Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **49**, 1-15; **http://www.math.u-szeged.hu/ejqtde/**

Oscillation and nonoscillation of two terms linear and half-linear equations of higher order

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Abstract. In this paper we investigate the properties of nonoscillation for the equation

$$(-1)^n(\rho(t)|y^{(n)}|^{p-2}y^{(n)})^{(n)} - v(t)|y|^{p-2}y = 0,$$

where $1 < p < \infty$ and v is a non-negative continuous function and ρ is a positive n-times continuously differentiable function on the half - line $[0, \infty)$. When the principle of reciprocity is used for the linear equation ($p = 2$) we suppose that the functions v and ρ are positive and n-times continuously differentiable on the half - line $[0, \infty)$.

Mathematics Subject Classification. 34C10.

Key words and phrases. Oscillation, nonoscillation, higher order half–linear differential equation, variational method, weighted Hardy type inequalities.

1. Introduction

Let $I = [0, \infty)$ and $1 < p < \infty$. We consider the following higher order differential equation

$$(-1)^n(\rho(t)|y^{(n)}(t)|^{p-2}y^{(n)}(t))^{(n)} - v(t)|y(t)|^{p-2}y(t) = 0 \tag{1}$$

on I, where v is a non-negative continuous function and ρ is a positive n-times continuously differentiable function on I. When the principle of reciprocity is used for the linear equation ($p = 2$) we suppose that the functions v and ρ are positive and n-times continuously differentiable on I.

A function $y : I \to R$ is said to be a solution of the equation (1), if $y(t)$ and $\rho(t)|y^{(n)}(t)|^{p-2}y^{(n)}(t)$ are n-times continuously differentiable and $y(t)$ satisfies the equation (1) on I.

The equation (1) is called oscillatory at infinity if for any $T \geq 0$ there exist points $t_1 > t_2 > T$ and a nonzero solution $y(\cdot)$ of the equation (1) such that $y^{(i)}(t_k) = 0$, $i = 0, 1, ..., n-1$, $k = 1, 2$; otherwise the equation (1) is called nonoscillatory.

If $p = 2$, then the equation (1) becomes a higher order linear equation

$$(-1)^{(n)}(\rho(t)y^{(n)}(t))^{(n)} - v(t)y(t) = 0. \tag{2}$$

In the case $n = 1$ the oscillatory properties of the equations (1) and (2) have been enough well studied and there are known various investigation methods (see [1] and the bibliography therein).

The variational method to investigate the oscillatory properties of higher order linear equations and their relations to spectral characteristics of the corresponding differential operators are well presented in the monograph [2]. Another method is the transition from a higher order linear equation to a Hamilton system of equations [3]. However, to obtain the conditions of oscillation or nonoscillation of a higher order linear equation by this method we need to find the principal solutions of a Hamilton system (see [4,5]) that is not an easy task.

However, the general method of the investigation of the oscillatory properties for the equation (1) has been not developed yet. In the monograph [1] by O. Došlý, one of the leading experts in the oscillation theory of half–linear differential equations, and his colleagues, the oscillation theory of half–linear equations of higher order is compared with "terra incognita".

In this book the authors mention that it is possible to use Hardy's inequality in the oscillation theory of differential equations. That was done by M. Otelbaev [6] who found the conditions of oscillation and nonoscillation of Sturm-Liouville's equation.

The main aim of this paper is to establish the conditions of oscillation and nonoscillation of the equations (1) and (2) in terms of their coefficients by applying the latest results in the theory of weighted Hardy type inequalities.

The paper is organized in the following way: In Section 2 we formulate the facts and statements, which are required for proofs of the main results. In Section 3 the main results with proofs are presented.

2. Preliminaries

Let $I_T = [T, \infty)$, $T \geq 0$ and $1 < p < \infty$. Suppose that $L_p \equiv L_p(\rho, I_T)$ is the space of measurable and finite almost everywhere functions f, for which the following norm

$$\|f\|_{p,\rho} = \left(\int\limits_T^\infty \rho(t) |f(t)|^p dt \right)^{\frac{1}{p}}$$

is finite.

