# Non-Simultaneous Blow-Up for a Reaction-Diffusion System with Absorption and Coupled Boundary Flux*

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**Abstract.** This paper deals with non-simultaneous blow-up for a reaction-diffusion system with absorption and nonlinear boundary flux. We establish necessary and sufficient conditions for the occurrence of non-simultaneous blow-up with proper initial data.

**Keywords.** non-simultaneous blow-up; reaction-diffusion system; nonlinear absorption; nonlinear boundary flux; blow-up rate
**Mathematics Subject Classification (2000)**. 35K55; 35B33

## 1 Introduction

In this paper, we study non-simultaneous blow-up for the following reaction-diffusion system

$$\begin{cases} u_t = u_{xx} - a_1 e^{\alpha_1 u}, \quad v_t = v_{xx} - a_2 e^{\beta_1 v}, & (x,t) \in (0,1)\times(0,T), \\ u_x(1,t) = e^{\alpha_2 u(1,t)+pv(1,t)}, \quad v_x(1,t) = e^{qu(1,t)+\beta_2 v(1,t)}, & t\in(0,T), \\ u_x(0,t)=0, \quad v_x(0,t)=0, & t\in(0,T), \\ u(x,0)=u_0(x), \quad v(x,0)=v_0(x), & x\in[0,1], \end{cases} \tag{1.1}$$

where $p, q, a_i > 0$, $\alpha_i, \beta_i \geq 0$, $i = 1, 2$. The initial data satisfy $u_0, v_0 \geq 0$, $u_0', v_0' \geq 0$, $u_0'' - a_1 e^{\alpha_1 u_0}, v_0'' - a_2 e^{\beta_1 v_0} \geq \delta > 0$, as well as the compatibility conditions on $[0,1]$. By comparison principle, it follows that $u_t, v_t > 0$, $u_x, v_x \geq 0$ and $u, v \geq 0$ for $(x,t) \in [0,1] \times [0,T)$.

The reaction-diffusion system (1.1) can be used to describe heat propagations in mixed solid media with nonlinear absorption and nonlinear boundary flux [1-3, 5, 9, 11, 16]. The

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nonlinear Neumann boundary values in (1.1), coupling the two heat equations, represent some cross-boundary flux.

The problem of heat equations

$$u_t = \Delta u,\ v_t = \Delta v, \quad (x,t) \in \Omega \times (0,t), \tag{1.2}$$

coupled via somewhat special nonlinear Neumann boundary conditions

$$\frac{\partial u}{\partial \nu} = v^p, \frac{\partial v}{\partial \nu} = u^q, \quad (x,t) \in \partial\Omega \times (0,T), \tag{1.3}$$

was studied by Deng [7] and Lin and Xie [11], who showed that the solutions globally exist if $pq \le 1$ and may blow up in finite time if $pq > 1$ with the blow-up rates $O\left((T-t)^{-(p+1)/2(pq-1)}\right)$ and $O\left((T-t)^{-(q+1)/2(pq-1)}\right)$. Similarly, the blow-up rates for the corresponding scalar case of (1.2) and (1.3) was shown to be $O\left((T-t)^{-1/2(p-1)}\right)$ in [10].

The system (coupled via a variational boundary flux of exponential type)

$$\begin{array}{ll} u_t = \Delta u, \quad v_t = \Delta v, & (x,t) \in \Omega \times (0,t), \\ \dfrac{\partial u}{\partial \nu} = e^{pv}, \quad \dfrac{\partial v}{\partial \nu} = e^{qu}, & (x,t) \in \partial\Omega \times (0,t), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{array} \tag{1.4}$$

was studied by Deng [7], and has blow-up rates

$$\begin{array}{l} -\dfrac{1}{2q}\log c(T-t) \le \max\limits_{\overline{\Omega}} u(\cdot,t) \le -\dfrac{1}{2q}\log C(T-t), \\ -\dfrac{1}{2p}\log c(T-t) \le \max\limits_{\overline{\Omega}} v(\cdot,t) \le -\dfrac{1}{2p}\log C(T-t), \end{array} \tag{1.5}$$

for $t \in (0,T)$. This is the special case with $\alpha_i = \beta_i = a_i = 0,\ i = 1,2$, in our system (1.1).

