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Global Existence and Controllability to a Stochastic Integro-differential Equation

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Abstract

In this paper, we are focused upon the global uniqueness results for a stochastic integro-differential equation in Fréchet spaces. The main results are proved by using the resolvent operators combined with a nonlinear alternative of Levay-Schauder type in Fréchet spaces due to Frigon and Granas. As an application, a controllability result with one parameter is given to illustrate the theory.

Keywords: Stochastic integro-differential equations, Resolvent operators, Fréchet spaces, Controllability.

Mathematics Subject Classification(2000): 34K14, 60H10, 35B15, 34F05.

1 Introduction

In this paper, we consider the uniqueness of mild solutions on a semi-infinite positive real interval $J := [0, +\infty)$ for a class of stochastic integro-differential equations in the abstract form

$$dx(t) = \left[Ax(t) + \int_0^t B(t-s)x(s)ds\right]dt + f(t, x(t))dw(t), \ t \in J, \tag{1.1}$$

$$x(0) = x_0, \tag{1.2}$$

where $A : D(A) \subset \mathbb{H} \to \mathbb{H}$, $B(t) : D(B(t)) \subset \mathbb{H} \to \mathbb{H}$, $t \geq 0$, are linear, closed, and densely defined operators in a Hilbert space $\mathbb{H}$, $f : J \times \mathbb{H} \to L_Q(\mathbb{K}, \mathbb{H})$ is an

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appropriate function specified later and $w(t), t \geq 0$ is a given $\mathbb{K}$-valued Brownian motion, which will be defined in Section 2. The initial data x_0 is an $\mathscr{F}_0$-adapted, $\mathbb{H}$-valued random variable independent of the Wiener process w.

Stochastic differential and integro-differential equations have attracted great interest due to their applications in characterizing many problems in physics, biology, mechanics and so on. Qualitative properties such as existence, uniqueness and stability for various stochastic differential and integro-differential systems have been extensively studied by many researchers, see for instance [1, 2, 3, 4, 5, 6, 7, 8, 9] and the references therein. The theory of nonlinear functional integro-differential equations with resolvent operators serves as an abstract formulation of partial integro-differential equations which arises in many physical phenomena [10, 11, 12, 13, 14]. Just as pointed out by Ouahab in [15], the investigation of many properties of solutions for a given equation, such as stability, oscillation, often needs to guarantee its global existence. Thus it is of great importance to establish sufficient conditions for global existence results for functional differential equations. The existence of unique global solutions for deterministic functional differential evolution equations with infinite delay in Fréchet spaces were studied by Baghli et al. [16, 17] and Benchohra et al. [18]. Motivated by the works [16, 17, 19, 18], the main purpose of this paper is to establish the global uniqueness of solutions for the problem (1.1)-(1.2). Our approach here is based on a recent Frigon and Granas nonlinear alternative of Leray-Schauder type in Fréchet spaces [20] combined with the resolvent operators theory. The obtained result can be seen as a contribution to this emerging field.

The rest of this paper is organized as follows: In section 2, we recall some basic definitions, notations, lemmas and theorems which will be needed in the sequel. In section 3, we prove the existence of the unique mild solutions for the problem (1.1)-(1.2). Section 4 is reserved for an application.

2 Preliminaries

This section is concerned with some basic concepts, notations, definitions, lemmas and preliminary facts which are used throughout this paper. For more details on this section, we refer the reader to [5, 21].

Throughout the paper, $(\mathbb{H}, \|\cdot\|, \langle\cdot,\cdot\rangle)$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle_{\mathbb{K}})$ denote two real separable Hilbert spaces. Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete probability space equipped with some filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is right continuous and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets). Let $\{e_i\}_{i=1}^{\infty}$ be a complete orthonormal basis of $\mathbb{K}$. We denote by $\{w(t), t \geq 0\}$ a cylindrical $\mathbb{K}$-valued Wiener process with a finite trace nuclear covariance operator $Q \geq 0$, denote $Tr(Q) = \sum_{i=1}^{\infty} \lambda_i = \lambda < \infty$, which satisfies that $Qe_i = \lambda_i e_i$, $i = 1, 2, \cdots$. So, actually, $w(t)$ is defined by

$$w(t) = \sum_{i=1}^{\infty} \sqrt{\lambda_i} w_i(t) e_i, \ t \geq 0,$$

where $\{w_i(t)\}_{i=1}^{\infty}$ are mutually independent one-dimensional standard Wiener processes. We then let $\mathscr{F}_t = \sigma\{w(s) : 0 \leq s \leq t\}$ be the σ-algebra generated by w.

