

# Existence of Time Periodic Solutions for One-Dimensional Newtonian Filtration Equation with Multiple Delays\*

Ying Yang<sup>a</sup>, Jingxue Yin<sup>b</sup> and Chunhua Jin<sup>b,c,\*†</sup>

<sup>a</sup> *Department of Mathematics, Jilin University, Changchun 130012, China*

<sup>b</sup> *School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China*

<sup>c</sup> *School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China*

## Abstract

In this paper, we study one-dimensional Newtonian filtration equation including unbounded sources with multiple delays. The existence of nonnegative non-trivial time periodic solutions will be established by the Leray-Schauder fixed point theorem based on some suitable Lyapunov functionals and some a priori estimates for all possible periodic solutions.

**Keywords:** Newtonian Filtration; Multiple Delays; Periodic Solution.

## 1 Introduction

Consider the following one-dimensional Newtonian filtration equation with multiple delays

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2} + au + f(u(x, t - \tau_1), \dots, u(x, t - \tau_n)) + g(x, t) + \gamma \int_{t-\tau_0}^t e^{-\alpha(t-s)} u(x, s) ds, \\ x \in (0, 1), t \in \mathbb{R}, \quad (1.1)$$

subject to the homogeneous Dirichlet boundary value condition

$$u(0, t) = u(1, t) = 0, \quad t \in \mathbb{R}, \quad (1.2)$$

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†Corresponding author. Email: [jinchhua@126.com](mailto:jinchhua@126.com)

where  $m > 1$ ,  $a$  and  $\alpha$  are constants,  $\gamma$  is a positive constant,  $f$  and  $g$  are the known functions satisfying some structure conditions.

This kind of equation arises from a variety of areas in applied mathematics, physics and mathematical ecology. For the case  $m = 1$ ,  $n = 1$  and  $\gamma = 0$  with  $f(r) = \alpha r / (1 + r^\beta)$ , which appears in the blood cell production model[1]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - au + f(u(x, t - \tau)) + g(x, t).$$

On the other hand, for the same case, that is  $m = 1$ ,  $n = 1$ ,  $\gamma = 0$ , with different  $f$ , the equation (1.1) also is known as the Hematopoiesis model ( $f(r) = e^{-kr}$  ( $k > 0$ )) as well as Nicholson's blowflies model ( $f(r) = re^{-kr}$  ( $k > 0$ )), see for example [2, 3]. While, it is worth noting that all the above models are linearly diffusive, but if nonlinear diffusion is introduced, the model will be more consistent with biologic phenomena in the real world. However, as far as we know, only a few works are concerned with time periodic solutions for degenerate parabolic equation with delay(s). For example, in [4], the authors investigated the existence of time periodic solutions for  $p$ -Laplacian with multiple delays. In [5], the authors studied the existence of periodic solutions for Nicholson's blowflies model with Newtonian diffusion

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^m}{\partial x^2} - \delta u + pu(x, t - \tau)e^{-\alpha u(x, t - \tau)} + g(x, t) + \beta \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds.$$

Nevertheless, in this paper, a more general source will be discussed, which is allowed to be the blood cell production model or other types.

In the present paper, we pay our attention to the existence of nonnegative time periodic solutions for (1.1). It is worth noticing that in the model of [5], the source with delay is a typical but quite special bounded source. However, in this paper, a more general source will be discussed, particularly, the source with delays is allowed to be unbounded, which caused us difficulties in making the maximum norm estimates and some other a priori estimates. On the other hand, the method used in[4] will also not work for the equation we consider, that is the coefficient matrix associated with Lyapunov function depends on solutions of the problem, and therefore the required estimates as did in [4] could not be obtained. So, we must try some other methods. By constructing some suitable Lyapunov functionals, the a priori estimates for all possible periodic solutions, and combining with Leray-Schauder fixed point theorem, we finally establish the existence of time periodic solutions.

The rest of this paper is organized as follows. In Section 2 we introduce some basic assumptions, preliminary lemmas and state the main results of this paper. Section 3 is devoted to investigating the existence of periodic solutions based on the a priori estimates obtained in Section 2 and Leray-Schauder fixed point theorem.

## 2 Preliminaries and the Main Result

Throughout this paper, we make the following assumptions:

- (H<sub>1</sub>)  $0 \leq g \in C(\overline{Q})$ ,  $g(x, t) \not\equiv 0$ ,  $g(x, t + T) = g(x, t)$ ;  
 (H<sub>2</sub>)  $f(0, \dots, 0) = 0$ ,  $f(r_1, \dots, r_n) \geq 0$ ,  $r_i \geq 0$  ( $i = 1, \dots, n$ ) and

$$|f(a_1, \dots, a_n) - f(b_1, \dots, b_n)| \leq \sum_{i=1}^n \beta_i |a_i - b_i|;$$

where  $T$  and  $\beta_i$  are positive constants,  $Q = (0, 1) \times (0, T)$ .

Since the equation (1.1) is degenerate parabolic and problem (1.1)–(1.2) usually admits solutions only in some generalized sense. Hence we introduce the following definition.

**Definition 2.1** *A function  $u$  is said to be a weak solution of the problem (1.1)–(1.2), if  $u \in \{w; w \in L^\infty, w^m \in L^\infty(0, T; W_0^{1,2}(0, 1)), \frac{\partial w^m}{\partial t} \in L^2(Q)\}$ , and for any  $\varphi \in C^\infty(\overline{Q})$  with  $\varphi(x, 0) = \varphi(x, T)$  and  $\varphi(0, t) = \varphi(1, t) = 0$ , the following integral equality holds*

$$\int_0^T \int_0^1 \left\{ u \varphi_t - \frac{\partial u^m}{\partial x} \frac{\partial \varphi}{\partial x} + au \varphi + f(u(x, t - \tau_1), \dots, u(x, t - \tau_n)) \varphi + g(x, t) \varphi + \left( \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds \right) \varphi \right\} dx dt = 0.$$

Now we state the main result of this paper.

**Theorem 2.1** *Suppose that (H<sub>1</sub>) and (H<sub>2</sub>) hold. Then the problem (1.1)–(1.2) admits at least one nonnegative  $T$ -periodic solution.*

To prove the existence of periodic solutions (1.1)–(1.2), let us first consider the regularized problem

$$\frac{\partial u_\varepsilon}{\partial t} = \frac{\partial^2}{\partial x^2} (\varepsilon u_\varepsilon + u_\varepsilon^m) + au_\varepsilon + f(u_\varepsilon(x, t - \tau_1), \dots, u_\varepsilon(x, t - \tau_n)) + g(x, t) + \gamma \int_{t-\tau_0}^t e^{-\alpha(t-s)} u_\varepsilon ds, \quad x \in (0, 1), t \in \mathbb{R}, \quad (2.1)$$

$$u_\varepsilon(0, t) = u_\varepsilon(1, t), \quad t \in \mathbb{R}, \quad (2.2)$$

$$u_\varepsilon(x, t) = u_\varepsilon(x, t + T), \quad x \in [0, 1], t \in \mathbb{R}. \quad (2.3)$$

The desired solution of the problem (1.1)–(1.2) will be obtained by the limit of some subsequence of solutions  $u_\varepsilon$  of the regularized problem. However, we need first to establish the existence of solutions  $u_\varepsilon$ , for which, we will make use of the Leray-Schauder fixed point theorem and our efforts center on obtaining the uniform boundness of  $u_\varepsilon$ . To this end, we prove the following lemmas.