We shall consider the weighted Hardy inequality

$$\left(\int\limits_T^\infty v(t) \left| \int\limits_T^t f(s) ds \right|^p dt \right)^{\frac{1}{p}} \leq C \left(\int\limits_T^\infty \rho(t) |f(t)|^p dt \right)^{\frac{1}{p}}, \quad f \in L_p, \tag{3}$$

where $C > 0$ does not dependent on f.

For about the last 50 years the inequality (3) has been intensively investigated and at the present there are numerous criteria for the validity of this inequality.

The history of this problem and the results of investigations of weighted Hardy type inequalities are exposed in the book [7].

Let

$$J(T) \equiv J(\rho, v; T) = \sup_{0\neq f\in L_p} \frac{\int_T^\infty v(t) \left| \int_T^t f(s)ds \right|^p dt}{\int_T^\infty \rho(t)|f(t)|^p dt}.$$

The criteria for $J(T)$ to be finite which is equivalent to the validity of the inequality (3) are given in Theorem A (see [7]).

Theorem A. *Let* $1 < p < \infty$.

Then $J(T) \equiv J(\rho, v; T) < \infty$ *if and only if* $A_1(T) < \infty$ *or* $A_2(T) < \infty$, *where*

$$A_1(T) \equiv A_1(\rho, v; T) = \sup_{x>T} \int_x^\infty v(t)dt \left(\int_T^x \rho^{1-p'}(s)ds \right)^{p-1},$$

$$A_2(T) \equiv A_2(\rho, v; T) = \sup_{x>T} \left(\int_T^x \rho^{1-p'}(s)ds \right)^{-1} \int_T^x v(t) \left(\int_T^t \rho^{1-p'}(s)ds \right)^p dt.$$

Moreover, $J(T)$ *can be estimated from above and from below, i.e.,*

$$A_1(T) \leq J(T) \leq p\left(\frac{p}{p-1}\right)^{p-1} A_1(T), \tag{4}$$

$$A_2(T) \leq J(T) \leq \left(\frac{p}{p-1}\right)^p A_2(T), \tag{5}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

In [8] it is shown that the constant $p\left(\frac{p}{p-1}\right)^{p-1}$ in (4) is the best possible.

Remark. *Here and further in theorems the conditions of the type* $A_1(T) \leq K < \infty$ *mean that there the integrals converge with respect to infinite interval, and the conditions of the type* $A_1(T) \geq K$ *allow the divergence of the integrals.*

Next, we consider the following expression

$$J_n(T) \equiv J_n(\rho, v; T) = \sup_{0\neq f\in L_2} \frac{\int_T^\infty \left| \int_T^t (t-s)^{n-1} f(s)ds \right|^2 dt}{\int_T^\infty \rho(t)|f(t)|^2 dt}.$$

We quote the following result proved in [9].

Theorem B. $J_n(T) \equiv J_n(\rho, v; T) < \infty$ *if and only if* $B_1(T) < \infty$ *and* $B_2(T) < \infty$, *where*

$$B_1(T) \equiv B_1(\rho, v; T) = \sup_{x>T} \int_x^\infty v(t)dt \int_T^x (x-s)^{2(n-1)} \rho^{-1}(s)ds,$$

$$B_2(T) \equiv B_2(\rho, v; T) = \sup_{x>T} \int_x^\infty v(t)(t-x)^{2(n-1)}dt \int_T^x \rho^{-1}(s)ds.$$

Moreover, there exists a constant $\beta \geq 1$ independent of ρ, v and T such that

$$B(T) \leq J_n(T) \leq \beta B(T), \tag{6}$$

where $B(T) = \max\{B_1(T), B_2(T)\}$.

Assume that $AC_p^{n-1}(\rho, I_T)$ is a set of all functions f that have absolutely continuous $n-1$ order derivatives on $[T, N]$ for any $N > T$ and $f^{(n)} \in L_p$. Let $AC_{p,L}^{n-1}(\rho, I_T) = \{f \in AC_p^{n-1}(\rho, I_T) : \ f^{(i)}(T) = 0, \ i = 0, 1, ..., n-1\}$.

Suppose that $A^0C_p^{n-1}(\rho, I_T)$ is a set of all functions from $AC_{p,L}^{n-1}(\rho, I_T)$ that are equal to zero in a neighborhood of infinity. The function f from $AC_{p,L}^{n-1}(\rho, I_T)$ is called nontrivial if $\|f^{(n)}\|_p \neq 0$; we write down that $f \neq 0$.