Let $L(\mathbb{K}, \mathbb{H})$ denote the space of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, equipped with the usual operator norm $\|\cdot\|_{L(\mathbb{K},\mathbb{H})}$. For $\phi \in L(\mathbb{K}, \mathbb{H})$, we define

$$\|\phi\|_Q^2 = Tr(\phi Q \phi^*) = \sum_{i=1}^{\infty} \|\sqrt{\lambda_i} \phi e_i\|^2.$$

If $\|\phi\|_Q^2 < \infty$, then ϕ is called a Q-Hillbert-Schmidt operator. Let $L_Q(\mathbb{K}, \mathbb{H})$ denote the space of all Q-Hillbert-Schmidt operators $\phi : \mathbb{K} \to \mathbb{H}$. The completion $L_Q(\mathbb{K}, \mathbb{H})$ of $L(\mathbb{K}, \mathbb{H})$ with respect to the topology induced by the norm $\|\cdot\|_Q$ where $\|\phi\|_Q^2 = \langle \phi, \phi \rangle$ is a Hilbert space with the above norm topology.

The collection of all strongly measurable, square integrable, $\mathbb{H}$-valued random variables, denoted by $L_2(\Omega, \mathbb{H})$, is a Banach space equipped with norm $\|x\|_{L^2(\Omega;\mathbb{H})} = (E\|x\|^2)^{\frac{1}{2}}$, where the expectation E is defined $E[x] = \int_\Omega x(w) d\mathbb{P}(w)$. An important subspace is given by $L_2^0(\Omega, \mathbb{H}) = \{f \in L_2(\Omega, \mathbb{H}) : f \text{ is } \mathscr{F}_0 - \text{measurable}\}$. Let $C_{\mathscr{F}_t}(J, \mathbb{H})$ denote the space of all continuous and $\mathscr{F}_t$-adapted measurable processes from J into $\mathbb{H}$.

A measurable function $x : [0, +\infty) \to \mathbb{H}$ is Bochner integrable if $\|x\|$ is Lebesgue integrable. (For details on the Bochner integral properties, see Yosida [22]).

Let $L^1([0, +\infty), \mathbb{H})$ be the space of measurable functions $x : [0, +\infty) \to \mathbb{H}$ which are Bochner integrable, equipped with the norm

$$\|x\|_{L^1} = \int_0^{+\infty} \|x(t)\| dt.$$

Consider the space

$$B_{+\infty} = \{x : J \to \mathbb{H} \in C_{\mathscr{F}_t}(J, \mathbb{H}) : x_0 \in L_2^0(\Omega, \mathbb{H})\}.$$

Throughout the rest of the paper, $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a resolvent operator $R(t), t \geq 0$ in the Hilbert space $\mathbb{H}$ and $B(t) : D(B(t)) \subset \mathbb{H} \to \mathbb{H}$, $t \geq 0$ is a bounded linear operator. To obtain our results, we assume that the abstract Cauchy problem

$$dx(t) = \left[Ax(t) + \int_0^t B(t-s) x(s) ds \right] dt, \ t \geq 0, \tag{2.1}$$

$$x(0) = x_0 \in \mathbb{H}, \tag{2.2}$$

has an associated resolvent operator of bounded linear operators $R(t), t \geq 0$ on $\mathbb{H}$.

Definition 2.1 *A family of bounded linear operators $R(t), t \geq 0$ from $\mathbb{H}$ into $\mathbb{H}$ is a resolvent operator family for the problem (2.1)-(2.2) if the following conditions are verified.*

(i) $R(0) = I$ (the identity operator on $\mathbb{H}$) and the map $t \to R(t)x$ is a continuous function on $[0, +\infty) \to \mathbb{H}$ for every $x \in \mathbb{H}$.

(ii) $AR(\cdot)x \in C([0, \infty), \mathbb{H})$ and $R(\cdot)x \in C^1([0, \infty), \mathbb{H})$ for every $x \in D(A)$.

(iii) For every $x \in D(A)$ and $t \geq 0$,

$$\frac{d}{dt}R(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds,$$

$$\frac{d}{dt}R(t)x = R(t)Ax + \int_0^t R(t-s)B(s)xds.$$

For more details on semigroup theory and resolvent operators, we refer [13, 14, 23, 24].

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}}$. Let $Y \subset X$, we say that Y is bounded if for every $n \in \mathbb{N}$, there exists $\overline{M}_n > 0$ such that

$$\|y\|_n \leq \overline{M}_n \quad \text{for all } y \in Y.$$

With X, we associate a sequence of Banach spaces $\{(X^n, \|\cdot\|_n)\}$ as follows: For every $n \in \mathbb{N}$, we consider the equivalence relation $x \sim_n y$ if and only if $\|x - y\|_n = 0$ for all $x, y \in X$. We denote $X^n = (X|_{\sim_n}, \|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y \subset X$, we associate a sequence the $\{Y^n\}$ of subsets $Y^n \subset X^n$ as follows: For every $x \in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n = \{[x]_n : x \in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and $\partial_n Y^n$, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1 \leq \|x\|_2 \leq \|x\|_3 \leq \cdots \quad \text{for every } x \in X.$$