**Lemma 2.1** Assume that  $(H_1)$ ,  $(H_2)$  hold and  $u$  is a nonnegative  $T$ -periodic solution of the equation

$$u_t = \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m) + \lambda \left( au + f(u(x, t - \tau_1), \dots, u(x, t - \tau_n)) + g(x, t) + \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds \right), \quad (2.4)$$

satisfying the boundary value condition (2.2), where  $\lambda \in [0, 1]$ ,  $0 < \varepsilon < 1$  is a constant which is arbitrary. Then for any  $r > 0$ , we have

$$\int_0^T \int_0^1 u^{m+r} dx dt \leq C_1(m, r),$$

where  $C_1(m, r) > 0$  is a constant which depend on  $m$  and  $r$ .

**Proof.** Note that, multiplying Eq.(2.4) by  $u^r$  and integrating over  $Q$ ,

$$\begin{aligned} \int_0^T \int_0^1 u_t u^r dx dt &= \int_0^T \int_0^1 \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m) \cdot u^r dx dt + \lambda \left( a \int_0^T \int_0^1 u^{r+1} dx dt \right. \\ &\quad + \int_0^T \int_0^1 f(u(x, t - \tau_1), \dots, u(x, t - \tau_n)) u^r dx dt + \int_0^T \int_0^1 g u^r dx dt \\ &\quad \left. + \int_0^T \int_0^1 u^r \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds dx dt \right). \end{aligned}$$

Since  $u$  is  $T$ -periodic,

$$\int_0^T \int_0^1 \frac{\partial u}{\partial t} u^r dx dt = \frac{1}{r+1} \int_0^T \int_0^1 \frac{\partial u^{r+1}}{\partial t} dx dt = 0,$$

it follows that

$$\begin{aligned} &\int_0^T \int_0^1 \frac{\partial}{\partial x}(\varepsilon u + u^m) \frac{\partial u^r}{\partial x} dx dt \\ &= \lambda a \int_0^T \int_0^1 u^{r+1} dx dt + \lambda \int_0^T \int_0^1 f(u(x, t - \tau_1), \dots, u(x, t - \tau_n)) u^r dx dt \\ &\quad + \lambda \int_0^T \int_0^1 g u^r dx dt + \lambda \int_0^T \int_0^1 u^r \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds dx dt \\ &\leq |a| \int_0^T \int_0^1 u^{r+1} dx dt + \sum_{i=1}^n \beta_i \int_0^T \int_0^1 u(x, t - \tau_i) u^r dx dt + K \int_0^T \int_0^1 u^r dx dt \\ &\quad + \int_0^T \int_0^1 u^r \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds dx dt \end{aligned}$$

$$\begin{aligned}
&\leq |a|\varepsilon_1 \int_0^T \int_0^1 u^{m+r} dxdt + |a|\varepsilon_1^{-(r+1)/(m-1)} T + \sum_{i=1}^n \beta_i \varepsilon_2 \int_0^T \int_0^1 u^{m+r} dxdt \\
&\quad + \sum_{i=1}^n \beta_i \varepsilon_2^{-r/m} \varepsilon_3 \int_0^T \int_0^1 |u(x, t - \tau_i)|^{m+r} dxdt + \sum_{i=1}^n \beta_i \varepsilon_2^{-r/m} \varepsilon_3^{-1/(m-1)} T \\
&\quad + K\varepsilon_4 \int_0^T \int_0^1 u^{m+r} dxdt + K\varepsilon_4^{-r/m} T + \int_0^T \int_0^1 u^r \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds dxdt \\
&\leq \left( |a|\varepsilon_1 + \sum_{i=1}^n \beta_i \varepsilon_2 + \sum_{i=1}^n \beta_i \varepsilon_2^{-r/m} \varepsilon_3 + K\varepsilon_4 \right) \int_0^T \int_0^1 u^{m+r} dxdt \\
&\quad + \left( \varepsilon_1^{-(r+1)/(m-1)} + \sum_{i=1}^n \beta_i \varepsilon_2^{-r/m} \varepsilon_3^{-1/(m-1)} + K\varepsilon_4^{-r/m} \right) T \\
&\quad + \int_0^T \int_0^1 u^r \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds dxdt.
\end{aligned}$$

On the other hand

$$\begin{aligned}
&\int_0^T \int_0^1 u^r \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds dxdt \\
&\leq e^{|\alpha|\tau} \gamma \int_0^T \int_0^1 u^r \int_{t-\tau}^t u(x, s) ds dxdt \leq e^{|\alpha|\tau} \gamma \int_0^T \int_0^1 u^r \int_{-\tau}^T u(x, s) ds dxdt \\
&\leq e^{|\alpha|\tau} \gamma \left[ \varepsilon_5 \int_0^T \int_0^1 u^{m+r} dxdt + \varepsilon_5^{-\frac{r}{m}} \int_0^T \int_0^1 \left( \int_{-\tau}^T u(x, s) ds \right)^{\frac{m+r}{m}} dxdt \right] \\
&= e^{|\alpha|\tau} \gamma \left[ \varepsilon_5 \int_0^T \int_0^1 u^{m+r} dxdt + \varepsilon_5^{-\frac{r}{m}} T \int_0^1 \left( \int_{-\tau}^T u(x, s) ds \right)^{\frac{m+r}{m}} dx \right] \\
&\leq e^{|\alpha|\tau} \gamma \left[ \varepsilon_5 \int_0^T \int_0^1 u^{m+r} dxdt + \varepsilon_5^{-\frac{r}{m}} T(T + \tau)^{\frac{r}{m}} \int_0^1 \int_{-\tau}^T u(x, s)^{\frac{m+r}{m}} ds dx \right] \\
&\leq e^{|\alpha|\tau} \gamma \left[ \varepsilon_5 \int_0^T \int_0^1 u^{m+r} dxdt + \varepsilon_5^{-\frac{r}{m}} T(T + \tau)^{\frac{r}{m}} \left( \left[ \frac{\tau}{T} \right] + 1 \right) \int_0^1 \int_0^T u(x, s)^{\frac{m+r}{m}} ds dx \right] \\
&\leq e^{|\alpha|\tau} \gamma \left[ \varepsilon_5 \int_0^T \int_0^1 u^{m+r} dxdt + \varepsilon_5^{-\frac{r}{m}} T(T + \tau)^{\frac{r}{m}} \left( \left[ \frac{\tau}{T} \right] + 1 \right) \varepsilon_6 \int_0^1 \int_0^T u(x, s)^{m+r} ds dx \right. \\
&\quad \left. + \varepsilon_5^{-\frac{r}{m}} T^2(T + \tau)^{\frac{r}{m}} \left( \left[ \frac{\tau}{T} \right] + 1 \right) \varepsilon_6^{-\frac{1}{m-1}} \right].
\end{aligned}$$