Zhao and Zheng [17] studied the following nonlinear parabolic system:

$$\begin{array}{ll} u_t = \Delta u, \quad v_t = \Delta v, & (x,t) \in \Omega \times (0,t), \\ \dfrac{\partial u}{\partial \nu} = e^{\alpha_2 u + pv}, \quad \dfrac{\partial v}{\partial \nu} = e^{qu + \beta_2 v}, & (x,t) \in \partial\Omega \times (0,t), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega. \end{array} \tag{1.6}$$

The blow-up rates for (1.6) were shown to be

$$\max_{\overline{\Omega}} u(\cdot,t) = O\left(\log(T-t)^{\alpha/2}\right), \quad \max_{\overline{\Omega}} v(\cdot,t) = O\left(\log(T-t)^{\beta/2}\right), \tag{1.7}$$

as $t \to T$, where $(\alpha,\beta)^T$ is the only positive solution of

$$\begin{pmatrix} \alpha_2 & p \\ q & \beta_2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$

namely,

$$\alpha = \frac{p-\beta_2}{pq-\alpha_2\beta_2}, \quad \beta = \frac{q-\alpha_2}{pq-\alpha_2\beta_2}.$$

Clearly the blow-up rate estimate (1.5) is just the special case of (1.7) with $\alpha_2 = \beta_2 = 0$.

The phenomenon of non-simultaneous blow-up is researched extensively [see 4, 13-15]. Recently Zheng and Qiao [20] consider the non-simultaneous blow-up phenomenon of following reaction-diffusion problem

$$\begin{array}{ll} u_t = u_{xx} - \lambda_1 u^{\alpha_1}, \quad v_t = v_{xx} - \lambda_2 v^{\beta_1}, & (x,t) \in (0,1) \times (0,T), \\ u_x(1,t) = u^{\alpha_2} v^p, \quad v_x(1,t) = u^q v^{\beta_2}, & t \in (0,T), \\ u_x(0,t) = 0, \quad v_x(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in [0,1], \end{array} \tag{1.8}$$

and they get the following conclusions:

(1) If $q < \alpha_2 - 1$ with either $\alpha_2 > \mu$ or $\alpha_2 = \mu > 1$, then there exists initial data $(u_0, v_0)$ such that $u$ blows up at a finite time $T$ while $v$ remains bounded.

(2) If $u$ blows up at time $T$ and $v$ remains bounded up to that time, then $q < \alpha_2 - 1$ with either $\alpha_2 > \mu$ or $\alpha_2 = \mu > 1$.

(3) Under the condition of (1), if in addition either (i) $\beta_2 \leq 1$, or (ii) $1 < \beta_2 < \gamma$ and $q < \frac{(\alpha_2-1)(\gamma-\beta_2)}{\gamma-1}$ hold, then any blow-up must be non-simultaneous, namely, $u$ blows up at a finite time $T$ while $v$ remains bounded.

The critical exponents for the system (1.1) were studied in [18] by Zheng and Li, where the following characteristic algebraic system was introduced:

$$\begin{pmatrix} \alpha_2 - \frac{1}{2}\alpha_1 & p \\ q & \beta_2 - \frac{1}{2}\beta_1 \end{pmatrix} \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \tag{1.9}$$

namely,

$$\tau_1 = \frac{p + \frac{1}{2}\beta_1 - \beta_2}{pq - (\frac{1}{2}\alpha_1 - \alpha_2)(\frac{1}{2}\beta_1 - \beta_2)}, \quad \tau_2 = \frac{q + \frac{1}{2}\alpha_1 - \alpha_2}{pq - (\frac{1}{2}\alpha_1 - \alpha_2)(\frac{1}{2}\beta_1 - \beta_2)}. \tag{1.10}$$

To state their main result, first we give some information about eigenfunction for Laplace's equation.