Definition 2.2 *A function $f : J \times \mathbb{H} \to L_Q(\mathbb{K}, \mathbb{H})$ is said to be an L^2-Carathéodory function if it satisfies:*

(i) for each $t \in J$ the function $f(t, \cdot) : \mathbb{H} \to L_Q(\mathbb{K}, \mathbb{H})$ is continuous;

(ii) for each $x \in \mathbb{H}$ the function $f(\cdot, x) : J \to L_Q(\mathbb{K}, \mathbb{H})$ is $\mathscr{F}_t$-measurable;

(iii) for every positive integer k there exists $\alpha_k \in L^1_{loc}(J, \mathbb{R}_+)$ such that

$$E\|f(t,x)\|^2 \leq \alpha_k(t) \quad \textit{for all} \quad E\|x\|^2 \leq k$$

and for almost all $t \in J$.

Definition 2.3 *[20] A function $G : X \to X$ is said to be a contraction if for each $n \in \mathbb{N}$ there exists $k_n \in (0,1)$ such that:*

$$\|G(x) - G(y)\|_n \le k_n \|x - y\|_n \ \ \textit{for all} \ \ x, y \in X.$$

Theorem 2.1 *(Nonlinear Alternative of Granas-Frigon, [20]). Let X be a Fréchet space and $Y \subset X$ a closed subset and $N : Y \to X$ be a contraction such that $N(Y)$ is bounded. Then one of the following statements hold:*

(C1) N has a unique fixed point;

(C2) There exists $\lambda \in [0,1)$, $n \in \mathbb{N}$, and $x \in \partial_n Y^n$ such that $\|x - \lambda N(x)\|_n = 0$.

3 Existence Results

In this section, we prove that there is a unique global mild solution for the problem (1.1)-(1.2). We begin introducing the following concept of mild solutions.

Definition 3.1 *An $\mathscr{F}_t$-adapted stochastic process $x : [0, +\infty) \to \mathbb{H}$ is called a mild solution of (1.1)-(1.2) if $x(0) = x_0 \in L_2^0(\Omega, \mathbb{H})$, $x(t)$ is continuous and satisfies the following integral equation*

$$x(t) = R(t)x_0 + \int_0^t R(t-s) f(s, x(s)) dw(s), \ \textit{for each} \ t \in [0, +\infty).$$

Let us list the following assumptions:

(H1) A is the infinitesimal generator of a resolvent operator $R(t)$, $t \ge 0$ in the Hilbert space $\mathbb{H}$ and there exists a constant $M > 0$ such that

$$\|R(t)\|^2 \le M, \quad t \ge 0.$$

(H2) The function $f : J \times \mathbb{H} \to L_Q(\mathbb{K}, \mathbb{H})$ is L^2-Carathéodory and satisfies the following conditions:

(i) There exist a function $p \in L^1_{loc}(J, \mathbb{R}_+)$ and a continuous nondecreasing function $\psi : J \to (0, +\infty)$ such that

$$E\|f(t,u)\|^2 \le p(t)\psi(E\|u\|^2),$$

for a.e. $t \in J$ and each $u \in \mathbb{H}$.

(ii) For all $\mathfrak{R} > 0$, there exists a function $l_{\mathfrak{R}} \in L^1_{loc}(J, \mathbb{R}_+)$ such that

$$E\|f(t,u) - f(t,v)\|^2 \le l_{\mathfrak{R}}(t) E\|u - v\|^2,$$

for all $u, v \in \mathbb{H}$ with $E\|u\|^2 \le \mathfrak{R}$ and $E\|v\|^2 \le \mathfrak{R}$.

Theorem 3.1 *Assume the conditions (H1)-(H2) are satisfied and moreover for each* $n \in \mathbb{N}$

$$\int_{c_n}^{+\infty} \frac{ds}{\psi(s)} > 2Tr(Q)M \int_0^n p(s)ds, \tag{3.1}$$

where $c_n = 2ME\|x_0\|^2$. *Then the problem (1.1)-(1.2) has a unique mild solution on* J.

Proof: Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms

$$\|x\|_n := \sup\{e^{-\tau L_n^*(t)} E\|x(t)\|^2 : t \in [0, n]\},$$

where $L_n^*(t) = \int_0^t \bar{l}_n(s)ds$, and $\bar{l}_n(t) = MTr(Q)l_n(t)$ and l_n is the function from (H2). Then $B_{+\infty}$ is a Fréchet space with the family of semi-norms $\|\cdot\|_{n\in\mathbb{N}}$.