Then we know that

$$\int_0^T \int_0^1 \frac{\partial}{\partial x} (\varepsilon u + u^m) \frac{\partial u^r}{\partial x} dxdt$$

$$\begin{aligned}
&\leq \left( |a|\varepsilon_1 + \sum_{i=1}^n \beta_i \varepsilon_2 + \sum_{i=1}^n \beta_i \varepsilon_2^{-r/m} \varepsilon_3 + K\varepsilon_4 \right) \int_0^T \int_0^1 u^{m+r} dx dt \\
&\quad + \left( \varepsilon_1^{-(r+1)/(m-1)} + \sum_{i=1}^n \beta_i \varepsilon_2^{-r/m} \varepsilon_3^{-1/(m-1)} + K\varepsilon_4^{-r/m} \right) T \\
&\quad + \int_0^T \int_0^1 u^r \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds dx dt \\
&\leq \left( |a|\varepsilon_1 + \sum_{i=1}^n \beta_i \varepsilon_2 + \sum_{i=1}^n \beta_i \varepsilon_2^{-r/m} \varepsilon_3 + K\varepsilon_4 + e^{|\alpha|\tau} \gamma \varepsilon_5 \right. \\
&\quad \left. + e^{|\alpha|\tau} \gamma \varepsilon_5^{-r/m} T (T + \tau)^{r/m} \left( \left[ \frac{\tau}{T} \right] + 1 \right) \varepsilon_6 \right) \int_0^T \int_0^1 u^{m+r} dx dt \\
&\quad + \left( \varepsilon_1^{-(r+1)/(m-1)} + \sum_{i=1}^n \beta_i \varepsilon_2^{-r/m} \varepsilon_3^{-1/(m-1)} + K\varepsilon_4^{-r/m} \right. \\
&\quad \left. + e^{|\alpha|\tau} \gamma \varepsilon_5^{-\frac{r}{m}} T (T + \tau)^{\frac{r}{m}} \left( \left[ \frac{\tau}{T} \right] + 1 \right) \varepsilon_6^{-\frac{1}{m-1}} \right) T \\
&\leq \left( |a|\varepsilon_1 + \sum_{i=1}^n \beta_i \varepsilon_2 + \sum_{i=1}^n \beta_i \varepsilon_2^{-r/m} \varepsilon_3 + K\varepsilon_4 + e^{|\alpha|\tau} \gamma \varepsilon_5 \right. \\
&\quad \left. + e^{|\alpha|\tau} \gamma \varepsilon_5^{-r/m} T (T + \tau)^{r/m} \left( \left[ \frac{\tau}{T} \right] + 1 \right) \varepsilon_6 \right) \int_0^T \int_0^1 u^{m+r} dx dt + C.
\end{aligned}$$

Here and below, we use  $C > 0$  to denote different positive constants depending only on the known quantities. In addition, it is easy to see that

$$\begin{aligned}
\int_0^T \int_0^1 \frac{\partial}{\partial x} (\varepsilon u + u^m) \frac{\partial u^r}{\partial x} dx dt &= \int_0^T \int_0^1 (\varepsilon + m u^{m-1}) r u^{r-1} u_x^2 dx dt \\
&\geq \int_0^T \int_0^1 m r u^{m+r-2} u_x^2 dx dt \\
&= \frac{4mr}{(m+r)^2} \int_0^T \int_0^1 \left| \frac{\partial}{\partial x} u^{(m+r)/2} \right|^2 dx dt,
\end{aligned}$$

which implies

$$\begin{aligned}
&\frac{4mr}{(m+r)^2} \int_0^T \int_0^1 \left| \frac{\partial}{\partial x} u^{(m+r)/2} \right|^2 dx dt \\
&\leq \left( |a|\varepsilon_1 + \sum_{i=1}^n \beta_i \varepsilon_2 + \sum_{i=1}^n \beta_i \varepsilon_2^{-r/m} \varepsilon_3 + K\varepsilon_4 + e^{|\alpha|\tau} \gamma \varepsilon_5 \right. \\
&\quad \left. + e^{|\alpha|\tau} \gamma \varepsilon_5^{-r/m} T (T + \tau)^{r/m} \left( \left[ \frac{\tau}{T} \right] + 1 \right) \varepsilon_6 \right) \int_0^T \int_0^1 u^{m+r} dx dt + C
\end{aligned}$$

$$\leq \left( |a|\varepsilon_1 + \sum_{i=1}^n \beta_i \varepsilon_2 + \sum_{i=1}^n \beta_i \varepsilon_2^{-r/m} \varepsilon_3 + K\varepsilon_4 + e^{|\alpha|\tau} \gamma \varepsilon_5 \right. \\ \left. + e^{|\alpha|\tau} \gamma \varepsilon_5^{-r/m} T(T + \tau)^{r/m} \left( \left[ \frac{\tau}{T} \right] + 1 \right) \varepsilon_6 \right) \mu \int_0^T \int_0^1 \left| \frac{\partial}{\partial x} u^{(m+r)/2} \right|^2 dxdt + C.$$

Thence if  $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$  and  $\varepsilon_6$  are appropriately small, we can get

$$\int_0^T \int_0^1 \left| \frac{\partial}{\partial x} u^{(m+r)/2} \right|^2 dxdt \leq C(m, r) \quad (2.5)$$

Using Poincaré inequality, we see that

$$\int_0^T \int_0^1 u^{m+r} dxdt \leq C_1(m, r),$$

which completes the proof of Lemma 2.1.  $\square$

**Lemma 2.2** *Assume that  $(H_1), (H_2)$  hold and  $u$  is a nonnegative  $T$ -periodic solution of the equation (2.4) satisfying the boundary value condition (2.2). Then we have*

$$\int_0^T \int_0^1 \left| \frac{\partial u^m}{\partial x} \right|^2 dxdt \leq C,$$

where  $C > 0$  is a constant.

**Proof.** In fact, choosing  $r = m$  in (2.5), we obtain

$$\int_0^T \int_0^1 \left| \frac{\partial u^m}{\partial x} \right|^2 dxdt \leq C.$$

$\square$

**Lemma 2.3** *Assume that  $(H_1), (H_2)$  hold and  $u$  is a nonnegative  $T$ -periodic solution of the equation (2.4) satisfying the boundary value condition (2.2). Then we have*

$$\int_0^T \int_0^1 \left| \frac{\partial}{\partial x} (\varepsilon u + u^m) \right|^2 dxdt \leq C,$$

where  $C > 0$  is a constant.

**Proof.** The proof is a direct verification. A simple calculation shows

$$\int_0^T \int_0^1 \left| \frac{\partial}{\partial x} (\varepsilon u + u^m) \right|^2 dxdt$$

$$\begin{aligned}
&= \int_0^T \int_0^1 \frac{\partial}{\partial x}(\varepsilon u + u^m) \frac{\partial}{\partial x}(\varepsilon u + u^m) dx dt \\
&= \int_0^T (\varepsilon u + u^m)(\varepsilon u_x + m u^{m-1} u_x)|_0^1 dt - \int_0^T \int_0^1 \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m)(\varepsilon u + u^m) dx dt \\
&= - \int_0^T \int_0^1 \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m)(\varepsilon u + u^m) dx dt \\
&= \int_0^T \int_0^1 (\varepsilon u + u^m) (-u_t + a \lambda u + \lambda f(u(x, t - \tau_1), \dots, u(x, t - \tau_n)) + \lambda g(x, t) \\
&\quad + \lambda \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds) dx dt \\
&= \int_0^T \int_0^1 [-(\varepsilon u + u^m) u_t + a \lambda u (\varepsilon u + u^m) + \lambda f(u(x, t - \tau_1), \dots, u(x, t - \tau_n)) (\varepsilon u + u^m) \\
&\quad + \lambda g(\varepsilon u + u^m) + \lambda \gamma (\varepsilon u + u^m) \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds] dx dt \\
&\leq |a| \int_0^T \int_0^1 u (\varepsilon u + u^m) dx dt + \sum_{i=1}^n \beta_i \int_0^T \int_0^1 |u(x, t - \tau_i)| (\varepsilon u + u^m) dx dt \\
&\quad + \int_0^T \int_0^1 g(\varepsilon u + u^m) dx dt + \gamma \int_0^T \int_0^1 (\varepsilon u + u^m) \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds dx dt \\
&\leq |a| \varepsilon \int_0^T \int_0^1 u^2 dx dt + |a| \int_0^T \int_0^1 u^{m+1} dx dt + K \varepsilon \int_0^T \int_0^1 u dx dt + K \int_0^T \int_0^1 u^m dx dt \\
&\quad + \sum_{i=1}^n \beta_i \varepsilon \int_0^T \int_0^1 u^2 dx dt + \sum_{i=1}^n \beta_i \varepsilon \int_0^T \int_0^1 |u(x, t - \tau_i)|^2 dx dt + \sum_{i=1}^n \beta_i \int_0^T \int_0^1 u^{2m} dx dt \\
&\quad + \sum_{i=1}^n \beta_i \int_0^T \int_0^1 |u(x, t - \tau_i)|^2 dx dt + \gamma \varepsilon \int_0^T \int_0^1 u \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds dx dt \\
&\quad + \gamma \int_0^T \int_0^1 u^m \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds dx dt \\
&\leq C.
\end{aligned}$$