From the variational method for higher order linear equations [2] we have:

Theorem C. *The equation (2)*

(i) is nonoscillatory if and only if there exists $T \geq 0$ such that

$$\int_T^\infty (\rho(t)|f^{(n)}(t)|^2 - v(t)|f(t)|^2)dt > 0 \tag{7}$$

for every nontrivial $f \in A^0C_2^{n-1}(\rho, I_T)$;

(ii) is oscillatory if and only if for every $T \geq 0$ there exists a nontrivial function $\tilde{f} \in A^0C_2^{n-1}(\rho, I_T)$ such that

$$\int_T^\infty (\rho(t)|\tilde{f}^{(n)}(t)|^2 - v(t)|\tilde{f}(t)|^2)dt \leq 0. \tag{8}$$

The following statement is due to Theorem 9.4.4 from [1]:

Theorem D. *Let $1 < p < \infty$. If there exists $T \geq 0$ such that*

$$\int_T^\infty (\rho(t)|f^{(n)}(t)|^p - v(t)|f(t)|^p)dt > 0 \tag{9}$$

for all nontrivial $f \in A^0C_p^{n-1}(\rho, I_T)$, then the equation (1) is nonoscillatory.

Suppose that $W_p^n \equiv W_p^n(\rho, I_T)$ is a set of functions f that have n order generalized derivatives on I_T and for which the norm

$$\|f\|_{W_p^n} = \|f^{(n)}\|_p + \sum_{i=0}^{n-1} |f^{(i)}(T)| \tag{10}$$

is finite.

It is obvious that $A^0C_p^{n-1}(\rho, I_T) \subset AC_{p,L}^{n-1}(\rho, I_T) \subset W_p^n(\rho, I_T)$. The closures of the sets $A^0C_p^{n-1}(\rho, I_T)$ and $AC_{p,L}^{n-1}(\rho, I_T)$ with respect to the norm (10) we denote by $\overset{\circ}{W}{}_p^n \equiv \overset{\circ}{W}{}_p^n(\rho, I_T)$ and $W_{p,L}^n \equiv W_{p,L}^n(\rho, I_T)$, respectively. Since $\rho(t) > 0$ for $t \geq 0$ we have that

$$f^{(i)}(T) = 0, \quad i = 0, 1, ..., n-1 \tag{11}$$

for any $f \in W_{p,L}^n(\rho, I_T)$.

3. Main results

In this section we consider nonoscillation of the equations (1) and (2) and oscillation of the equation (2).

Theorem 1. *Let $1 < p < \infty$. Suppose that v is a non-negative continuous function and ρ is a positive and n-times continuously differentiable function on I. If one of the following conditions*

$$\lim_{T\to\infty} \sup_{x>T} \left(\int_T^x \rho^{1-p'}(s)ds \right)^{p-1} \int_x^\infty v(t)(t-T)^{p(n-1)}dt <$$
$$< \frac{1}{p-1} \left[\frac{(n-1)!(p-1)}{p} \right]^p \tag{12}$$

or

$$\lim_{T\to\infty} \sup_{x>T} \left(\int_T^x \rho^{1-p'}(s)ds \right)^{-1} \int_T^x v(t)(t-T)^{p(n-1)} \left(\int_T^t \rho^{1-p'}(s)ds \right)^p dt <$$
$$< \left[\frac{(n-1)!(p-1)}{p} \right]^p \tag{13}$$

holds, then the equation (1) is nonoscillatory.

Nonoscillation of the equation (2) follows from Theorem 1 with $p = 2$:

Theorem 2. *Suppose that v is a non-negative continuous function and ρ is a positive and n-times continuously differentiable function on I. If one of the following conditions*

$$\lim_{T\to\infty} \sup_{x>T} \int_T^x \rho^{-1}(s)ds \int_x^\infty v(t)(t-T)^{2(n-1)}dt < \left[\frac{(n-1)!}{2} \right]^2$$

or

$$\lim_{T\to\infty} \sup_{x>T} \left(\int_T^x \rho^{-1}(s)ds \right)^{-1} \int_T^x v(t)(t-T)^{2(n-1)} \left(\int_T^t \rho^{-1}(s)ds \right)^2 dt < \left[\frac{(n-1)!}{2} \right]^2$$

holds, then the equation (2) is nonoscillatory.

