

# Multiple positive solutions for a nonlinear $2n$ -th order $m$ -point boundary value problems <sup>\*†</sup>

Youyu Wang <sup>‡</sup>    Yantao Tang    Meng Zhao

Department of Mathematics, Tianjin University of Finance and Economics

Tianjin 300222, P. R. China

**Abstract** In this paper, we consider the existence of multiple positive solutions for the  $2n$ -th order  $m$ -point boundary value problems:

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x''(t), \dots, x^{(2(n-1))}(t)), & 0 \leq t \leq 1, \\ x^{(2i+1)}(0) = \sum_{j=1}^{m-2} \alpha_{ij} x^{(2i+1)}(\xi_j), \quad x^{(2i)}(1) = \sum_{j=1}^{m-2} \beta_{ij} x^{(2i)}(\xi_j), & 0 \leq i \leq n-1, \end{cases}$$

where  $\alpha_{ij}, \beta_{ij}$  ( $0 \leq i \leq n-1, 1 \leq j \leq m-2$ )  $\in [0, \infty)$ ,  $\sum_{j=1}^{m-2} \alpha_{ij}, \sum_{j=1}^{m-2} \beta_{ij} \in (0, 1)$ ,  $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$ . Using Leggett-Williams fixed point theorem, we provide sufficient conditions for the existence of at least three positive solutions to the above boundary value problem.

**Keywords** Higher order  $m$ -point boundary value problem, Leggett-Williams fixed point theorem, Green's function, Positive solution.

## 1. Introduction

The multi-point boundary value problems for ordinary differential equations arises in a variety of different areas of applied mathematics and physics. Linear and nonlinear second order multi-point boundary value problems have also been studied by several authors. We refer the reader to

---

\*This research was supported by grants from Tianjin Municipal Education Commission (20081005)

†2000 Mathematics Subject Classification 34B10, 34B15

‡E-mail: wang\_youyu@163.com

[2-8] and references therein. Davis et al. [9,10] studied the following 2n-th Lidstone BVP

$$\begin{cases} x^{(2n)} = f(x(t), x''(t), \dots, x^{(2(n-1))}(t)), & t \in [0, 1], \\ x^{(2i)}(0) = x^{(2i)}(1) = 0, & 0 \leq i \leq n-1, \end{cases} \quad (1)$$

where  $(-1)^n f : R^n \rightarrow [0, \infty)$  is continuous. They obtained the existence of three symmetric positive solutions of the BVP (1).

Y. Guo et al. [11] studied the following 2n-th BVP

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x''(t), \dots, x^{(2(n-1))}(t)), & 0 \leq t \leq 1, \\ x^{(2i)}(0) - \beta_i x^{(2i+1)}(0) = 0, \quad x^{(2i)}(1) = \sum_{j=1}^{m-2} k_{ij} y^{(2i)}(\xi_j), & 0 \leq i \leq n-1. \end{cases} \quad (2)$$

They obtained the existence of at least two positive solution for the above BVP.

Recently, Y. Guo et al. [13] studied the following 2n-th BVP

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x''(t), \dots, x^{(2(n-1))}(t)), & 0 \leq t \leq 1, \\ x^{(2i)}(0) = 0, \quad x^{(2i)}(1) = \sum_{j=1}^{m-2} k_{ij} y^{(2i)}(\xi_j), & 0 \leq i \leq n-1. \end{cases} \quad (3)$$

By using Leggett-Williams fixed point theorem, they got at least three positive solutions for the BVP(3).

The authors [14,15] investigated the following two BVPs

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x''(t), \dots, x^{(2(n-1))}(t)), & 0 \leq t \leq 1, \\ x^{(2i)}(0) = \sum_{j=1}^{m-2} \alpha_{ij} x^{(2i)}(\xi_j), \quad x^{(2i)}(1) = \sum_{j=1}^{m-2} \beta_{ij} x^{(2i)}(\xi_j), & 0 \leq i \leq n-1, \end{cases} \quad (4)$$

and

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x''(t), \dots, x^{(2(n-1))}(t)), & 0 \leq t \leq 1, \\ x^{(2i)}(0) - a_i x^{(2i+1)}(0) = \sum_{j=1}^{m-2} \alpha_{ij} x^{(2i)}(\xi_j), & \\ x^{(2i)}(1) + b_i x^{(2i+1)}(1) = \sum_{j=1}^{m-2} \beta_{ij} x^{(2i)}(\xi_j), & 0 \leq i \leq n-1, \end{cases} \quad (5)$$