□

For the convenience of further discussion, we denote

$$u(t) := u(x, t), \quad u(t + \theta) := u(x, t + \theta),$$

and have the following result

**Lemma 2.4** *Assume that  $(H_1)$ ,  $(H_2)$  hold and  $u$  is a nonnegative  $T$ -periodic solution of the equation (2.4) satisfying the boundary value condition (2.2). Then we have*

$$\|u\|_{L^\infty(Q)} \leq C,$$



where  $C > 0$  is a constant.

**Proof.** Define

$$V(t) = \int_0^1 \left( \frac{1}{2}u^2 + \frac{1}{2} \left| \frac{\partial}{\partial x}(\varepsilon u + u^m) \right|^2 \right) dx + \sum_{i=1}^n \beta_i^2 \int_{-\tau_i}^0 \int_0^1 u^2(t + \theta) dx d\theta, \quad (2.6)$$

then by the Cauchy inequality and assumption  $(H_1)$ , it follows that

$$\begin{aligned} V'(t) &= \int_0^1 \left[ uu_t + \frac{\partial}{\partial x}(\varepsilon u + u^m) \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x}(\varepsilon u + u^m) \right) \right] dx + \sum_{i=1}^n \beta_i^2 \int_0^1 [u^2 - u^2(t - \tau_i)] dx \\ &= \int_0^1 \left[ uu_t - \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m)(\varepsilon + mu^{m-1})u_t \right] dx + \sum_{i=1}^n \beta_i^2 \int_0^1 [u^2 - u^2(t - \tau_i)] dx \\ &= \int_0^1 u \left[ \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m) + \lambda au + \lambda f(u(x, t - \tau_1), \dots, u(x, t - \tau_n)) + \lambda g(t, x) \right. \\ &\quad \left. + \lambda \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds \right] dx \\ &\quad + \int_0^1 (\varepsilon + mu^{m-1})u_t [(-u_t + \lambda au + \lambda f(u(x, t - \tau_1), \dots, u(x, t - \tau_n)) + \lambda g(t, x) \\ &\quad + \lambda \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds)] dx + \sum_{i=1}^n \beta_i^2 \int_0^1 [u^2 - u^2(t - \tau_i)] dx \\ &\leq \int_0^1 \left[ -\frac{\partial}{\partial x}(\varepsilon u + u^m) \frac{\partial u}{\partial x} + |a|u^2 + \sum_{i=1}^n \beta_i u(x, t - \tau_i)u + |u||g(x, t)| \right] dx \\ &\quad + \int_0^1 \left[ -\tilde{f}u_t^2 + |a|\tilde{f}|u_t||u| + \tilde{f} \sum_{i=1}^n \beta_i u(t - \tau_i)|u_t| + \tilde{f}|u_t||g| \right] dx \\ &\quad + \sum_{i=1}^n \beta_i^2 \int_0^1 [u^2 - u^2(t - \tau_i)] dx \\ &\quad + \gamma \int_0^1 u \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds dx + \gamma \int_0^1 \tilde{f}|u_t| \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds dx \\ &\leq \int_0^1 \left( -mu^{m-1} \left| \frac{\partial u}{\partial x} \right|^2 + (|a| + \sum_{i=1}^n \beta_i^2)u^2 \right) dx \\ &\quad + \int_0^1 \left[ \sum_{i=1}^n \beta_i u(t - \tau_i)u + ug - \tilde{f}u_t^2 + |a|\tilde{f}u|u_t| + \sum_{i=1}^n \beta_i \tilde{f}u(t - \tau_i)|u_t| + \tilde{f}|u_t|g \right] dx dt \\ &\quad + \gamma \int_0^1 u \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds dx + \gamma \int_0^1 \tilde{f}|u_t| \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds dx \end{aligned}$$

$$\begin{aligned}
&= \int_0^1 \left( -mu^{m-1} \left| \frac{\partial u}{\partial x} \right|^2 + (|a| + \sum_{i=1}^n \beta_i^2) u^2 \right) dx + \int_0^1 F dx \\
&\quad \left( \text{since } \int_0^1 u^2 dx \leq \varepsilon_1 \int_0^1 u^{m+1} dx + \varepsilon_1^{-\frac{2}{(m-1)}} \leq \varepsilon_1 \mu \int_0^1 \left| \frac{\partial u^{(m+1)/2}}{\partial x} \right|^2 dx + \varepsilon_1^{-\frac{2}{(m-1)}} \right) \\
&\leq -c_1 \left( \int_0^1 u^2 dx + \int_0^1 \left| \frac{\partial u^{(m+1)/2}}{\partial x} \right|^2 dx \right) + (|a| + \sum_{i=1}^n \beta_i^2 + c_1) \varepsilon_1^{-2/(m-1)} + \int_0^1 F dx,
\end{aligned}$$

where

$$\begin{aligned}
0 < \varepsilon_1 < \frac{1}{(|a| + \sum_{i=1}^n \beta_i^2) \mu} \frac{4m}{(m+1)^2}, \\
c_1 &= \frac{1}{1 + \varepsilon_1 \mu} \left( \frac{4m}{(m+1)^2} - (|a| + \sum_{i=1}^n \beta_i^2) \varepsilon_1 \mu \right) > 0 \\
\tilde{f} &= \varepsilon + mu^{m-1}, \\
F &= -\tilde{f}u_t^2 + |a|\tilde{f}u|u_t| + \sum_{i=1}^n \beta_i \tilde{f}u(t - \tau_i)|u_t| + \tilde{f}|u_t|g + \sum_{i=1}^n \beta_i u(t - \tau_i)u \\
&\quad + ug + \gamma u \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds + \gamma \tilde{f}|u_t| \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds.
\end{aligned}$$

Letting

$$c_2 = (|a| + \sum_{i=1}^n \beta_i^2 + c_1) \varepsilon_1^{-2/(m-1)},$$

it is obvious that

$$V'(t) \leq -c_1 \left( \int_0^1 u^2 dx + \int_0^1 \left| \frac{\partial u^{(m+1)/2}}{\partial x} \right|^2 dx \right) + c_2 + \int_0^1 F dx. \quad (2.7)$$