Let $\varphi_0$ be the first eigenfunction of

$$\varphi'' + \lambda\varphi = 0 \;\; in \;\; (-1,1); \quad \varphi(-1) = \varphi(1) = 0, \tag{1.11}$$

with the first eigenvalue $\lambda_0$, normalized by $\|\varphi_0\|_\infty = 1$. It is well know that [6] $\varphi_0 > 0$ in $(-1,1)$, and there are positive constants $c_i$ (i=1,2,3,4) and $\varepsilon_0$ such that

$$\begin{array}{l} c_1 \leq \varphi_0'(-1), -\varphi_0'(1) \leq c_2 \leq \max\limits_{[-1,1]} |\varphi_0'| = c_4, \\ |\varphi_0'| \geq \dfrac{c_1}{2} \;\; on \; \{x \in (-1,1) : \; dist(x,-1) \leq \varepsilon_0\} \cup \{x \in (-1,1) : \; dist(x,1) \leq \varepsilon_0\}, \\ \varphi_0 \geq c_3 \;\; on \; \{x \in (-1,1) : \; dist(x,-1) \geq \varepsilon_0\} \cap \{x \in (-1,1) : \; dist(x,1) \geq \varepsilon_0\}. \end{array} \tag{1.12}$$

Now we can state the main result of [18]

**Proposition 1.1** *(1) If* $1/\tau_1 > 0$ *or* $1/\tau_2 > 0$*, then the solutions of (1.1) blow up in finite time with large initial data.*

*(2) If* $1/\tau_i < 0$*, i=1,2, then the solutions of (1.1) are globally bounded.*

*(3) Assume that* $1/\tau_1 = 1/\tau_2 = 0$*.*

*(i) If* $\alpha_2 > \frac{1}{2}\alpha_1$ *and* $\beta_2 > \frac{1}{2}\beta_1$*, then the solutions of (1.1) blow up in finite time with large initial data.*

*(ii) If* $a_1 \geq 2^{\alpha_1}\left(\frac{\lambda_0}{c_1} + \frac{3c_4^2}{c_1^2}\right)$*,* $a_2 \geq 2^{\beta_1}\left(\frac{\lambda_0}{c_1} + \frac{3c_4^2}{c_1^2}\right)$ *with* $\alpha_2 < \frac{1}{2}\alpha_1$*,* $\beta_2 < \frac{1}{2}\beta_1$*, then the solutions of (1.1) are globally bounded.*

*(iii) If* $a_1 \leq \min\left\{\frac{c_1^2 M^2}{4\alpha_1}, \frac{\lambda_0 c_3^2 M^2}{\alpha_1}\right\}$*,* $a_2 \leq \min\left\{\frac{c_1^2 M^2}{4\beta_1}, \frac{\lambda_0 c_3^2 M^2}{\beta_1}\right\}$ *with* $\alpha_2 < \frac{1}{2}\alpha_1$*,* $\beta_2 < \frac{1}{2}\beta_1$*,* $M = \min\{\alpha_1/(2c_2), \beta_1/(2c_2)\}$*, then the solutions of (1.1) blow up in finite time for large initial data.*

Intrigued by [18-20], we consider the non-simultaneous blow-up of (1.1). The main results of this paper are the following two Theorems for non-simultaneous blow-up. Without loss of generality, we only deal with the case where $u$ blows up while $v$ remains bounded.

**Theorem 1.1** *If* $q < \alpha_2$ *with either* $2\alpha_2 > \alpha_1$ *or* $2\alpha_2 = \alpha_1$*,* $a_1 \leq \frac{\alpha_1}{4}\min\{c_1^2/(4c_2^2), \lambda_0 c_3^2/c_2^2\}$*, then there exists initial data* $(u_0, v_0)$ *such that* $u$ *blows up at a finite time* $T$ *while* $v$ *remains bounded up to that time.*

**Theorem 1.2** *If* $u$ *blows up at a finite time* $T$ *while* $v$ *remains bounded up to that time, then* $q < \alpha_2$ *with either* $2\alpha_2 > \alpha_1$ *or* $2\alpha_2 = \alpha_1$*,* $a_1 \leq \frac{\alpha_1}{4}\min\{c_1^2/(4c_2^2), \lambda_0 c_3^2/c_2^2\}$*.*

We will prove Theorem 1.1 and 1.2 in the next two sections.