Motivated by the above results, in this paper, we study the existence of multiple positive solutions for the following 2n-th order  $m$ -point boundary value problem

$$\begin{cases} x^{(2n)}(t) = f(t, x(t), x''(t), \dots, x^{(2(n-1))}(t)), & 0 \leq t \leq 1, \\ x^{(2i+1)}(0) = \sum_{j=1}^{m-2} \alpha_{ij} x^{(2i+1)}(\xi_j), \quad x^{(2i)}(1) = \sum_{j=1}^{m-2} \beta_{ij} x^{(2i)}(\xi_j), & 0 \leq i \leq n-1, \end{cases} \quad (6)$$

To the best of our knowledge, existence results for positive solutions of above boundary value problems have not been studied previously.

Throughout the paper, we assume the following conditions satisfied:

- (H<sub>1</sub>)  $\alpha_{ij}, \beta_{ij}$  ( $0 \leq i \leq n-1, 1 \leq j \leq m-2$ )  $\in [0, \infty)$ ,  $\sum_{j=1}^{m-2} \alpha_{ij}, \sum_{j=1}^{m-2} \beta_{ij} \in (0, 1)$ , and  $0 < \xi_1 < \xi_2 < \dots < \xi_{m-2} < 1$ ;
- (H<sub>2</sub>)  $(-1)^n f : [0, 1] \times R^n \rightarrow [0, \infty)$  is continuous;

## 2. Preliminaries

Our main results will depend on the Leggett-Williams fixed point theorem. For convenience, we present here the necessary definitions from the theory of cones in Banach spaces.

**Definition 2.1** Let  $E$  be a real Banach space. A nonempty convex closed set  $P \subset E$  is said to be a cone provided that

- (i)  $au \in P$  for all  $u \in P$  and all  $a \geq 0$  and
- (ii)  $u, -u \in P$  implies  $u = 0$ .

Note that every cone  $P \subset E$  induces an ordering in  $E$  given by  $x \leq y$  if  $y - x \in P$ .

**Definition 2.2** The map  $\alpha$  is said to be a nonnegative continuous **concave** functional on a cone  $P$  of a real Banach space  $E$  provided that  $\alpha : P \rightarrow [0, \infty)$  is continuous and

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all  $x, y \in P$  and  $0 \leq t \leq 1$ .

Similarly, we say the map  $\beta$  is a nonnegative continuous **convex** functional on a cone  $P$  of a real Banach space  $E$  provided that  $\beta : P \rightarrow [0, \infty)$  is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all  $x, y \in P$  and  $0 \leq t \leq 1$ .

**Definition 2.3** An operator is called completely continuous if it is continuous and maps bounded sets into pre-compact sets.

For positive real numbers  $a, b$ , we define the following convex sets:

$$P_r = \{x \in P \mid \|x\| < r\},$$

$$P(\alpha, a, b) = \{x \in P \mid a \leq \alpha(x), \|x\| \leq b\},$$

**Theorem 2.1 [1]** (Leggett-Williams Fixed Point Theorem) Let  $A : \overline{P}_c \rightarrow \overline{P}_c$  be a completely continuous operators and let  $\alpha$  be a nonnegative continuous concave function on  $P$  such that  $\alpha(x) \leq \|x\|$  for all  $x \in \overline{P}_c$ . Suppose there exists  $0 < a < b < d \leq c$  such that

$$(C1) \{x \in P(\alpha, b, d) \mid \alpha(x) > b\} \neq \emptyset \quad \text{and} \quad \alpha(Ax) > b \quad \text{for } x \in P(\alpha, b, d),$$

$$(C2) \|Ax\| < a \quad \text{for } \|x\| \leq a, \quad \text{and}$$

$$(C3) \alpha(Ax) > b \quad \text{for } x \in P(\alpha, b, c) \quad \text{with } \|Ax\| > d.$$

Then  $A$  has at least three fixed points  $x_1, x_2$  and  $x_3$  such that  $\|x_1\| < a$ ,  $b < \alpha(x_2)$ , and  $\|x_3\| > a$  with  $\alpha(x_3) < b$ .

