 1,$$

$$\cdots\cdots,$$

$$\sigma_1(t) = \int_0^1 G(t,s)g(s)w_1(s)ds, \quad 0 \leq t \leq 1,$$

$$\sigma_2(t) = \int_0^1 G(t,s)g(s)\sigma_1(s)ds, \quad 0 \leq t \leq 1,$$

$$\cdots\cdots.$$

Next, for each $n \geq 1$, we define the constants

$$m_n = \left(\sup_{(n-2)/(n-1) \leq t \leq 1} \theta_n(t) \right)^{-1/n},$$

$$M_n = \left(\sup_{(n-2)/(n-1) \leq t \leq 1} \sigma_n(t) \right)^{-1/n}.$$

Theorem 3.5. *Let $r(L)$ be the radius of the spectrum of L and let $\mu_1 = 1/r(L)$. For each $n \geq 1$, we have $m_n \leq \mu_1 \leq M_n$.*

Proof. Let $\phi \in P \setminus \{0\}$ be a positive eigenfunction corresponding to the principal eigenvalue $r(L)$ of L. By Lemma 2.2, we have $\phi(t) > 0$ for $0 < t < 1$. Let $n \geq 1$. We have

$$\phi = \mu_1 T\phi = \mu_1^2 T^2 \phi = \cdots = \mu_1^n T^n \phi.$$

For each $0 \leq t \leq 1$ we have

$$\begin{aligned} \phi(t) &= \mu_1^n (T^n \phi)(t) \\ &\leq \|\phi\| \mu_1^n (T^n w_2)(t) \\ &= \|\phi\| \mu_1^n \theta_n(t) \\ &\leq \|\phi\| \mu_1^n \sup_{0 \leq t \leq 1} \theta_n(t). \end{aligned}$$

This implies that $\|\phi\| \leq \|\phi\| \mu_1^n \|\theta_n\|$. Thus we have

$$\mu_1 \geq \|\theta_n\|^{-1/n}.$$

By Lemma 2.4, the maximum of $\theta_n(t)$ must occur at a point in the interval $[(n-2)/(n-1), 1]$. Therefore we have

$$\|\theta_n\| = \sup_{(n-2)/(n-1) \leq t \leq 1} \theta_n(t).$$

Thus we have proved that $m_n \leq \mu_1$. In a similar fashion, we can show that $\mu_1 \leq M_n$. The proof is complete. □

The next example was first considered in [4]. We now reconsider it to illustrate some of our results.

Example 3.6. Consider the $(3, 1)$ conjugate boundary value problem

$$u''''(t) + \mu g(t) u(t) = 0, \quad 0 < t < 1, \tag{3.1}$$

$$u^{(i)}(0) = u(1) = 0, \quad 0 \leq i \leq 2, \tag{3.2}$$

where $g(t) \equiv 1$, $0 \le t \le 1$. Here $\mu > 0$ is a parameter. This problem (3.1)-(3.2) is equivalent to the equality

$$u = \mu L u,$$

where $L : X \to X$ is defined as

$$Lu(t) = \int_0^1 G(t,s)g(s)u(s)ds, \quad 0 \le t \le 1, \quad \forall u \in X.$$

Let $r(L)$ be the radius of the spectrum of L and let $\mu_1 = 1/r(L)$. In other words, we let μ_1 be the smallest eigenvalue of the boundary value problem (3.1)-(3.2). According to Webb [4], we have

$$227.557 \approx m \le \mu_1 \le M \approx 2859.530.$$

With the software *Maple*, we can easily calculate the numbers m_i and M_i, $i = 1,2,3,4$. Our calculations indicate that

$$m_1 \approx 437.107, \quad M_1 \approx 2783.13,$$
$$m_2 \approx 598.3, \quad M_2 \approx 1658.2,$$
$$m_3 \approx 693.5, \quad M_3 \approx 1379.2,$$
$$m_4 \approx 751.6, \quad M_4 \approx 1260.9.$$

By Theorem 3.5, we have $m_i \le \mu_1 \le M_i$, $i = 1,2,3,4$. This example shows that our lower bounds m_i $(1 \le i \le 4)$ for μ_1 improve m significantly, and our upper bounds M_i $(1 \le i \le 4)$ improve M significantly.

If we combine Theorems 3.3 and 3.5, we get the following existence result.

Theorem 3.7. *If there exist positive integers k and l such that either*

$$0 \le F_0 < m_k \quad \textit{and} \quad M_l < f_\infty \le \infty$$

or

$$0 \le F_\infty < m_k \quad \textit{and} \quad M_l < f_0 \le \infty,$$

then the problem (1.1)-(1.2) *has at least one positive solution.*

Example 3.8. Consider the $(3,1)$ conjugate boundary value problem

$$u''''(t) + g(t)f(u(t)) = 0, \quad 0 < t < 1, \tag{3.3}$$

$$u^{(i)}(0) = u(1) = 0, \quad 0 \le i \le 2, \tag{3.4}$$

where $g(t) \equiv 1$, $0 \le t \le 1$,

$$f(x) = \frac{\lambda x(1+9x)}{1+x}, \quad x \ge 0.$$

Here $\lambda > 0$ is a parameter. It is easy to see that $F_0 = f_0 = \lambda$ and $f_\infty = F_\infty = 9\lambda$. We see from Example 3.6 that

$$m_4 \approx 751.6, \quad M_4 \approx 1260.9.$$

By Theorem 3.7, we see that if

$$140.1 \approx M_4/9 < \lambda < m_4 \approx 751.6,$$

then the boundary value problem (3.3)-(3.4) has at least one positive solution.

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