## 2 Proof of Theorem 1.1

At first, we consider the scalar problem of the form

$$\begin{array}{ll} u_t = u_{xx} - a_1 e^{\alpha_1 u}, & (x,t) \in (0,1)\times(0,T), \\ u_x(1,t) = e^{\alpha_2 u(1,t)} e^{ph(t)},\ u_x(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,1], \end{array} \tag{2.1}$$

with $a_1, \alpha_i$ in (1.1), i=1,2 and $h(t)$ continuous, non-decreasing, $0 \leq h(t) \leq K$. Similarly to Theorem 3.2 in [19], we can prove the following Lemma, where and in the sequel $C$ is used to represent positive constants independent of $t$, and may change from line to line.

**Lemma 2.1** *Let* $u$ *be a solution of (2.1). Assume (i)* $2\alpha_2 > \alpha_1$ *or (ii)* $2\alpha_2 = \alpha_1$ *with* $a_1 \leq \frac{\alpha_1}{4}\min\{c_1^2/(4c_2^2), \lambda_0 c_3^2/c_2^2\}$*. Then* $u$ *blows up in a finite time* $T$ *for sufficiently large* *initial value, and moreover*

$$u(1,t)=\max_{[0,1]}u(\cdot,t)\leq \log C(T-t)^{-\frac{1}{2\alpha_2}},\quad 0<t<T. \tag{2.2}$$

**Proof.** Let $w$ slove

$$\begin{array}{ll} w_t=w_{xx}-a_1e^{\alpha_1 w}, & (x,t)\in(0,1)\times(0,T),\\ w_x(1,t)=e^{\alpha_2 w(1,t)},\ w_x(0,t)=0, & t\in(0,T),\\ w(x,0)=u_0(x), & x\in[0,1]. \end{array} \tag{2.3}$$

Then, $w\leq u$ in $(0,1)\times[0,T)$ by the comparison principle. Notice that the two assumptions $2\alpha_2>\alpha_1$ or $2\alpha_2=\alpha_1$ with $a_1\leq\frac{\alpha_1}{4}\min\{c_1^2/(4c_2^2),\lambda_0c_3^2/c_2^2\}$ are corresponding to the blow-up conditions by Proposition 1.1 with $\alpha_1=\beta_1,\ \alpha_2=\beta_2,\ p=q=0$ and $u_0=v_0,\ a_1=a_2$ in (1.1). So there exists initial data such that $w$ blows up in finite time $T'$. Then $u$ blows up in finite time $t=T$.

To establish the desired blow-up rate, we exploit the method used in [19]. From the assumptions on initial data, we know that $w_t>0$ and $w_x\geq 0$ for $(x,t)\in[0,1)\times[0,T)$. Set $J(x,t)=\sqrt{w_t}-\varepsilon w_x$ for $(x,t)\in(0,1)\times[0,T)$. Let $\varepsilon$ be sufficiently small such that

$$J(x,0)=\sqrt{w_t(x,0)}-\varepsilon w_x(x,0)\geq 0,\quad x\in[0,1], \tag{2.4}$$

a simple computation yields

$$\begin{aligned} &J_x(1,t)-\left[\left(\frac{1}{2}\alpha_2e^{\alpha_2 w}-\varepsilon w_t^{\frac{1}{2}}-\varepsilon^2e^{\alpha_2 w}\right)J\right](1,t)\\ &=\varepsilon\left(\frac{1}{2}\alpha_2e^{2\alpha_2 w}-a_1e^{\alpha_1 w}-\varepsilon^2e^{2\alpha_2 w}\right)(1,t)\geq 0,\quad t\in(0,T), \end{aligned} \tag{2.5}$$

when (i) $2\alpha_2>\alpha_1$ or (ii) $2\alpha_2=\alpha_1$ with $a_1\leq\frac{\alpha_1}{4}\min\{c_1^2/(4c_2^2),\lambda_0c_3^2/c_2^2\}$.