Proof of Theorem 1. If we show that from one of conditions (12) or (13) it follows that there exists $T \geq 0$ such that

$$F_{p,0}(T) \equiv F_{p,0}(\rho, v; T) = \sup_{0 \neq f \in A^0 C_p^{n-1}(\rho, I_T)} \frac{\int_T^\infty v(t)|f(t)|^p dt}{\int_T^\infty \rho(t)|f^{(n)}(t)|^p dt}$$

$$= \sup_{0 \neq f \in \overset{\circ}{W}{}_p^n} \frac{\int_T^\infty v(t)|f(t)|^p dt}{\int_T^\infty \rho(t)|f^{(n)}(t)|^p dt} < 1, \tag{14}$$

then by Theorem D the equation (1) is nonoscillatory.

We define

$$F_{p,L}(T) \equiv F_{p,L}(\rho, v; T) = \sup_{0 \neq f \in W_{p,L}^n} \frac{\int_T^\infty v(t)|f(t)|^p dt}{\int_T^\infty \rho(t)|f^{(n)}(t)|^p dt}. \tag{15}$$

Since $\overset{\circ}{W}{}_p^n \subset W_{p,L}^n$, then

$$F_{p,0}(T) \leq F_{p,L}(T). \tag{16}$$

From (11) the mapping

$$f^{(n)} = g, \quad f(t) = \frac{1}{(n-1)!} \int_T^t (t-s)^{n-1} g(s) ds \tag{17}$$

gives one-to-one correspondence of $W_{p,L}^n$ and L_p. Therefore, replacing $f \in W_{p,L}^n$ by $g \in L_p$ we have

$$F_{p,L}(T) = \frac{1}{[(n-1)!]^p} \sup_{0 \neq g \in L_p} \frac{\int_T^\infty v(t) \left| \int_T^t (t-s)^{n-1} g(s) ds \right|^p dt}{\int_T^\infty \rho(t)|g(t)|^p dt} \leq$$

$$\leq \frac{1}{[(n-1)!]^p} \sup_{0 \neq g \in L_p} \frac{\int_T^\infty v(t)(t-T)^{p(n-1)} \left| \int_T^t g(s) ds \right|^p dt}{\int_T^\infty \rho(t)|g(t)|^p dt} = \frac{J(\rho, \tilde{v}; T)}{[(n-1)!]^p}, \tag{18}$$

where $\tilde{v} = v(t)(t-T)^{p(n-1)}$.

Thus, from the estimates (4) and (5) of Theorem A, we have

$$\frac{J(\rho, \tilde{v}; T)}{[(n-1)!]^p} \leq (p-1) \left[\frac{(n-1)!(p-1)}{p} \right]^{-p} \times$$

$$\times \sup_{x>T} \int_x^\infty v(t)(t-T)^{p(n-1)} dt \left(\int_T^x \rho^{1-p'}(s) ds \right)^{p-1} \tag{19}$$

and

$$\frac{J(\rho,\tilde{v};T)}{[(n-1)!]^p} \le \left[\frac{(n-1)!(p-1)}{p} \right]^{-p} \times$$

$$\times \sup_{x>T} \left(\int_T^x \rho^{1-p'}(s) ds \right)^{-1} \int_T^x v(t)(t-T)^{p(n-1)} \left(\int_T^t \rho^{1-p'}(s) ds \right)^p dt. \tag{20}$$

If (12) or (13) is satisfied, then there exists $T \ge 0$ such that the left–hand side of (19) or (20) respectively becomes less than one. Under the assumptions of Theorem 1 there exists $T \ge 0$ such that

$$\frac{J(\rho,\tilde{v};T)}{[(n-1)!]^p} < 1.$$

Then (14) follows from (18) and (16). The proof of Theorem 1 is completed.

Example. We consider the equation

$$(-1)^n (|y^{(n)}|^{p-2} y^{(n)})^{(n)} - \frac{\gamma}{t^{np}} |y|^{p-2} y = 0, \tag{21}$$

where $\gamma \in R$.