We transform (1.1)-(1.2) into a fixed point problem. Consider the operator $\Gamma : B_{+\infty} \to B_{+\infty}$ defined by

$$\Gamma(x)(t) = R(t)x_0 + \int_0^t R(t-s)f(s, x(s))dw(s), \ t \in J.$$

Clearly fixed points of the operator Γ are mild solutions of the problem (1.1)-(1.2). For convenience, we set for $n \in \mathbb{N}$

$$c_n = 2ME\|x_0\|^2, \ m(t) = 2Tr(Q)Mp(t).$$

Let $x \in B_{+\infty}$ be a possible fixed point of the operator Γ. By the hypotheses (H1) and (H2), we have for each $t \in [0, n]$

$$\begin{aligned} E\|x(t)\|^2 &\leq 2E\|R(t)x_0\|^2 + 2E\left\|\int_0^t R(t-s)f(s, x(s))dw(s)\right\|^2 \\ &\leq 2ME\|x_0\|^2 + 2Tr(Q)M\int_0^t p(s)\psi(E\|x(s)\|^2)ds. \end{aligned}$$

We consider the function u defined by

$$u(t) = \sup\{E\|x(s)\|^2 : 0 \leq s \leq t\}, \ 0 \leq t < +\infty.$$

Let $t^* \in [0, t]$ be such that

$$u(t) = E\|x(t^*)\|^2.$$

By the previous inequality, we have

$$u(t) \leq 2ME\|x_0\|^2 + 2Tr(Q)M\int_0^t p(s)\psi(u(s))ds.$$

Let us take the right-hand side of the above inequality as $v(t)$. Then, we have

$$u(t) \le v(t) \text{ for all } t \in [0,n] \text{ and } v(0) = c_n = 2ME\|x_0\|^2$$

and

$$v'(t) = 2Tr(Q)Mp(t)\psi(u(t)) \text{ a.e. } t \in [0,n].$$

Using the nondecreasing character of ψ, we get

$$v'(t) = 2Tr(Q)Mp(t)\psi(v(t)) \text{ a.e. } t \in [0,n].$$

This implies that for each $t \in [0,n]$, we have

$$\int_{c_n}^{v(t)} \frac{ds}{\psi(s)} \le \int_0^n m(s)ds < \int_{c_n}^{+\infty} \frac{ds}{\psi(s)}.$$

Thus by (3.1), for every $t \in [0,n]$, there exists a constant Λ_n, such that $v(t) \le \Lambda_n$ and hence $u(t) \le \Lambda_n$. Since $\|x\|_n \le u(t)$, we have $\|x\|_n \le \Lambda_n$. Set

$$\mathfrak{X} = \{x \in B_{+\infty} : \sup\{E\|x(t)\|^2 : 0 \le t \le n\} \le \Lambda_n + 1 \text{ for all } n \in \mathbb{N}\}.$$

Clearly, $\mathfrak{X}$ is a closed subset of $B_{+\infty}$.

We shall show that $\Gamma : \mathfrak{X} \to B_{+\infty}$, is a contraction operator. Indeed, consider $x, \overline{x} \in B_{+\infty}$, thus using (H1) and (H2) for each $t \in [0,n]$ and $n \in \mathbb{N}$

$$\begin{aligned} E\|\Gamma(x)(t) - \Gamma(\overline{x})(t)\|^2 &= E\left\|\int_0^t R(t-s)[f(s,x(s)) - f(s,\overline{x}(s))]dw(s)\right\|^2 \\ &\le Tr(Q)M\int_0^t l_n(s)E\|x(s) - \overline{x}(s)\|^2 ds \\ &\le \int_0^t [\overline{l}_n(s)e^{\tau L_n^*(s)}][e^{-\tau L_n^*(s)}E\|x(s) - \overline{x}(s)\|^2]ds \\ &\le \int_0^t [\overline{l}_n(s)e^{\tau L_n^*(s)}]ds\|x - \overline{x}\|_n \\ &\le \int_0^t \frac{1}{\tau}[e^{\tau L_n^*(s)}]' ds\|x - \overline{x}\|_n \\ &\le \frac{1}{\tau}e^{\tau L_n^*(t)}\|x - \overline{x}\|_n. \end{aligned}$$

Therefore

$$\|\Gamma(x) - \Gamma(\overline{x})\|_n \le \frac{1}{\tau}\|x - \overline{x}\|_n.$$

So, the operator Γ is a contraction for all $n \in \mathbb{N}$. From the choice of $\mathfrak{X}$ there is no $x \in \partial\mathfrak{X}^n$ such that $x = \lambda\Gamma(x)$ for some $\lambda \in (0,1)$. Then the statement (C2) in theorem 2.1 does not hold. A consequence of the nonlinear alternative of Frigon and Granas shows that (C1) holds. We deduce that the operator Γ has a unique fixed point x, which is the unique mild solution of the problem (1.1)-(1.2). The proof is completed.