On the other hand

$$\begin{aligned}
&\int_0^T \int_0^1 F dx dt \\
&\leq \int_0^T \int_0^1 [-\tilde{f}u_t^2 + |a|\tilde{f}u|u_t| + \sum_{i=1}^n \beta_i \tilde{f}u(t - \tau_i)|u_t| + \tilde{f}|u_t|g + \sum_{i=1}^n \beta_i u(t - \tau_i)u + u|g| \\
&\quad + \gamma u \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds + \gamma \tilde{f}u_t \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds] dx dt
\end{aligned}$$

$$\begin{aligned} &\leq \int_0^T \int_0^1 \left[ -\tilde{f}u_t^2 + \frac{\varepsilon_2|a|}{2}\tilde{f}u_t^2 + \frac{|a|}{2\varepsilon_2}\tilde{f}u^2 + \frac{\varepsilon_3}{2}\sum_{i=1}^n\beta_i\tilde{f}u_t^2 + \frac{1}{2\varepsilon_3}\sum_{i=1}^n\beta_i\tilde{f}u^2(t-\tau_i) + \frac{\varepsilon_4}{2}\tilde{f}u_t^2 \right. \\ &\quad \left. + \frac{1}{2\varepsilon_4}f|g|^2 + \sum_{i=1}^n\beta_iu^2(t-\tau_i) + \sum_{i=1}^n\beta_iu^2 + u^2 + g^2 + \gamma\tilde{f}|u_t| \int_{t-\tau}^t e^{-\alpha(t-s)}u(s)ds \right] dxdt \\ &\quad + C \int_0^T \int_0^1 u^{m+1}dxdt + C. \end{aligned}$$

Since

$$\begin{aligned} &\int_0^T \int_0^1 \gamma\tilde{f}|u_t| \int_{t-\tau}^t e^{-\alpha(t-s)}u(s)dsdxdt \\ &\leq e^{|\alpha|\tau}\gamma \int_0^T \int_0^1 \tilde{f}|u_t| \int_{t-\tau}^t u(s)dsdxdt \\ &\leq e^{|\alpha|\tau}\gamma \left[ \frac{\varepsilon_5}{2} \int_0^T \int_0^1 \tilde{f}u_t^2dxdt + \frac{1}{2\varepsilon_5} \int_0^T \int_0^1 \tilde{f} \left( \int_{t-\tau}^t u(s)ds \right)^2 dxdt \right] \\ &\leq e^{|\alpha|\tau}\gamma \left[ \frac{\varepsilon_5}{2} \int_0^T \int_0^1 \tilde{f}u_t^2dxdt + \frac{\tau}{2\varepsilon_5} \int_0^T \int_0^1 \tilde{f} \int_{t-\tau}^t u^2(s)dsdxdt \right] \\ &\leq e^{|\alpha|\tau}\gamma \left[ \frac{\varepsilon_5}{2} \int_0^T \int_0^1 \tilde{f}u_t^2dxdt + \frac{\tau}{2\varepsilon_5} \int_0^T \int_0^1 \int_{-\tau}^T u^2(s)dsdxdt + \frac{m\tau}{2\varepsilon_5} \int_0^T \int_0^1 u^{m-1} \int_{-\tau}^T u^2(s)dsdxdt \right] \\ &\leq e^{|\alpha|\tau}\gamma \left[ \frac{\varepsilon_5}{2} \int_0^T \int_0^1 \tilde{f}u_t^2dxdt + \frac{\tau T}{2\varepsilon_5} \left( \left[ \frac{\tau}{T} \right] + 1 \right) \int_0^T \int_0^1 u^2dxdt + \frac{m\tau}{2\varepsilon_5} \int_0^T \int_0^1 u^{2m}dxdt \right. \\ &\quad \left. + \frac{m\tau}{2\varepsilon_5} \int_0^T \int_0^1 \left( \int_{-\tau}^t u^2(s)ds \right)^2 dxdt \right] \\ &\leq e^{|\alpha|\tau}\gamma \left[ \frac{\varepsilon_5}{2} \int_0^T \int_0^1 \tilde{f}u_t^2dxdt + \frac{\tau T}{2\varepsilon_5} \left( \left[ \frac{\tau}{T} \right] + 1 \right) \int_0^T \int_0^1 u^2dxdt + \frac{m\tau}{2\varepsilon_5} \int_0^T \int_0^1 u^{2m}dxdt \right. \\ &\quad \left. + \frac{m\tau T}{2\varepsilon_5} (T + \tau) \left( \left[ \frac{\tau}{T} \right] + 1 \right) \int_0^T \int_0^1 u^4dxdt \right], \end{aligned}$$

choosing  $\varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5$  small appropriately, we have

$$\begin{aligned} &\int_0^T \int_0^1 F dxdt \\ &\leq \iint_Q \left[ \left( -1 + \frac{|a|\varepsilon_2}{2} + \frac{\varepsilon_3}{2} \sum_{i=1}^n \beta_i + \frac{\varepsilon_4}{2} + \frac{\varepsilon_5 e^{|\alpha|\tau}}{2} \gamma \right) \tilde{f}u_t^2 + \frac{|a|}{2\varepsilon_2} \tilde{f}u^2 + \frac{1}{2\varepsilon_3} \sum_{i=1}^n \beta_i \tilde{f}u^2(t-\tau_i) \right. \\ &\quad \left. + \frac{1}{2\varepsilon_4} \tilde{f}|g|^2 + \sum_{i=1}^n \beta_i u^2(t-\tau_i) + \left( \sum_{i=1}^n \beta_i + 1 \right) u^2 + g^2 + e^{|\alpha|\tau} \gamma \frac{\tau T}{2\varepsilon_5} \left( \left[ \frac{\tau}{T} \right] + 1 \right) u^2 \right] dxdt \end{aligned}$$

$$+e^{|\alpha|\tau}\gamma\frac{m\tau}{2\varepsilon_5}u^{2m} + e^{|\alpha|\tau}\gamma\frac{m\tau T}{2\varepsilon_5}(T + \tau) \left( \left[ \frac{\tau}{T} \right] + 1 \right) u^4 \Big] dxdt + C \int_0^T \int_0^1 u^{m+1} dxdt + C.$$

Applying Lemma 2.1,  $\|u\|_{L^r} \leq C$ , for any  $r > 0$ , yields

$$\int_0^T \int_0^1 F dxdt \leq C. \tag{2.8}$$

Since  $V(t)$  and  $u$  are  $T$ -periodic and from (2.7), (2.8), we have

$$\begin{aligned} 0 &= \int_0^T V'(t) dt \\ &\leq -c_1 \left( \int_0^T \int_0^1 u^2 dxdt + \int_0^T \int_0^1 \left| \frac{\partial u^{(m+1)/2}}{\partial x} \right|^2 dxdt \right) + c_2 T + \int_0^T \int_0^1 F dxdt, \end{aligned}$$

which implies

$$\int_0^T \int_0^1 u^2 dxdt \leq C, \quad \int_0^T \int_0^1 \left| \frac{\partial u^{(m+1)/2}}{\partial x} \right|^2 dxdt \leq C.$$

Applying Lemma 2.1 and Lemma 2.3 yields

$$\begin{aligned} &\int_0^T V(t) dt \\ &\leq \int_0^T \int_0^1 \left( \frac{1}{2} u^2 + \frac{1}{2} \left| \frac{\partial}{\partial x} (\varepsilon u + u^m) \right|^2 \right) dxdt + \sum_{i=1}^n \beta_i \int_0^T \int_{-\tau_i}^0 \int_0^1 u^2(t + \theta) dx d\theta dt \\ &\leq C. \end{aligned}$$