For $(x,t)\in(0,1)\times[0,T)$, a simple computation shows

$$J_t-J_{xx}+\frac{1}{2}a_1\alpha_1e^{\alpha_1 w}J=\frac{1}{4}w_t^{-\frac{3}{2}}w_{tx}^2+\frac{1}{2}\varepsilon a_1\alpha_1e^{\alpha_1 w}w_x\geq 0. \tag{2.6}$$

By the comparison principle [12], we have $J\geq 0$ and hence

$$w_t(1,t)\geq\varepsilon^2w_x^2(1,t)=\varepsilon^2e^{2\alpha_2 w(1,t)},\quad t\in[0,T). \tag{2.7}$$

Integrating (2.7) from $t$ to $T$, we get (2.2) immediately. □

**Proof of Theorem 1.1.** It suffices to choose initial data $(u_0,v_0)$ such that $u$ blows up while $v$ remains bounded. At first, fix $v_0\geq 0$ and take $K=\max_{[0,1]}v_0=v_0(1)$, $N=\frac{1}{K}e^{2\beta_2K}+3$. Thus, $w(x,t)$ which solves (2.3) is a subsolution of $u$. Since Proposition 1.1, there exists initial data $u_0$ such that $w$ blows up at a finite time $T'$. Now, for the fixed $v_0$, retake $u_0(x)=w(x,T'-\varepsilon)$, the $u$ blows up in a finite time $T\leq\varepsilon$.

If $v$ remain bounded by $v < 2K$ for $t \in [0, T]$, the proof is complete.

Otherwise, Let $t_0$ be the first time such that $\max_{[0,1]} v(\cdot, t_0) = v(1, t_0) = 2K$. Now, we introduce the following cut-off function:

$$\widetilde{v}(x,t) = \begin{cases} v(x,t), & (x,t) \in [0,1] \times [0,t_0], \\ 2K, & (x,t) \in [0,1] \times [t_0, T]. \end{cases} \tag{2.8}$$

Corresponding, let $\widetilde{u}(x,t)$ solve

$$\begin{array}{ll} \widetilde{u}_t = \widetilde{u}_{xx} - a_1 e^{\alpha_1 \widetilde{u}}, & (x,t) \in (0,1) \times (0, \widetilde{T}), \\ \widetilde{u}_x(1,t) = e^{\alpha_2 \widetilde{u}(1,t)} e^{p \widetilde{v}(1,t)},\ \widetilde{u}_x(0,t) = 0, & t \in (0, \widetilde{T}), \\ \widetilde{u}(x,0) = u_0(x), & x \in [0,1], \end{array} \tag{2.9}$$

where $\widetilde{T}$ is the blow-up time of $\widetilde{u}$ satisfying $\widetilde{T} \geq T$. By Lemma 2.1,

$$\widetilde{u}(1,t) = \max_{[0,1]} \widetilde{u}(\cdot, t) \leq \log C(\widetilde{T} - t)^{-\frac{1}{2\alpha_2}}, \quad 0 < t < \widetilde{T}. \tag{2.10}$$

Therefore,

$$u(1,t) = \widetilde{u}(1,t) \leq \log C(\widetilde{T} - t)^{-\frac{1}{2\alpha_2}} \leq \log C(T-t)^{-\frac{1}{2\alpha_2}}, \quad 0 < t \leq t_0. \tag{2.11}$$

Let $\Gamma(x,t)$ be the fundamental solution of the heat equation in $[0,1]$, namely

$$\Gamma(x,t) = \frac{1}{2\sqrt{\pi t}} \exp\left\{ \frac{-x^2}{4t} \right\}. \tag{2.12}$$