By the proof of Theorem 1 it follows that if

$$\gamma F_{p,L}(0) = \frac{\gamma}{[(n-1)!]^p} \sup_{0 \ne g \in L_p} \frac{\int_0^\infty \left| \frac{1}{t^n} \int_0^t (t-s)^{n-1} g(s) ds \right|^p dt}{\int_0^\infty |g(t)|^p dt} < 1,$$

then the equation (21) is nonoscillatory.

By Theorem 329 from [10] we have

$$\gamma F_{p,L}(0) = \gamma \left[\frac{\Gamma(1-\frac{1}{p})}{\Gamma\left(n+1-\frac{1}{p}\right)} \right]^p < 1. \tag{22}$$

Here $\Gamma(\cdot)$ is the gamma–function. Using the reduction formula $\Gamma(q+1) = q\Gamma(q)$, $q > 0$, we have

$$\Gamma\left(n+1-\frac{1}{p}\right) = \prod_{k=1}^n \left(k - \frac{1}{p}\right) \Gamma\left(1-\frac{1}{p}\right).$$

Taking into account (22) we obtain that the equation (21) is nonoscillatory if

$$\gamma < \prod_{k=1}^n \left(k-\frac{1}{p}\right)^p = p^{-np} \prod_{k=1}^n (kp-1)^p. \tag{23}$$

Let us notice that the condition (23) is obtained in Theorem 9.4.5 in [1] by another way.

Now, we consider the problem of oscillation of the equation (2).

By Theorem 2 it is easy to prove that if both integrals

$$\int_T^\infty \rho^{-1}(s)ds$$

and

$$\int_T^\infty v(t)(t-T)^{2(n-1)}dt$$

are finite, then the equation (2) is nonoscillatory.

Therefore, we are interested in the case when at least one of these integrals is infinite.

We start with the case

$$\int_T^\infty \rho^{-1}(s)ds = \infty. \tag{24}$$

Theorem 3. *Let (24) hold. If one of the inequalities*

$$\lim_{T\to\infty} \sup_{x>T} \int_T^x \rho^{-1}(s)ds \int_x^\infty v(t)(t-x)^{2(n-1)}dt > [(n-1)!]^2$$

or

$$\lim_{T\to\infty} \sup_{x>T} \int_T^x \rho^{-1}(s)(x-s)^{2(n-1)}ds \int_x^\infty v(t)dt > [(n-1)!]^2$$

holds, then the equation (2) is oscillatory.

Proof of Theorem 3. If we show that

$$F_{2,0}(T) > 1 \tag{25}$$

for any $T \geq 0$, then the equation (2) is oscillatory.

Indeed, from (25) it follows that for every $T \geq 0$ there exists a nontrivial function $\tilde{f} \in A^0C_p^{n-1}(\rho, I_T)$ such that the inequality (8) holds. Consequently, by Theorem C the equation (2) is oscillatory.

According to the results of [11] the condition (24) implies that $\overset{\circ}{W}{}_2^n = W_{2,L}^n$. Then

$$F_{2,0}(T) = F_{2,L}(T) \tag{26}$$

and from (17) we have

$$F_{2,0}(T) = \sup_{0\neq f\in W_{2,L}^n} \frac{\int_T^\infty v(t)|f(t)|^2dt}{\int_T^\infty \rho(t)|f^{(n)}(t)|^2dt} =$$

$$= \frac{1}{[(n-1)!]^2} \sup_{0 \neq g \in L_2} \frac{\int\limits_T^\infty v(t) \left| \int\limits_T^t (t-s)^{n-1} g(s) ds \right|^2 dt}{\int\limits_T^\infty \rho(t) |g(t)|^2 dt} = \frac{J_n(T)}{[(n-1)!]^2}. \tag{27}$$

From the estimate (6) of Theorem B it follows that

$$\frac{B(T)}{[(n-1)!]^2} \leq F_{2,0}(T) \leq \beta \frac{B(T)}{[(n-1)!]^2}. \tag{28}$$

From the left–hand side of the inequality (28) and the assumptions of Theorem it follows that the inequality (25) holds. Thus, the equation (2) is oscillatory.

The proof of Theorem 3 is completed.

Let us turn to the equation (2) with parameter $\lambda > 0$ in the form:

$$(-1)^n (\rho(t) y^{(n)})^{(n)} - \lambda v(t) y = 0. \tag{29}$$

If the equation (29) for any $\lambda > 0$ is oscillatory or nonoscillatory, then the equation (29) is called strongly oscillatory or strongly nonoscillatory, respectively.