Example 3.1 Consider the following nonlinear stochastic functional differential equations

$$\left\{\begin{array}{l} \dfrac{\partial v}{\partial t}(t,x) = \dfrac{\partial^2}{\partial x^2}\left[v(t,x) + \int_0^t b(t-s)v(s,x)\,ds\right] + k(t,v(t,x))dw(t), t \geq 0, x \in [0,\pi], \\ v(t,0) = v(t,\pi) = 0, t \geq 0, \\ v(0,x) = u_0(x), x \in [0,\pi], \end{array}\right. \tag{3.2}$$

where $w(t)$ denotes a $\mathbb{K}$-valued Brownian motion, $k : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, $u_0(\cdot) \in L^2([0,\pi])$ is $\mathcal{F}_0$-measurable and satisfies $E\|u_0\|^2 < \infty$.

Let $\mathbb{H} = L^2([0,\pi])$ and define $A : \mathbb{H} \to \mathbb{H}$ by $Az = z''$ with domain

$$D(A) = \{z \in \mathbb{H}, z, z' \text{ are absolutely continuous, } z'' \in \mathbb{H}, z(0) = z(\pi) = 0\}.$$

Then A generates a strongly continuous semigroup and resolvent operator $R(t)$ can be extracted from this semigroup [13].

Hence let $f(t,v)(\cdot) = k(t,v(\cdot))$. Then the system (3.2) takes the abstract form as (1.1)-(1.2). Further, we can impose some suitable conditions on the above-defined functions to verify the assumptions on Theorem 3.1, we can conclude that the system (3.2) admits a unique mild solution on J.

4 Controllability Results

As an application of Theorem 3.1, we consider the following controllability for stochastic functional integro-differential evolution equations of the form

$$dx(t) = \left[Ax(t) + \int_0^t B(t-s)x(s)ds\right]dt + Cu(t)dt + f(t,x(t))dw(t),\ t \in J = [0,+\infty), \tag{4.1}$$

$$x(0) = x_0, \tag{4.2}$$

where the control function $u(\cdot)$ is given in $L^2(J,\mathbb{U})$, the Banach space of admissible control functions with $\mathbb{U}$ is real separable Hilbert space with the norm $|\cdot|$, C is a bounded linear operator from $\mathbb{U}$ into $\mathbb{H}$. And the functions $A, B(t-s), f$ and x_0 are as in problem (1.1)-(1.2). For more results on the controllability defined on a compact interval, we refer to [25, 26, 27, 28, 29, 30, 31] and the references therein.

Definition 4.1 *An $\mathscr{F}_t$-adapted stochastic process $x : [0, +\infty) \to \mathbb{H}$ is called a mild solution of the problem (4.1)-(4.2) if $x(0) = x_0 \in L_2^0(\Omega, \mathbb{H})$, $x(t)$ is continuous and satisfies the following integral equation*

$$x(t) = R(t)x_0 + \int_0^t R(t-s)Cu(s)ds + \int_0^t R(t-s)f(s, x(s))dw(s),\ t \in J = [0, +\infty).$$

Definition 4.2 *The system (4.1)-(4.2) is said to controllable if for every initial random variable $x_0 \in L_2^0(\Omega, \mathbb{H})$, $x^* \in \mathbb{H}$, and $n \in \mathbb{N}$, there is some $\mathscr{F}_t$-adapted stochastic control $u \in L^2([0, n], \mathbb{U})$ such that the mild solution $x(\cdot)$ of (4.1)-(4.2) satisfies the terminal condition $x(n) = x^*$.*

We need the following assumption besides the conditions (H1)-(H2):
(H3) For each $n \in \mathbb{N}$, the linear operator $W : L^2([0, n], \mathbb{U}) \to L_2(\Omega, \mathbb{H})$ is defined by

$$Wu = \int_0^n R(n-s)Cu(s)ds,$$

has a pseudo invertible operator $\widetilde{W}^{-1}$ which takes values in $L^2([0, n], \mathbb{U})/KerW$ and there exist positive constants M_1 and M_2 such that

$$\|C\|^2 \le M_1, \quad \text{and} \quad \|\widetilde{W}^{-1}\|^2 \le M_2.$$

Remark 4.1 *For the construction of $\widetilde{W}^{-1}$ see the paper of Quinn and Carmichael [32].*

Theorem 4.1 *Assume the conditions (H1)-(H3) are satisfied and moreover for each $n \in \mathbb{N}$, there exists a constant $\Lambda_n > 0$ such that*

$$\frac{\Lambda_n}{\beta_n + 3Tr(Q)M[3MM_1M_2n^2 + 1]\psi(\Lambda_n)\|p\|_{L^1_{[0,n]}}} > 1, \tag{4.3}$$

with

$$\beta_n = \beta_n(x^*, x_0) = 3M[3MM_1M_2n^2 + 1]E\|x_0\|^2 + 9MM_1M_2n^2E\|x^*\|^2.$$

Then the system (4.1)-(4.2) is controllable on J.