It is know that $\Gamma$ satisfies (see [8])

$$\begin{array}{l} \displaystyle\int_0^1 \Gamma(x-y, t-z) dy \leq 1, \\ \displaystyle\int_z^t \Gamma(1, t-\tau) \frac{1}{2(t-\tau)} d\tau \leq C^* \sqrt{t-z}, \int_z^t \Gamma(0, t-\tau) d\tau = \frac{1}{\sqrt{\pi}} \sqrt{t-z}, \\ \displaystyle\frac{\partial \Gamma}{\partial \nu_y}(x-y, t-\tau) = \frac{x-y}{2(t-\tau)} \Gamma(x-y, t-\tau), \quad x, y \in [0,1], \quad 0 \leq z < t. \end{array} \tag{2.13}$$

By the Greeen's identity with (1.1) for $v$,

$$\begin{aligned} v(x,t) = & \int_0^1 \Gamma(x-y, t-z) v(y,z) dy + \int_z^t \int_0^1 \Gamma(x-y, t-\tau) \left(-a_2 e^{\beta_1 v(y,\tau)}\right) dy d\tau \\ & + \int_z^t \frac{\partial v}{\partial x}(1,\tau) \Gamma(x-1, t-\tau) d\tau - \int_z^t \frac{\partial \Gamma}{\partial \nu_y}(x-1, t-\tau) v(1,\tau) d\tau \\ & + \int_z^t \frac{\partial \Gamma}{\partial \nu_y}(x, t-\tau) v(0,\tau) d\tau, \end{aligned} \tag{2.14}$$

where $0 \leq z < t < T,\ 0 < x < 1$. With $z = 0$ and $x \to 1$, it follows that

$$\begin{aligned} v(x,t) = & \int_0^1 \Gamma(1-y, t) v(y,0) dy + \int_0^t \int_0^1 \Gamma(x-y, t-\tau) \left(-a_2 e^{\beta_1 v(y,\tau)}\right) dy d\tau \\ & + \int_0^t e^{qu(1,\tau) + \beta_2 v(1,\tau)} \Gamma(0, t-\tau) d\tau + \int_0^t v(0,\tau) \Gamma(1, t-\tau) \frac{1}{2(t-\tau)} d\tau. \end{aligned} \tag{2.15}$$

By (2.11), we have furthermore

$$v(1,t_0) \le v_0(1) + C_0 e^{\beta_2 v(1,t_0)} \int_0^{t_0} (t_0-\tau)^{-\frac{q}{2\alpha_2}-\frac{1}{2}} d\tau + C^* \sqrt{t_0}\, v(1,t_0). \tag{2.16}$$

Since $q < \alpha_2$, the integral term in (2.16) is smaller than $1/(NC_0)$ with $\sqrt{t_0} \le \sqrt{T} \le 1/(NC^*)$ if we choose $u_0$ large to make $T$ sufficiently small. This yields

$$\frac{N-1}{N} v(1,t_0) \le v_0(1) + \frac{1}{N} e^{\beta_2 v(1,t_0)}. \tag{2.17}$$

Consequently,

$$\frac{2(N-1)}{N} K \le K + \frac{1}{N} e^{2\beta_2 K}, \tag{2.18}$$

and hence

$$N \le \frac{1}{K} e^{2\beta_2 K} + 2, \tag{2.19}$$

a contradiction. □

# 3 Proof of Theorem 1.2

We begin with a Lemma to prove Theorem 1.2.

**Lemma 3.1** *Let $u$ be a solution of*

$$\begin{array}{ll} u_t = u_{xx} - a_1 e^{\alpha_1 u}, & (x,t) \in (0,1)\times(0,T), \\ u_x(1,t) \le L e^{\alpha_2 u(1,t)}, \ u_x(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,1], \end{array} \tag{3.1}$$

*where $a_1 > 0$, $\alpha_i \ge 0$, i=1,2 and $L$ is a positive constant. If $u$ blows up at a finite time, then either $2\alpha_2 > \alpha_1$ or $2\alpha_2 = \alpha_1$, $a_1 \le \frac{\alpha_1}{4}\min\{c_1^2/(4c_2^2), \lambda_0 c_3^2/c_2^2\}$. Furthermore,*

$$u(1,t) = \max_{[0,1]} u(\cdot,t) \ge \log C(T-t)^{-\frac{1}{2\alpha_2}}, \quad \text{as } t \to T. \tag{3.2}$$

**Proof.** The blow-up of $u$ implies either $2\alpha_2 > \alpha_1$ or $2\alpha_2 = \alpha_1$, $a_1 \le \frac{\alpha_1}{4}\min\{c_1^2/(4c_2^2), \lambda_0 c_3^2/c_2^2\}$ by Proposition 1.1.