Theorem 4. *If the condition (24) is satisfied, then the equation (29)*

(i) is strongly nonoscillatory if and only if

$$\lim_{x \to \infty} \int\limits_0^x \rho^{-1}(s) ds \int\limits_x^\infty v(t)(t-x)^{2(n-1)} dt = 0 \tag{30}$$

and

$$\lim_{x \to \infty} \int\limits_0^x \rho^{-1}(s)(x-s)^{2(n-1)} ds \int\limits_x^\infty v(t) dt = 0; \tag{31}$$

(ii) is strongly oscillatory if and only if at least one of the following conditions

$$\limsup_{x \to \infty} \int\limits_0^x \rho^{-1}(s) ds \int\limits_x^\infty v(t)(t-x)^{2(n-1)} dt = \infty \tag{32}$$

or

$$\limsup_{x \to \infty} \int\limits_0^x \rho^{-1}(s)(x-s)^{2(n-1)} ds \int\limits_x^\infty v(t) dt = \infty. \tag{33}$$

holds.

Proof of Theorem 4. Let the equation (29) be nonoscillatory for any $\lambda > 0$. Then by the criterion of nonoscillation (7) of Theorem C for every $\lambda > 0$ there exists $T_\lambda \geq 0$ such that $\lambda F_{2,0}(T_\lambda) \leq 1$. Then $\lim\limits_{\lambda \to \infty} F_{2,0}(T_\lambda) = 0$. However, if the equation (29) is nonoscillatory for $\lambda = \lambda_0 > 0$, then by (7) it is nonoscillatory for any $0 < \lambda \leq \lambda_0$. Therefore, T_λ does not decrease. Hence

$$\lim_{T \to \infty} F_{2,0}(T) = 0. \tag{34}$$

Thus, from the left–hand side of the inequality (28) and from (34) it follows that $\lim\limits_{T\to\infty} B(T) = 0$, where $B(T) = \max\{B_1(T), B_2(T)\}$ and

$$B_1(T) = \sup_{x>T} \int\limits_x^\infty v(t)dt \int\limits_T^x (x-s)^{2(n-1)} \rho^{-1}(s)ds,$$

$$B_2(T) = \sup_{x>T} \int\limits_x^\infty v(t)(t-x)^{2(n-1)}dt \int\limits_T^x \rho^{-1}(s)ds.$$

Then for any $\varepsilon > 0$ there exists $T_\varepsilon^1 > 0$ such that for every $x \geq T_\varepsilon^1$ we have

$$\int\limits_{T_\varepsilon^1}^x \rho^{-1}(s)ds \int\limits_x^\infty v(t)(t-x)^{2(n-1)}dt \leq \frac{\varepsilon}{2}$$

and there exists $T_\varepsilon \geq T_\varepsilon^1$ such that for every $x \geq T_\varepsilon$ we have

$$\int\limits_0^{T_\varepsilon^1} \rho^{-1}(s)ds \int\limits_x^\infty v(t)(t-x)^{2(n-1)}dt \leq \frac{\varepsilon}{2}$$

since $\lim\limits_{x\to\infty} \int\limits_x^\infty v(t)(t-x)^{2(n-1)}dt = 0$.

Therefore, for every $x \geq T_\varepsilon$ we have

$$\int\limits_0^x \rho^{-1}(s)ds \int\limits_x^\infty v(t)(t-x)^{2(n-1)}dt \leq \varepsilon,$$

which means that the equality (30) is satisfied. The equality (31) can be proved similarly.

Now, we shall prove that if the equalities (30) and (31) hold, then the equation (29) is strongly nonoscillatory.

Since the equalities (30) and (31) hold, then $\lim\limits_{T\to\infty} B(T) = 0$. Therefore, from the right–hand side of the inequality (28) we have the equality (34). Hence for every $\lambda > 0$ there exists $T_\lambda \geq 0$ such that $\lambda F_{2,0}(T_\lambda) < 1$. Then the equation (29) is strongly nonoscillatory. Thus, (i) is proved.