Proof: Let us fix $\tau > 1$. For every $n \in \mathbb{N}$, we define in $B_{+\infty}$ the semi-norms

$$\|x\|_n := \sup\{e^{-\tau L_n^*(t)}E\|x(t)\|^2 : t \in [0, n]\},$$

where $L_n^*(t) = \int_0^t \bar{l}_n(s)ds$, and $\bar{l}_n(t) = 2Tr(Q)Ml_n(t)[MM_1M_2n^2 + 1]$ and l_n is the function from (H2). Then $B_{+\infty}$ is a Fréchet space with the family of semi-norms $\|\cdot\|_{n\in\mathbb{N}}$.

We transform (4.1)-(4.2) into a fixed point problem. Consider the operator $\Xi : B_{+\infty} \to B_{+\infty}$ defined by

$$\Xi(x)(t) = R(t)x_0 + \int_0^t R(t-s)Cu_x(s)ds + \int_0^t R(t-s)f(s,x(s))dw(s),\ t \in J.$$

Using the condition (H3), for arbitrary function $x(\cdot)$, we define the control

$$u_x(t) = \widetilde{W}^{-1}\left[x^* - R(n)x_0 - \int_0^n R(n-s)f(s,x(s))dw(s)\right](t).$$

Noting that, we have

$$E\|u_x(t)\|^2 \leq \|\widetilde{W}^{-1}\|^2 E\left\|x^* - R(t)x_0 - \int_0^n R(n-\tau)f(\tau,x(\tau))dw(\tau)\right\|^2.$$

Applying (H1)-(H3), we get

$$E\|u_x(t)\|^2 \leq 3M_2\left[E\|x^*\|^2 + ME\|x_0\|^2 + Tr(Q)M\int_0^n p(\tau)\psi(E\|x(\tau)\|^2)d\tau\right].$$

We shall show that using this control the operator Ξ has a fixed point $x(\cdot)$. Then $x(\cdot)$ is a mild solution of the system (4.1)-(4.2).

Let $x \in B_{+\infty}$ be a possible fixed point of the operator Ξ. By the conditions (H1)-(H3), we have for each $t \in [0,n]$

$$\begin{aligned}
E\|x(t)\|^2 &\leq 3E\|R(t)x_0\|^2 + 3E\left\|\int_0^t R(t-s)Cu_x(s)ds\right\|^2 \\
&\quad +3E\left\|\int_0^t R(t-s)f(s,x(s))dw(s)\right\|^2 \\
&\leq 3ME\|x_0\|^2 + 3Tr(Q)M\int_0^t p(s)\psi(E\|x(s)\|^2)ds \\
&\quad +9MM_1M_2n\int_0^t\left[E\|x^*\|^2 + ME\|x_0\|^2 + Tr(Q)M\int_0^n p(\tau)\psi(E\|x(\tau)\|^2)d\tau\right]ds \\
&\leq 3ME\|x_0\|^2 + 9MM_1M_2n^2[E\|x^*\|^2 + ME\|x_0\|^2] \\
&\quad +9Tr(Q)M_1M_2M^2n^2\int_0^n p(s)\psi(E\|x(s)\|^2)ds \\
&\quad +3Tr(Q)M\int_0^t p(s)\psi(E\|x(s)\|^2)ds.
\end{aligned}$$

Set

$$\beta_n = 3ME\|x_0\|^2 + 9MM_1M_2n^2[E\|x^*\|^2 + ME\|x_0\|^2].$$

So

$$\begin{aligned} E\|x(t)\|^2 &\le \beta_n + 9Tr(Q)M_1M_2M^2n^2\int_0^n p(s)\psi(E\|x(s)\|^2)ds \\ &\quad +3Tr(Q)M\int_0^t p(s)\psi(E\|x(s)\|^2)ds. \end{aligned}$$

We consider the function μ defined by

$$\mu(t)=\sup\{E\|x(s)\|^2 : 0\le s\le t\},\ 0\le t<+\infty.$$

Let $t^*\in[0,t]$ be such that$\mu(t)=E\|x(t^*)\|^2$. If $t^*\in[0,n]$, by the previous inequality, we have for $t\in[0,n]$

$$\mu(t) \le \beta_n + 9Tr(Q)M_1M_2M^2n^2\int_0^n p(s)\psi(\mu(s))ds + 3Tr(Q)M\int_0^t p(s)\psi(\mu(s))ds.$$

Then, we have

$$\mu(t)\le \beta_n + 3Tr(Q)M[3MM_1M_2n^2+1]\int_0^n p(s)\psi(\mu(s))ds.$$

Consequently,

$$\frac{\|x\|_n}{\beta_n + 3Tr(Q)M[3MM_1M_2n^2+1]\psi(\|x\|_n)\|p\|_{L^1_{[0,n]}}}\le 1.$$

Then by the condition (4.3), there exists Λ_n such that $\mu(t)\le\Lambda_n$. Since $\|x\|_n\le\mu(t)$, we have $\|x\|_n\le\Lambda_n$.