By the Green's identity, similarly to (2.14)

$$\begin{aligned} u(x,t) \le& \int_0^1 \Gamma(x-y,t-z)u(y,z)dy + L\int_z^t e^{\alpha_2 u(1,\tau)}\Gamma(x-1,t-\tau)d\tau \\ &- \int_z^t \frac{\partial\Gamma}{\partial\nu_y}(x-1,t-\tau)u(1,\tau)d\tau + \int_z^t \frac{\partial\Gamma}{\partial\nu_y}(x,t-\tau)u(0,\tau)d\tau, \end{aligned} \tag{3.3}$$

where $0 < z < t < T$, $0 < x < 1$. Let $x \to 1$ with the jumping relations to obtain

$$\begin{aligned} \frac{1}{2}u(1,t) \le & \int_0^1 \Gamma(1-y,t-z)u(y,z)dy + L\int_z^t e^{\alpha_2 u(1,\tau)}\Gamma(0,t-\tau)d\tau \\ & + \int_z^t \frac{\partial\Gamma}{\partial\nu_y}(1,t-\tau)u(0,\tau)d\tau \\ \le & \; u(1,z) + \frac{L}{\sqrt{\pi}}\sqrt{T-z}e^{\alpha_2 u(1,t)} + C^*\sqrt{T-z}u(1,t). \end{aligned} \tag{3.4}$$

For any $z \in (0,T)$ with $C^*\sqrt{T-z} \le 1/4$, choose $t \in (z,T)$ such that $\frac{1}{4}u(1,t) - u(1,z) \ge C_0 > 0$. Then

$$C_0 \le \frac{L}{\sqrt{\pi}}\sqrt{T-t}e^{\alpha_2 u(1,t)}, \tag{3.5}$$

which implies (3.2). □

**Proof of Theorem 1.2.** Since $v \le K$ for $(x,t) \in [0,1] \times [0,T)$, we have

$$\begin{array}{ll} u_t = u_{xx} - a_1 e^{\alpha_1 u}, & (x,t) \in (0,1) \times (0,T), \\ u_x(1,t) \le e^{pK}e^{\alpha_2 u(1,t)}, \; u_x(0,t) = 0, & t \in (0,T), \\ u(x,0) = u_0(x), & x \in [0,1]. \end{array} \tag{3.6}$$

Then, we obtain from Lemma 3.1 that $2\alpha_2 > \alpha_1$ or $2\alpha_2 = \alpha_1$ with $a_1 \le \frac{\alpha_1}{4}\min\left\{\frac{c_1^2}{4c_2^2}, \frac{\lambda_0 c_3^2}{c_2^2}\right\}$, and moreover,

$$u(1,t) = \max_{[0,1]} u(\cdot,t) \ge \log C(T-t)^{-\frac{1}{2\alpha_2}}, \quad as\ t \to T. \tag{3.7}$$

Next, let us show $q < \alpha_2$. Due to (2.14), we have by letting $x \to 1$ that

$$v(1,t) \ge \int_z^t e^{qu(1,\tau)+\beta_2 v(1,\tau)}\Gamma(0,t-\tau)d\tau - a_2\int_z^t\int_0^1 \Gamma(1-y,t-\tau)e^{\beta_1 v(y,\tau)}dyd\tau, \tag{3.8}$$

and so,

$$v(1,t) \ge C_1\int_z^t (T-\tau)^{-\frac{q}{2\alpha_2}-\frac{1}{2}}d\tau - a_2 e^{\beta_1 v(1,\tau)}. \tag{3.9}$$

The boundedness of $v(1,t)$ as $t \to T$ requires that $q < \alpha_2$. □

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