Let us prove (ii). Let the equation (29) be strongly oscillatory. By Theorem C we have that $\lambda F_{2,0}(T) \geq 1$ for every $\lambda > 0$ and for every $T \geq 0$. Therefore, $F_{2,0}(T) \geq \sup\limits_{\lambda>0} \frac{1}{\lambda} = \infty$ for every $T \geq 0$.

Thus, from the right–hand side of the inequality (28) it follows that $B(T) = \infty$ for every $T \geq 0$, so at least $B_1(T) = \infty$ or $B_2(T) = \infty$. This means that the equality (32) or (33) holds.

Suppose that for every $T \geq 0$ one of the conditions (32) or (33) holds. Then either $B_1(T) = \infty$ or $B_2(T) = \infty$. Therefore, $B(T) = \infty$ for any $T \geq 0$. Then from the left–hand side of the inequality (28) it follows $F_{2,0}(T) = \infty$ for any $T \geq 0$. Consequently, $\lambda F_{2,0}(T) > 1$ for any $\lambda > 0$ and $T \geq 0$, which by (8) means the oscillation of the equation (29) for $\lambda > 0$.

The proof of Theorem 4 is completed.

Corollary 1. *Let $T \geq 0$. If the conditions (24) and*

$$\int_T^\infty v(t)(t-T)^{2(n-1)}dt = \infty$$

are satisfied, then the equation (2) is strongly oscillatory.

As an example let us consider the equation

$$(-1)^n \left(t^{-\alpha} y^{(n)(t)}\right)^{(n)} - \lambda v(t)y(t) = 0, \tag{35}$$

where $\alpha \geq 0$ and v is a non–negative continuous function on I. Since $\alpha \geq 0$, then the conditions (24) for the equation (35) is valid.

Since

$$\int_0^x s^\alpha (x-s)^{2(n-1)}ds = x^{2n-1+\alpha} \int_0^1 s^\alpha (1-s)^{2(n-1)}ds,$$

then the conditions (31) and (33) for the equation (35) are respectively equivalent to the conditions

$$\lim_{x\to\infty} x^{2n-1+\alpha} \int_x^\infty v(t)dt = 0, \tag{36}$$

$$\lim_{x\to\infty} \sup x^{2n-1+\alpha} \int_x^\infty v(t)dt = \infty. \tag{37}$$

Using the L'Hospital rule $2(n-1)$ times it is easy to see that from (36) it follows the condition (30)

$$\lim_{x\to\infty} x^{\alpha+1} \int_x^\infty v(t)(t-x)^{2(n-1)}dt = 0$$

for the equation (35).

Thus, by Theorem 4 the equation (35) is strongly nonoscillatory if and only if (36) is correct. Moreover, it is strongly oscillatory if and only if (37) is correct. This yields for $\alpha = 0$ the validity of Theorems 15 and 16 from the monograph [2].

Now, we use Theorem 3 to the equation (35) for $\lambda = 1$. Let $k = \lim_{T\to\infty} \sup_{x>T} \int_T^x s^\alpha (x-s)^{2(n-1)}ds \int_x^\infty v(t)dt$ and $\gamma > 1$.

Then

$$\sup_{x>T}\int_T^x s^\alpha(x-s)^{2(n-1)}ds\int_x^\infty v(t)dt \geq \int_T^{\gamma T} s^\alpha(\gamma T-s)^{2(n-1)}ds\int_{\gamma T}^\infty v(t)dt =$$

$$=\frac{1}{\gamma^{2n-1+\alpha}}\int_1^\gamma s^\alpha(\gamma-s)^{2(n-1)}ds(\gamma T)^{2n-1+\alpha}\int_{\gamma T}^\infty v(t)dt.$$

If

$$\sup_{\gamma>1}\frac{1}{\gamma^{2n-1+\alpha}}\int_1^\gamma s^\alpha(\gamma-s)^{2(n-1)}ds\lim_{x\to\infty}x^{2n-1+\alpha}\int_x^\infty v(t)dt>[(n-1)!]^2, \tag{38}$$

then $k>[(n-1)!]^2$ and by Theorem 3 the equation (35) is oscillatory.