### 3. Multiple positive solutions of (6)

In order to apply Theorem 2.1, we must define an appropriate operator on a Banach space. We first consider the the unique solution of the following second order boundary value problem:

**Lemma 3.1[12]** Let  $(1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \sum_{i=1}^{m-2} \beta_i) \neq 0$ . Then for  $f(t) \in C[0, 1]$ , the problem

$$\begin{cases} x''(t) + f(t) = 0, & 0 \leq t \leq 1 \\ x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i), & x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases} \quad (7)$$

has a unique solution

$$x(t) = - \int_0^t (t-s)f(s)ds + At + B,$$

where

$$\begin{aligned} A &= - \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \left( \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} f(s)ds \right), \\ B &= \frac{1}{1 - \sum_{i=1}^{m-2} \beta_i} \left[ \int_0^1 (1-s)f(s)ds - \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} (\xi_i - s)f(s)ds \right. \\ &\quad \left. + \frac{1 - \sum_{i=1}^{m-2} \beta_i \xi_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \left( \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} f(s)ds \right) \right]. \end{aligned}$$

**Lemma 3.2[12]** Suppose  $\alpha_i, \beta_i > 0$  ( $i = 1, 2, \dots, m-2$ ),  $0 < \sum_{i=1}^{m-2} \alpha_i < 1$ ,  $0 < \sum_{i=1}^{m-2} \beta_i < 1$ .

If  $f(t) \in C[0, 1]$  and  $f \geq 0$ , then the unique solution of (7) satisfies

$$\inf_{t \in [0, 1]} x(t) \geq \gamma \|x\|,$$

where

$$\gamma = \frac{\sum_{i=1}^{m-2} \beta_i (1 - \xi_i)}{1 - \sum_{i=1}^{m-2} \beta_i \xi_i}.$$

**Lemma 3.3** Suppose  $\alpha_i, \beta_i > 0$  ( $i = 1, 2, \dots, m-2$ ),  $0 < \sum_{i=1}^{m-2} \alpha_i < 1$ ,  $0 < \sum_{i=1}^{m-2} \beta_i < 1$ , and let  $M = (1 - \sum_{i=1}^{m-2} \alpha_i)(1 - \sum_{i=1}^{m-2} \beta_i)$ . Then the Green's function for the boundary value problem

$$\begin{cases} -x''(t) = 0, & 0 \leq t \leq 1, \\ x'(0) = \sum_{i=1}^{m-2} \alpha_i x'(\xi_i), & x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \end{cases}$$

is given by

$$G^*(t, s) = \frac{1}{M} \begin{cases} (1 - \sum_{j=1}^{m-2} \beta_j \xi_j) - t(1 - \sum_{j=1}^{m-2} \beta_j), \\ 0 \leq t \leq 1, \quad 0 \leq s \leq \xi_1, \quad s \leq t; \\ \sum_{j=1}^{m-2} \alpha_j \left[ (1 - \sum_{j=1}^{m-2} \beta_j \xi_j) - t(1 - \sum_{j=1}^{m-2} \beta_j) \right] \\ + (1 - \sum_{j=1}^{m-2} \alpha_j) \left[ (1 - \sum_{j=1}^{m-2} \beta_j \xi_j) - s(1 - \sum_{j=1}^{m-2} \beta_j) \right], \\ 0 \leq t \leq 1, \quad 0 \leq s \leq \xi_1, \quad t \leq s; \\ \sum_{j=i}^{m-2} \alpha_j \left[ (1 - \sum_{j=1}^{m-2} \beta_j \xi_j) - t(1 - \sum_{j=1}^{m-2} \beta_j) \right] \\ + (1 - \sum_{j=1}^{m-2} \alpha_j) \left[ (1 - \sum_{j=i}^{m-2} \beta_j \xi_j) - s(1 - \sum_{j=i}^{m-2} \beta_j) \right], \\ \xi_{i-1} \leq s \leq \xi_i, \quad 2 \leq i \leq m-2, \quad t \leq s; \\ -M(t-s) + \sum_{j=i}^{m-2} \alpha_j \left[ (1 - \sum_{j=1}^{m-2} \beta_j \xi_j) - t(1 - \sum_{j=1}^{m-2} \beta_j) \right] \\ + (1 - \sum_{j=1}^{m-2} \alpha_j) \left[ (1 - \sum_{j=i}^{m-2} \beta_j \xi_j) - s(1 - \sum_{j=i}^{m-2} \beta_j) \right], \\ \xi_{i-1} \leq s \leq \xi_i, \quad 2 \leq i \leq m-2, \quad s \leq t; \\ (1 - \sum_{j=1}^{m-2} \alpha_j) \left[ (1-t) + \sum_{j=1}^{m-2} \beta_j(t-s) \right], \\ \xi_{m-2} \leq s \leq 1, \quad s \leq t; \\ (1 - \sum_{j=1}^{m-2} \alpha_j)(1-s), \\ 0 \leq t \leq 1, \quad \xi_{m-2} \leq s \leq 1, \quad t \leq s. \end{cases}$$