Since  $V$  is continuous, there exists a  $t_0 \in [0, T]$  satisfies

$$V(t_0) \leq \frac{C}{T} \leq C.$$

Hence, if  $t_0 \leq t \leq t_0 + T$ , we obtain

$$\begin{aligned} V(t) &= V(t_0) + \int_{t_0}^t V'(s) ds \\ &\leq C + \int_{t_0}^t \left[ -c_1 \int_0^1 \left( u^2 + \left| \frac{\partial u^{(m+1)/2}}{\partial x} \right|^2 \right) dx + c_2 + \int_0^1 F dx \right] ds \\ &\leq C + \int_0^T \int_0^1 \left[ c_1 \left( u^2 + \left| \frac{\partial u^{(m+1)/2}}{\partial x} \right|^2 \right) + F \right] dxdt + c_2 T \end{aligned}$$

$$\leq C.$$

A simple calculation yields

$$\begin{aligned} \int_0^1 \left| \frac{\partial}{\partial x}(\varepsilon u + u^m) \right|^2 dx &= \int_0^1 |(\varepsilon + mu^{m-1})u_x|^2 dx \\ &\geq \int_0^1 |mu^{m-1}u_x|^2 dx = \int_0^1 \left| \frac{\partial u^m}{\partial x} \right|^2 dx. \end{aligned}$$

By the definition of  $V$ , we see that

$$\int_0^1 \left( u^2 + \left| \frac{\partial u^m}{\partial x} \right|^2 \right) dx \leq \int_0^1 \left( u^2 + \left| \frac{\partial}{\partial x}(\varepsilon u + u^m) \right|^2 \right) dx \leq 2V(t) \leq C,$$

which implies

$$\sup_{0 \leq t \leq T} \int_0^1 \left( u^2 + \left| \frac{\partial u^m}{\partial x} \right|^2 \right) dx \leq C.$$

It follows from the definition of  $u$  that

$$\begin{aligned} |u(x, t)|^m = |u^m(x, t)| &= \left| u^m(0, t) + \int_0^x \frac{\partial u^m(s, t)}{\partial s} ds \right| = \left| \int_0^x \frac{\partial u^m(s, t)}{\partial s} ds \right| \\ &\leq \int_0^1 \left| \frac{\partial u^m}{\partial x} \right| dx \\ &\leq \left( \sup_{0 \leq t \leq T} \int_0^1 \left| \frac{\partial u^m}{\partial x} \right|^2 dx \right)^{1/2} \leq C^{1/2}, \end{aligned}$$

that is

$$\|u\|_{L^\infty(Q)} \leq C^{1/(2m)} := C_0.$$

The proof is complete. □

In addition, we can also easily get the  $L^2$  boundedness of  $\frac{\partial u^m}{\partial t}$  as follows.

**Lemma 2.5** *Assume that  $(H_1)$ ,  $(H_2)$  hold and  $u$  is a nonnegative  $T$ -periodic solution of the equation (2.1) satisfying the boundary value condition (2.2). Then there exists a constant  $C > 0$  such that*

$$\iint_Q \left| \frac{\partial u^m}{\partial t} \right|^2 dx dt \leq C.$$

**Proof.** Multiplying the equation (2.1) by  $\frac{\partial}{\partial t}(\varepsilon u + u^m)$  and integrating over  $Q$ , we obtain

$$\iint_Q \frac{\partial u}{\partial t} \frac{\partial}{\partial t}(\varepsilon u + u^m) dx dt$$

$$\begin{aligned}
&= \iint_Q \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m) \frac{\partial}{\partial t}(\varepsilon u + u^m) dxdt + a \iint_Q u \frac{\partial}{\partial t}(\varepsilon u + u^m) dxdt \\
&\quad + \iint_Q [f(u(x, t - \tau_1), \dots, u(x, t - \tau_n)) + g(x, t)] \frac{\partial}{\partial t}(\varepsilon u + u^m) dxdt \\
&\quad + \iint_Q \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds \frac{\partial}{\partial t}(\varepsilon u + u^m) dxdt.
\end{aligned}$$

A simple calculation yields

$$\int_0^T \int_0^1 \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m) \frac{\partial}{\partial t}(\varepsilon u + u^m) dxdt = -\frac{1}{2} \int_0^T \int_0^1 \frac{\partial}{\partial t} \left( \frac{\partial}{\partial x}(\varepsilon u + u^m) \right)^2 dxdt = 0.$$

Therefore, by Lemma 2.4, we obtain

$$\begin{aligned}
&\varepsilon \int_0^T \int_0^1 u_t^2 dxdt + \int_0^T \int_0^1 mu^{m-1} u_t^2 dxdt \\
&\leq \sum_{i=1}^n \beta_i \int_0^T \int_0^1 u(t - \tau_i) \left| \frac{\partial}{\partial t}(\varepsilon u + u^m) \right| dxdt + \int_0^T \int_0^1 g(x, t) \frac{\partial}{\partial t}(\varepsilon u + u^m) dxdt \\
&\quad + \iint_Q \gamma \left( \int_{t-\tau}^t e^{-\alpha(t-s)} u(s) ds \right) \frac{\partial}{\partial t}(\varepsilon u + u^m) dxdt \\
&\leq C \int_0^T \int_0^1 \left| \frac{\partial}{\partial t}(\varepsilon u + u^m) \right| dxdt \\
&\leq \frac{\varepsilon^2 C \varepsilon_1}{2} \int_0^T \int_0^1 |u_t|^2 dxdt + \frac{C}{2\varepsilon_1} + \frac{C\varepsilon_2}{2} \int_0^T \int_0^1 |mu^{m-1} u_t|^2 dxdt + \frac{C}{2\varepsilon_2} \\
&\leq \varepsilon \int_0^T \int_0^1 |u_t|^2 dxdt + C + \frac{C\varepsilon_2}{2} \int_0^T \int_0^t mu^{m-1} |u_t|^2 dxdt,
\end{aligned}$$

where  $\varepsilon_1 = 2/(C\varepsilon)$ . Letting  $\varepsilon_2$  be appropriately small,

$$\int_0^T \int_0^1 mu^{m-1} u_t^2 dxdt \leq C,$$

then we can see

$$\begin{aligned}
\int_0^T \int_0^1 \left| \frac{\partial u^m}{\partial t} \right|^2 dxdt &= \int_0^T \int_0^1 m^2 u^{2(m-1)} u_t^2 dxdt \\
&\leq mC^{m-1} \int_0^T \int_0^1 mu^{m-1} u_t^2 dxdt \leq C,
\end{aligned}$$

which completes the proof of Lemma 2.5. □

We will show the Hölder norm estimate of solutions in the following lemma.

**Lemma 2.6** Assume that  $(H_1)$ ,  $(H_2)$  hold and  $u$  is a nonnegative  $T$ -periodic solution of the equation (2.1) satisfying the boundary value condition (2.2). Then there exists a constant  $C > 0$  such that  $u \in C^{\alpha, \alpha/2}(Q)$  with  $0 < \alpha < 1/2$ .