In [12] the exact values of the oscillation constants of the equation (35) are obtained for the different values $\alpha\in R$. Moreover, there in Proposition 2.2 the main oscillation conditions found before are collected. If we compare the conditions (38) and the conditions from Proposition 2.2 for $\alpha\geq 0$, we can see that the conditions (38) are better than the conditions from Proposition 2.2. For example, when $n=2$ and $\alpha=0$ we have that

$$\sup_{\gamma>1}\frac{1}{\gamma^3}\int_1^\gamma(\gamma-s)^2ds=\frac{1}{3}\sup_{\gamma>1}\left(1-\frac{1}{\gamma}\right)^3=\frac{1}{3}.$$

Therefore, from (38) it follows that the equation $y^{IV}(t)=v(t)y(t)$ is oscillatory if $\lim_{x\to\infty}x^3\int_x^\infty v(t)dt>3$. The analogous condition from Proposition 2.2 has the form $\lim_{x\to\infty}x^3\int_x^\infty v(t)dt>12$.

Next, we assume that the functions v and ρ are positive and n-times continuously differentiable on I. Then by the principle of reciprocity [4] the equation (2) and the reciprocal equation

$$(-1)^n(v^{-1}(t)y^{(n)})^{(n)}-\rho^{-1}(t)y=0 \tag{39}$$

are simultaneously oscillatory or nonoscillatory. Applying the principle of reciprocity we obtain the following theorems.

Theorem 5. *Let functions v and ρ be positive and n-times continuously differentiable on I. Then, if one of the following conditions*

$$\lim_{T\to\infty}\sup_{x>T}\int_T^x v(t)dt\int_x^\infty\rho^{-1}(s)(s-T)^{2(n-1)}ds<\left[\frac{(n-1)!}{2}\right]^2,$$

or

$$\lim_{T\to\infty}\sup_{x>T}\left(\int_T^x v(t)dt\right)^{-1}\int_T^x\rho^{-1}(s)(s-T)^{2(n-1)}\left(\int_T^s v(t)dt\right)^2ds<\left[\frac{(n-1)!}{2}\right]^2$$

holds, then the equation (2) is nonoscillatory.

Indeed, if the condition of Theorem 5 is satisfied, then by Theorem 2 the equation (39) is nonoscillatory. Therefore, the equation (2) is also nonoscillatory.

In the case of

$$\int_T^\infty v(t)dt = \infty \tag{40}$$

the following theorem is valid.

Theorem 6. *Suppose that v and ρ are positive n-times continuously differentiable functions on I. Let the condition (40) hold. Then, if one of the following inequalities*

$$\lim_{T\to\infty} \sup_{x>T} \int_T^x v(t)dt \int_x^\infty \rho^{-1}(s)(s-x)^{2(n-1)}ds > [(n-1)!]^2$$

or

$$\lim_{T\to\infty} \sup_{x>T} \int_T^x v(t)(x-t)^{2(n-1)}dt \int_x^\infty \rho^{-1}(s)ds > [(n-1)!]^2$$

holds, then the equation (2) is oscillatory.

The proofs of this theorem and the following theorem are based on the principle of reciprocity.

Theorem 7. *Suppose that v and ρ is positive and n-times continuously differentiable functions on I. Let the condition (40) hold. Then the equation (29)*

(i) is strongly nonoscillatory if and only if

$$\lim_{x\to\infty} \int_0^x v(t)dt \int_x^\infty \rho^{-1}(s)(s-x)^{2(n-1)}ds = 0$$

and

$$\lim_{x\to\infty} \int_0^x v(t)(x-t)^{2(n-1)}dt \int_x^\infty \rho^{-1}(s)ds = 0;$$

(ii) is strongly oscillatory if and only if one of the following conditions

$$\lim_{x\to\infty} \sup \int_0^x v(t)dt \int_x^\infty \rho^{-1}(s)(s-x)^{2(n-1)}ds = \infty$$

or

$$\lim_{x\to\infty} \sup \int_0^x v(t)(x-t)^{2(n-1)}dt \int_x^\infty \rho^{-1}(s)ds = \infty$$

holds.

Corollary 2. *Let $T \geq 0$. If the conditions (40) and*

$$\int\limits_T^\infty \rho^{-1}(t)(t-T)^{2(n-1)}dt = \infty$$

are satisfied, then the equation (2) is strongly oscillatory.

Acknowledgements. The authors express deep gratitude to professor T. Tararykova for the help and advices in preparing of this paper. We also thank the Referee for some valuable suggestions, which have improved this paper.

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(Received April 2, 2010)