**Lemma 3.4** Suppose  $\alpha_i, \beta_i > 0$  ( $i = 1, 2, \dots, m-2$ ),  $0 < \sum_{i=1}^{m-2} \alpha_i < 1$ ,  $0 < \sum_{i=1}^{m-2} \beta_i < 1$ . Then

$$G^*(t, s) \geq 0 \quad \text{for } (t, s) \in [0, 1] \times [0, 1].$$

**Proof.** We only check that if  $s \leq t$ , then

$$\begin{aligned} Q &= -M(t-s) + \sum_{j=i}^{m-2} \alpha_j \left[ (1 - \sum_{j=1}^{m-2} \beta_j \xi_j) - t(1 - \sum_{j=1}^{m-2} \beta_j) \right] \\ &\quad + (1 - \sum_{j=1}^{m-2} \alpha_j) \left[ (1 - \sum_{j=i}^{m-2} \beta_j \xi_j) - s(1 - \sum_{j=i}^{m-2} \beta_j) \right] \geq 0. \end{aligned}$$

In fact

$$Q = \sum_{j=i}^{m-2} \alpha_j \left( 1 - \sum_{j=1}^{m-2} \beta_j \right) (1-t) + \sum_{j=i}^{m-2} \alpha_j \left( \sum_{j=1}^{m-2} \beta_j - \sum_{j=1}^{m-2} \beta_j \xi_j \right)$$

$$\begin{aligned}
& + \left(1 - \sum_{j=1}^{m-2} \alpha_j\right) \left(1 - \sum_{j=i}^{m-2} \beta_j\right) (1-s) + \left(1 - \sum_{j=1}^{m-2} \alpha_j\right) \left(\sum_{j=i}^{m-2} \beta_j - \sum_{j=i}^{m-2} \beta_j \xi_j\right) \\
& - \left(1 - \sum_{j=1}^{m-2} \alpha_j\right) \left(1 - \sum_{j=1}^{m-2} \beta_j\right) (t-s) \\
\geq & \left(1 - \sum_{j=1}^{m-2} \alpha_j\right) \left(1 - \sum_{j=i}^{m-2} \beta_j\right) (1-s) - \left(1 - \sum_{j=1}^{m-2} \alpha_j\right) \left(1 - \sum_{j=1}^{m-2} \beta_j\right) (t-s) \\
\geq & \left(1 - \sum_{j=1}^{m-2} \alpha_j\right) \left(1 - \sum_{j=i}^{m-2} \beta_j\right) (t-s) - \left(1 - \sum_{j=1}^{m-2} \alpha_j\right) \left(1 - \sum_{j=1}^{m-2} \beta_j\right) (t-s) \\
= & \left(1 - \sum_{j=1}^{m-2} \alpha_j\right) \sum_{j=1}^{i-1} \beta_j (t-s) \\
\geq & 0.
\end{aligned}$$

**Lemma 3.5** Suppose  $(H_1)$  holds. Then  $g_i(t, s) \leq 0$  ( $0 \leq i \leq n-1$ ), where  $g_i(t, s)$  is the Green's function for the BVP

$$\begin{cases} x''(t) = 0, & 0 \leq t \leq 1, \\ x'(0) = \sum_{j=1}^{m-2} \alpha_{ij} x'(\xi_j), & x(1) = \sum_{j=1}^{m-2} \beta_{ij} x(\xi_j). \end{cases}$$