**Proof.** In fact, through a similar discussion of [6] (see Chapter 2), we know that  $u \in L^\infty(0, T; H_0^1(0, 1))$  and  $\frac{\partial u}{\partial t} \in L^2(Q)$ . By direct computations, for any  $x_1 < x_2 \in (0, 1)$ , we conclude that

$$\begin{aligned} |u(x_2, t) - u(x_1, t)| &= \left| \int_{x_1}^{x_2} \frac{\partial u(x, t)}{\partial x} dx \right| \leq \int_{x_1}^{x_2} \left| \frac{\partial u(x, t)}{\partial x} \right| dx \\ &\leq \left( \int_0^1 \left| \frac{\partial u}{\partial x} \right|^2 dx \right)^{1/2} |x_2 - x_1|^{1/2} \\ &\leq C|x_2 - x_1|^{1/2}. \end{aligned} \tag{2.9}$$

On the other hand, to prove

$$|u(x, t_2) - u(x, t_1)| \leq C|t_2 - t_1|^{1/4}, \tag{2.10}$$

we need only consider the case that  $0 \leq x \leq \frac{1}{2}$ ,  $\Delta t = t_2 - t_1 > 0$ ,  $(\Delta t)^\alpha \leq \frac{1}{4}$ , where  $\alpha$  is determined. Integrating (2.2) over  $(y, y + (\Delta t)^\alpha) \times (t_1, t_2)$  gives

$$\begin{aligned} &\int_y^{y+(\Delta t)^\alpha} [u(z, t_2) - u(z, t_1)] dz \\ &= \int_{t_1}^{t_2} \left[ \frac{\partial}{\partial x} (\varepsilon u(y + (\Delta t)^\alpha, s) + u^m(y + (\Delta t)^\alpha, s)) - \frac{\partial}{\partial x} (\varepsilon u(y, s) + u^m(y, s)) \right] ds \\ &\quad + \int_{t_1}^{t_2} \int_y^{y+(\Delta t)^\alpha} [au(z, s) + f(u(z, s - \tau_1), \dots, u(z, s - \tau_n)) + g(z, s)] dz ds \\ &\quad + \int_{t_1}^{t_2} \int_y^{y+(\Delta t)^\alpha} \gamma \int_{s-\tau_0}^s e^{-\alpha(s-\sigma)} u(x, s) d\sigma dz ds, \end{aligned}$$

i.e.

$$\begin{aligned} &(\Delta t)^\alpha \int_0^1 (u(y + \theta(\Delta t)^\alpha, t_2) - u(y + \theta(\Delta t)^\alpha, t_1)) d\theta \\ &= \int_{t_1}^{t_2} \left[ \frac{\partial}{\partial x} (\varepsilon u(y + (\Delta t)^\alpha, s) + u^m(y + (\Delta t)^\alpha, s)) - \frac{\partial}{\partial x} (\varepsilon u(y, s) + u^m(y, s)) \right] ds \\ &\quad + \int_{t_1}^{t_2} \int_y^{y+(\Delta t)^\alpha} [au(z, s) + f(u(z, s - \tau_1), \dots, u(z, s - \tau_n)) + g(z, s)] dz ds \\ &\quad + \int_{t_1}^{t_2} \int_y^{y+(\Delta t)^\alpha} \gamma \int_{s-\tau_0}^s e^{-\alpha(s-\sigma)} u(x, s) d\sigma dz ds, \end{aligned}$$

Integrating the above equality with respect to  $y$  over  $(x, x + (\Delta t)^\alpha)$ , we conclude that

$$\begin{aligned}
 & (\Delta t)^\alpha \int_x^{x+(\Delta t)^\alpha} \int_0^1 (u(y + \theta(\Delta t)^\alpha, t_2) - u(y + \theta(\Delta t)^\alpha, t_1)) d\theta dy \\
 = & \int_x^{x+(\Delta t)^\alpha} \int_{t_1}^{t_2} \left[ \frac{\partial}{\partial x} (\varepsilon u(y + (\Delta t)^\alpha, s) + u^m(y + (\Delta t)^\alpha, s)) - \frac{\partial}{\partial x} (\varepsilon u(y, s) + u^m(y, s)) \right] ds dy \\
 & + \int_x^{x+(\Delta t)^\alpha} \int_{t_1}^{t_2} \int_y^{y+(\Delta t)^\alpha} [au(z, s) + f(u(z, s - \tau_1), \dots, u(z, s - \tau_n)) + g(z, s)] dz ds dy \\
 & + \int_x^{x+(\Delta t)^\alpha} \int_{t_1}^{t_2} \int_y^{y+(\Delta t)^\alpha} \gamma \int_{s-\tau_0}^s e^{-\alpha(s-\sigma)} u(x, s) d\sigma dz ds dy, \\
 \leq & C|(\Delta t)^{\alpha+1}|^{1/2} + C(\Delta t)^{2\alpha+1}.
 \end{aligned}$$

Hence, by a simple calculations, we have

$$|u(x^*, t_2) - u(x^*, t_1)| \leq C(\Delta t) + C(\Delta t)^{(\alpha+1)/2-2\alpha} \leq C|t_2 - t_1|^{1/4},$$

where  $x^* = y^* + \theta^*(\Delta t)^\alpha$ ,  $y^* \in (x, x + (\Delta t)^\alpha)$ ,  $\theta^* \in (0, 1)$ , and choose  $\alpha = 1/6$  specially, from which we see that (2.10) holds. Therefore, by (2.9) and (2.10), we obtain that

$$\begin{aligned}
 & |u(x_2, t_2) - u(x_1, t_1)| \\
 \leq & |u(x_2, t_2) - u(x^*, t_2)| + |u(x^*, t_2) - u(x^*, t_1)| + |u(x^*, t_1) - u(x_1, t_1)| \\
 \leq & C(|x_2 - x^*|^{1/2} + |t_2 - t_1|^{1/4} + |x^* - x_1|^{1/2}) \\
 \leq & C(|x_2 - x_1| + |t_2 - t_1|^{1/2})^{1/2},
 \end{aligned}$$

which completes the proof of Lemma 2.6. □

### 3 Proof of the Main Result

By means of the above proved lemmas and the Leray-Schauder fixed point theorem, we can obtain the existence of solutions  $u_\varepsilon$  of the regularized problem as follows.

**Proposition 3.1** *Assume that  $(H_1)$  and  $(H_2)$  hold. Then the regularized problem (2.2)–(2.4) has a nonnegative  $T$ -periodic solution.*

**Proof.** Denote by  $C_T(\overline{Q})$  the set of all continuous functions  $u$  with the  $T$ -periodicity in  $t$ . We study the following regularized equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} (\varepsilon u + |u|^{m-1} u) + g(x, t) \tag{3.1}$$



where  $0 \leq g \in C_T(\overline{Q})$ . We claim that, if the problem (3.1),(2.2),(2.3) has a unique  $T$ -periodic solution  $u$ , then  $u$  must be nonnegative. In fact, multiplying (3.1) by  $u_-$  and integrating over  $Q$ , we obtain

$$\iint_Q \frac{\partial u}{\partial t} u_- dxdt = \iint_Q \frac{\partial^2}{\partial x^2} (\varepsilon u + |u|^{m-1} u) u_- dxdt + \iint_Q g u_- dxdt,$$

where  $u_- = \min\{0, u(x, t), (x, t) \in Q\}$ . Making use of integration by parts, we have

$$\iint_Q \frac{\partial}{\partial x} (\varepsilon u_- + |u_-|^{m-1} u_-) \frac{\partial u_-}{\partial x} dxdt = \iint_Q g u_- dxdt \leq 0.$$

Since

$$\iint_Q \frac{\partial}{\partial x} (\varepsilon u_- + |u_-|^{m-1} u_-) \frac{\partial u_-}{\partial x} dxdt = \iint_Q (\varepsilon + m|u_-|^{m-1}) \left| \frac{\partial u_-}{\partial x} \right|^2 dxdt,$$

then we get

$$\iint_Q |u_-|^{m-1} \left| \frac{\partial u_-}{\partial x} \right|^2 dxdt \leq 0.$$

Therefore,

$$u_- = 0, \quad a.e. \text{ in } Q$$

By the definition of  $u_-$ , we see that

$$u \geq 0, \quad a.e. \text{ in } Q.$$

Consequently, we can rewrite the equation (3.1) as

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} (\varepsilon u + u^m) + g(x, t), \quad x \in (0, 1), t \in \mathbb{R}. \quad (3.2)$$

Hence, we know that, if the problem (3.2),(2.2),(2.3) has a  $T$ -periodic solution, it must be nonnegative.