Set

$$\mathfrak{X}=\{x\in B_{+\infty} : \sup\{E\|x(t)\|^2 : 0\le t\le n\}\le\Lambda_n+1 \ \text{ for all } n\in\mathbb{N}\}.$$

Clearly, $\mathfrak{X}$ is a closed subset of $B_{+\infty}$.

We shall show that $\Xi:\mathfrak{X}\to B_{+\infty}$ is a contraction operator. Indeed, consider $x,\overline{x}\in B_{+\infty}$. By (H1)-(H3) for each $t\in[0,n]$ and $n\in\mathbb{N}$

$$\begin{aligned} &E\|\Xi(x)(t)-\Xi(\overline{x})(t)\|^2 \\ \le\ & 2E\left\|\int_0^t R(t-s)C[u_x(s)-u_{\overline{x}}(s)]ds\right\|^2 \\ & +2E\left\|\int_0^t R(t-s)[f(s,x(s))-f(s,\overline{x}(s))]dw(s)\right\|^2 \\ \le\ & 2MM_1n\int_0^t E\left\|\widetilde{W}^{-1}\left[x^*-R(n)x_0-\int_0^n R(n-s)f(\tau,x(\tau))dw(\tau)\right]\right. \end{aligned}$$

$$
\begin{aligned}
& -\widetilde{W}^{-1}\left[x^* - R(n)x_0 - \int_0^n R(n-s)f(\tau,\overline{x}(\tau))dw(\tau)\right]\Big\|^2 ds \\
& +2Tr(Q)M\int_0^t l_n(s)E\|x(s)-\overline{x}(s)\|^2 ds \\
\leq\ & 2MM_1M_2 n\int_0^t Tr(Q)M\int_0^n E\|f(\tau,x(\tau))-f(\tau,\overline{x}(\tau))\|^2_{L_Q(\mathbb{K},\mathbb{H})}d\tau ds \\
& +2Tr(Q)M\int_0^t l_n(s)E\|x(s)-\overline{x}(s)\|^2 ds \\
\leq\ & 2Tr(Q)M_1M_2M^2n^2\int_0^t l_n(s)E\|x(s)-\overline{x}(s)\|^2 ds \\
& +2Tr(Q)M\int_0^t l_n(s)E\|x(s)-\overline{x}(s)\|^2 ds \\
\leq\ & \int_0^t [\bar{l}_n(s)e^{\tau L_n^*(s)}][e^{-\tau L_n^*(s)}E\|x(s)-\overline{x}(s)\|^2]ds \\
\leq\ & \int_0^t [\bar{l}_n(s)e^{\tau L_n^*(s)}]ds\|x-\overline{x}\|_n \\
\leq\ & \int_0^t \frac{1}{\tau}[e^{\tau L_n^*(s)}]'ds\|x-\overline{x}\|_n \\
\leq\ & \frac{1}{\tau}e^{\tau L_n^*(t)}\|x-\overline{x}\|_n.
\end{aligned}
$$

Therefore

$$\|\Xi(x)-\Xi(\overline{x})\|_n \leq \frac{1}{\tau}\|x-\overline{x}\|_n.$$

So, the operator Ξ is a contraction for all $n\in\mathbb{N}$. From the choice of $\mathfrak{X}$ there is no $x\in\partial\mathfrak{X}^n$ such that $x=\lambda\Xi(x)$ for some $\lambda\in(0,1)$. Then the statement (C2) in Theorem 2.1 does not hold. A consequence of the nonlinear alternative of Frigon and Granas shows that (C1) holds. We deduce that the operator Ξ has a unique fixed point x, which is the unique mild solution of the problem (4.1)-(4.2). The proof is completed.

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References

[1] Y. Ren, L. Chen, A note on the neutral stochastic functional differential equation with infinite delay and Poisson jumps in an abstract space, J. Math. Phys. 50 (2009) 082704.

[2] Y. Ren, N. Xia, Existence, uniqueness and stability of the solutions to neutral stochastic functional differential equations with infinite delay, Appl. Math. Comput. 210 (2009), 72-79.

[3] L. Hu, Y. Ren, Existence results for impulsive neutral stochastic functional integrodifferential equations with infinite delay, Acta Appl. Math. (2009), doi:10.1007/s10440-009-9546-x

[4] A. H. Lin, L. Y. Hu, Existence results for impulsive neutral stochastic functional integro-differential inclusions with nonlocal initial conditions, Comp. Math. Appl 59 (2010), 64-73.