**Proof.** It is easy to see that  $g_i(t, s) \leq 0$  by using Lemma 3.4.

Let  $G_1(t, s) = g_{n-2}(t, s)$ , then for  $2 \leq j \leq n-1$  we recursively define

$$G_j(t, s) = \int_0^1 g_{n-j-1}(t, r) G_{j-1}(r, s) dr.$$

**Lemma 3.6** Suppose  $(H_1)$  holds. If  $f(t) \in C[0, 1]$ , then the boundary value problem

$$\begin{cases} u^{(2l)}(t) = f(t), & 0 \leq t \leq 1, \\ u^{(2i+1)}(0) = \sum_{j=1}^{m-2} \alpha_{n-l+i-1, j} u^{(2i+1)}(\xi_j), \\ u^{(2i)}(1) = \sum_{j=1}^{m-2} \beta_{n-l+i-1, j} u^{(2i)}(\xi_j), & 0 \leq i \leq l-1, \end{cases} \quad (8)$$

has a unique solution for each  $1 \leq l \leq n-1$ ,  $G_l(t, s)$  is the associated Green's function for the boundary value problem (8).

**Proof.** We prove the result by using induction. Obviously, the result holds by using Lemma 3.3 for  $l = 1$ .

We assume that the result holds for  $l-1$ . Now we consider the case for  $l$ . Let  $u''(t) = v(t)$ ,

then (8) is equivalent to

$$\begin{cases} u''(t) = v(t), & 0 \leq t \leq 1, \\ u'(0) = \sum_{j=1}^{m-2} \alpha_{n-l-1,j} u'(\xi_j), \\ u(1) = \sum_{j=1}^{m-2} \beta_{n-l-1,j} u(\xi_j), \end{cases} \quad (9)$$

and

$$\begin{cases} v^{(2(l-1))}(t) = f(t), & 0 \leq t \leq 1, \\ v^{(2i+1)}(0) = \sum_{j=1}^{m-2} \alpha_{n-l+i,j} v^{(2i+1)}(\xi_j), \\ v^{(2i)}(1) = \sum_{j=1}^{m-2} \beta_{n-l+i,j} v^{(2i)}(\xi_j), & 0 \leq i \leq l-2. \end{cases} \quad (10)$$

Lemma 3.3 implies that (9) has a unique solution  $u(t) = \int_0^1 g_{n-l-1}(t,r)v(r)dr$ , and (10) has also a unique solution  $v(t) = \int_0^1 G_{l-1}(t,s)f(s)ds$  by the inductive hypothesis. Thus, (8) has a unique solution

$$\begin{aligned} u(t) &= \int_0^1 g_{n-l-1}(t,r) \int_0^1 G_{l-1}(r,s)f(s)dsdr \\ &= \int_0^1 \left( \int_0^1 g_{n-l-1}(t,r)G_{l-1}(r,s)dr \right) f(s)ds \\ &= \int_0^1 G_l(t,s)f(s)ds \end{aligned}$$

Therefore, the result hold for  $l$ . Lemma 3.6 is now completed.

For each  $1 \leq l \leq n-1$ , we define  $A_l : C[0,1] \rightarrow C[0,1]$  by

$$A_l v(t) = \int_0^1 G_l(t,\tau)v(\tau)d\tau.$$

With the use of Lemma 3.6, for each  $1 \leq l \leq n-1$ , we have

$$\begin{cases} (A_l v)^{(2l)}(t) = v(t), & 0 \leq t \leq 1, \\ (A_l v)^{(2i+1)}(0) = \sum_{j=1}^{m-2} \alpha_{n-l+i-1,j} (A_l v)^{(2i+1)}(\xi_j), \\ (A_l v)^{(2i)}(1) = \sum_{j=1}^{m-2} \beta_{n-l+i-1,j} (A_l v)^{(2i)}(\xi_j), & 0 \leq i \leq l-1. \end{cases}$$

Therefore (6) has a solution if and only if the boundary value problem

$$\begin{cases} v''(t) = f(t, A_{n-1}v(t), A_{n-2}v(t), \dots, A_1v(t), v(t)), & 0 \leq t \leq 1, \\ v'(0) = \sum_{j=1}^{m-2} \alpha_{n-1,j} v'(\xi_j), \quad v(1) = \sum_{j=1}^{m-2} \beta_{n-1,j} v(\xi_j), \end{cases} \quad (11)$$

has a solution. If  $x$  is a solution of (6), then  $v = x^{(2(n-1))}$  is a solution of (11). Conversely, if  $v$  is a solution of (11), then  $x = A_{n-1}v$  is a solution of (6).