Similarly, we have the same consequence for the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} (\varepsilon u + u^m) - \zeta u + g(x, t), \quad (3.3)$$

with the conditions (2.2) and (2.3), where  $\zeta \geq 0$ .

On the other hand, with an argument similar to [7], we claim that for any  $g \in C_T(\overline{Q})$ , the problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2} (\varepsilon u + u^m) + g(x, t), \quad x \in (0, 1), t \in \mathbb{R},$$

$$\begin{aligned} u(0, t) &= u(1, t), & t \in \mathbb{R}, \\ u(x, t) &= u(x, t + T), & x \in (0, 1), t \in \mathbb{R}. \end{aligned}$$

has a unique solution  $u \in C^{\alpha, \alpha/2}(\overline{Q})$ . By constructing a homotopy, it is easy to obtain that the problem (3.3),(2.2),(2.3) also admits a solution  $u \in C^{\alpha, \alpha/2}(\overline{Q})$ .

Next, we will obtain the existence of periodic solutions for the regularized problem (2.1),(2.2),(2.3).

In case that  $a \geq 0$ , we consider the periodic problem of the homotopy equation for regularized problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m) + \lambda G(x, t), \quad x \in (0, 1), t \in \mathbb{R}, \quad (3.4)$$

$$u(0, t) = u(1, t) = 0, \quad t \in \mathbb{R}, \quad (3.5)$$

where for any  $v(x, t) \in C_T(\overline{Q})$ ,

$$G(x, t) = av(x, t) + f(v(x, t - \tau_1), \dots, v(x, t - \tau_n)) + g(x, t) + \gamma \int_{t-\tau_0}^t e^{-\alpha(t-s)} v(x, s) ds.$$

Then problem (3.4)–(3.5) admits a unique solution  $u \in C_T^{\alpha, \alpha/2}(\overline{Q})$ . Define the mapping

$$\begin{aligned} L : C_T(\overline{Q}) \times [0, 1] &\longrightarrow C_T(\overline{Q}), \\ (v, \lambda) &\longmapsto u. \end{aligned}$$

Since  $C_T^{\alpha, \alpha/2}(\overline{Q})$  can be compactly embedded into  $C_T(\overline{Q})$ ,  $L$  is compact. By Lemma 2.4, we know that for any fixed point  $u_\lambda$  of the mapping  $L$ , there is a constant  $C_0$  independent of  $\varepsilon$  and  $\lambda$ , such that

$$\|u_\lambda\|_{L^\infty} \leq C_0.$$

Then in applying Leray-Schauder's fixed point theorem, we know that the problem (2.1)–(2.3) admits a solution  $u_\varepsilon$ .

In case that  $a < 0$ , we consider the periodic problem of the homotopy equation for regularized problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(\varepsilon u + u^m) + au + \lambda G(x, t), \quad x \in (0, 1), t \in \mathbb{R}, \quad (3.6)$$

$$u(0, t) = u(1, t) = 0, \quad t \in \mathbb{R}, \quad (3.7)$$

where for any  $v(x, t) \in C_T(\overline{Q})$ ,

$$G(x, t) = f(v(x, t - \tau_1), \dots, v(x, t - \tau_n)) + g(x, t) + \gamma \int_{t-\tau_0}^t e^{-\alpha(t-s)} v(x, s) ds.$$

Then problem (3.6)–(3.7) admits a unique solution  $u \in C_T^{\alpha, \alpha/2}(\overline{Q})$ . The following progress is the same as above case, then we get that the problem (2.1)–(2.3) admits a solution  $u_\varepsilon$ .  $\square$

Now, we turn to the proof of the our main result based on the above lemmas and Proposition 3.1

**The Proof of Theorem 2.1.** Let  $\varepsilon = 1/h$  ( $h = 1, 2, \dots$ ) and we note  $u_h$  for the solution of the problem (2.2)–(2.4). Clearly, according to Lemma 2.2, Lemma 2.4 and Lemma 2.5,

$$\begin{aligned} \|u_h\|_{L^\infty(Q)} &\leq C_0, \\ \left\| \frac{\partial u_h^m}{\partial x} \right\|_{L^2(Q)} &\leq C_1 T, \\ \left\| \frac{\partial u_h^m}{\partial t} \right\|_{L^2(Q)}^2 &\leq C. \end{aligned}$$

Hence there exists a subsequence  $\{u_h\}_{h=1}^\infty$ , supposed to be  $\{u_h\}_{h=1}^\infty$  itself, and a function  $u \in \{u; u \in L^\infty; u^m \in L^\infty(0, T; W_0^{1,2}(0, 1)); \frac{\partial u^m}{\partial t} \in L^2(Q)\}$ , such that

$$\begin{aligned} u_h(x, t) &\rightarrow u(x, t), & \text{in } L^2(Q), \\ \frac{\partial u_h^m(x, t)}{\partial x} &\rightarrow \frac{\partial u^m(x, t)}{\partial x}, & \text{in } L^2(Q), \\ f(u_h(x, t - \tau_1), \dots, u_h(x, t - \tau_n)) &\rightarrow f(u(x, t - \tau_1), \dots, u(x, t - \tau_n)), & \text{in } L^2(Q). \end{aligned}$$

Letting  $h \rightarrow \infty$  in

$$\begin{aligned} \int_0^T \int_0^1 \left\{ u_h \varphi_t + \frac{1}{h} u_h \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial u_h^m}{\partial x} \frac{\partial \varphi}{\partial x} + a u_h \varphi + f(u_h(x, t - \tau_1), \dots, u_h(x, t - \tau_n)) \varphi \right. \\ \left. + g(x, t) \varphi + \left( \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u_h(x, s) ds \right) \varphi \right\} dx dt = 0, \end{aligned}$$

we have

$$\begin{aligned} \int_0^T \int_0^1 \left\{ u \varphi_t - \frac{\partial u^m}{\partial x} \frac{\partial \varphi}{\partial x} + a u \varphi + f(u(x, t - \tau_1), \dots, u(x, t - \tau_n)) \varphi \right. \\ \left. + g(x, t) \varphi + \left( \gamma \int_{t-\tau}^t e^{-\alpha(t-s)} u(x, s) ds \right) \varphi \right\} dx dt = 0. \end{aligned}$$

That shows  $u$  satisfies the integral identity in the definition of weak solutions. Therefore, the problem (1.1)–(1.2) admits a nonnegative  $T$ -periodic solution  $u \in \{u; u \in L^\infty; u^m \in L^\infty(0, T; W_0^{1,2}(0, 1)); \frac{\partial u^m}{\partial t} \in L^2(Q)\}$ .

Since  $g(x, t) \not\equiv 0$ , we see that the  $T$ -periodic solution is nontrivial. The proof of Theorem 2.1 is complete.  $\square$

**Remark.** In equation (1.1), if the delays depend on time, we may construct a similar Lyapunov-type functional

$$V(t) = \int_0^1 \left( \frac{1}{2}u^2 + \frac{1}{2} \left| \frac{\partial}{\partial x}(\varepsilon u + u^m) \right|^2 \right) dx + \sum_{i=1}^n \beta_i^2 \int_{-\tau_i(t)}^0 \int_0^1 u^2(t + \theta) dx d\theta.$$

Under suitable assumptions, by the similar arguments, the corresponding existence of time periodic solution should be established for evolution equations with variable delays.

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