[5] A. H. Lin, Y. Ren, N. M. Xia, On neutral impulsive stochastic integro-differential equations with infinite delays via fractional operators, Math. Comput. Modelling 51 (2010) 413-424.

[6] R. Sakthivel, J. Luo, Asymptotic stability of nonlinear impulsive stochastic differential equations, Stat. Probabil. Lett. 79 (2009), 1219-1223.

[7] H. Bao, J. Cao, Existence and uniqueness of solutions to neutral stochastic functional differential equations with infinite delay, Appl. Math. Comput. 215 (2010), 1732-1743.

[8] J. Bao, Z. Zhou, Existence of mild solutions to stochastic neutral partial functional differential equations with non-Lipschitz coefficients, Comp. Math. Appl 59 (2010), 207-214.

[9] J. Luo, T. Taniguchi, Fixed point and stability of stochastic neutral partial differential equations with infinite delays, Stoch. Anal. Appl. 27 (2009), 1163-1173.

[10] R. Grimmer, Resolvent operators for integral equations in a Banach space, Trans. Amer. Math. Soc., 273 (1982), 333-349.

[11] R. Grimmer, A. J. Pritchard, Analytic resolvent operators for integral equations, J. Differential Equations, 50 (1983) 234-259.

[12] R. Ravi Kumar, Nonlocal Cauchy problem for analytic resolvent integrodifferential equations in Banach spaces, Appl. Math. Comput. 204 (2008), 352-362.

[13] J. Liang, J. H. Liu, T. J. Xiao, Nonlocal problems for integrodifferential equations, Dynamics Contin. Discr. Impulsive Sys., Series A, 15 (2008), 815-824.

[14] J. H. Liu, K. Ezzinbi, Non-Autonomous integrodifferential equations with nonlocal conditions, J. Integral Equations, 15 (2003), 79-93.

[15] A. Ouahab, Local and global existence and uniqueness results for impulsive functional differential equations with multiple delay, J. Math. Anal. Appl. 323 (2006) 456-472.

[16] S. Baghli, M. Benchohra, Perturbed functional and neutral functional evolution equations with infinite delay in Fréchet spaces, Electronic J. Differential Equations, 69 (2008), 1-19.

[17] S. Baghli, M. Benchohra, Global uniqueness results for partial functional and neutral functional evolution equations with infinite delay, Differential and Integral Equations, 23 (2010), 31-50.

[18] M. Benchohra, A. Ouahab, Controllability results for functional semilinear differential inclusions in Fréchet spaces, Nonlinear Anal., 61 (2005), 405-423.

[19] S. Baghli, M. Benchohra, Multivalued evolution equations with infinite delay in Fréchet spaces, Electron. J. Qual. Theory Differ. Equ., 33 (2008), 1-24.

[20] M. Frigon, A. Granas, Resultats de type Leray-Schauder pour des contractions sur des espaces de Fréchet, Ann. Sci. Math. Quebec, 22 (1998), 161-168.

[21] G. Da Prato, J. Zabczyk, Stochastic equations in infinite dimensions, Cambridge University Press, Cambridge, 1992.

[22] K. Yosida, Functional Analysis, 6th edn. Springer-Verlag, Berlin, 1980.

[23] A. Pazy, Semigroups of linear operators and applications to partial differential equations, Springer-Verlag, New York, 1983.

[24] K. J. Engel, R. Nagel, One-parameter Semigroups for Linear Evolution Equations, Springer-Verlag, New York, 2000.

[25] Y. K. Chang, J. J. Nieto and W. S. Li, Controllability of semilinear differential systems with nonlocal initial conditions in Banach spaces, J. Optim. Theory Appl. 142 (2009), 267-273.

[26] R. Sakthivel, N. I. Mahmudov and S. G. Lee, Controllability of non-linear impulsive stochastic systems, International Journal of Control, 82 (2009), 801-807.

[27] R. Sakthivel, Approximate controllability of impulsive stochastic evolution equations, Funkcialaj Ekvacioj, 52 (2009), 381-393.

[28] R. Sakthivel, E. R. Anandhi, Approximate controllability of impulsive differential equations with state-dependent delay, International Journal of Control, 83 (2010), 387-393.

[29] R. Sakthivel, Y. Ren and N. I. Mahmudov, Approximate controllability of second order stochastic differential equations with impulsive effects, Modern Physics Letters B, 24 (2010), 1-14.

[30] J. Klamka, J. Wyrwat, Controllability of second-order infinite-dimensional systems, Systems and Control Letters, 57 (2008), 386-391.

[31] J. Klamka, Constrained controllability of semilinear systems with delays, Nonlinear Dynamics, 56 (2009), 169-177.

[32] M. D. Quinn, N. Carmichael, An approach to nonlinear control problems using the fixed point methods, degree theory and pseudo-inverses, Numer. Funct. Anal. Optim., 7 (1984-1985), 197-219.

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