Define  $A : C[0, 1] \rightarrow C[0, 1]$  by

$$Av(t) = \int_0^1 g_{n-1}(t, s) f(s, A_{n-1}v(s), A_{n-2}v(s), \dots, A_1v(s), v(s)) ds.$$

It now follows that there exists a solution of BVP (6) if, and only if, there exists a continuous fixed point of  $A$ . Moreover, the relationship between a solution of BVP (6) and a fixed point of  $A$  is given by  $x = A_{n-1}v(t)$ , or equivalently,  $x^{(2(n-1))} = v$ .

Note that  $x$  is a positive solution of (6) if, and only if,  $(-1)^{n-1}x^{(2(n-1))} = (-1)^{n-1}v$  is positive, where  $v$  is the corresponding continuous fixed point of  $A$ .

For each  $0 \leq t \leq 1, 0 \leq i \leq n - 1$ , there are only finitely many points  $s$  such that  $g_i(t, s) = 0$ .

Let

$$M_i = \max_{0 \leq t \leq 1} \int_0^1 |g_i(t, s)| ds, \quad m_i = \min_{0 \leq t \leq 1} \int_0^1 |g_i(t, s)| ds,$$

obviously,  $M_i > m_i > 0$ .

Let  $X = C[0, 1]$  with the maximum norm  $\|x\| = \max_{0 \leq t \leq 1} |x(t)|$  and define the cone  $P \subset X$  by

$$P = \left\{ x \in X : (-1)^{n-1}x(t) \geq 0, (-1)^{n-1}x \text{ is concave on } [0, 1], \text{ and } \min_{t \in [0, 1]} (-1)^{n-1}x(t) \geq \gamma \|x\| \right\}.$$

Let  $\alpha : P \rightarrow [0, \infty)$  be the nonnegative continuous concave functional

$$\alpha(x) = \min_{t \in [0, 1]} (-1)^{n-1}x(t) \quad \text{for } x \in P.$$

We now present our main result.

**Theorem 3.1.** Suppose  $(H_1) - (H_2)$  hold. In addition there exist nonnegative numbers  $a, b$ , and  $c$  such that  $0 < a < b \leq \min\{\gamma, m_{n-1}/M_{n-1}\}c$  and  $f(t, u_{n-1}, u_{n-2}, \dots, u_1, u_0)$  satisfies the following growth conditions:

$$(H_3) \quad (-1)^n f(t, u_{n-1}, \dots, u_0) < a/M_{n-1} \quad \text{for } (t, |u_{n-1}|, |u_{n-2}|, \dots, |u_0|) \in [0, 1] \times$$

$$\prod_{j=n-1}^1 [0, \prod_{i=2}^{j+1} M_{n-i} a] \times [0, a];$$

$$(H_4) \quad (-1)^n f(t, u_{n-1}, \dots, u_0) < c/M_{n-1} \quad \text{for } (t, |u_{n-1}|, |u_{n-2}|, \dots, |u_0|) \in [0, 1] \times$$

$$\prod_{j=n-1}^1 [0, \prod_{i=2}^{j+1} M_{n-i} c] \times [0, c];$$

$$(H_5) \quad (-1)^n f(t, u_{n-1}, \dots, u_0) \geq b/m_{n-1} \quad \text{for } (t, |u_{n-1}|, |u_{n-2}|, \dots, |u_0|) \in [0, 1] \times$$

$$\prod_{j=n-1}^1 [\prod_{i=2}^{j+1} m_{n-i} b, \prod_{i=2}^{j+1} M_{n-i} b/\gamma] \times [b, b/\gamma].$$



Then the boundary value problem (6) has at least three positive solutions  $x_1$ ,  $x_2$  and  $x_3$  such that

$$\|x_1^{(2(n-1))}\| < a, \quad b < \min_{0 \leq t \leq 1} (-1)^{n-1} x_2^{(2(n-1))}(t),$$

and

$$\|x_3^{(2(n-1))}\| > a \quad \text{with} \quad \min_{0 \leq t \leq 1} (-1)^{n-1} x_3^{(2(n-1))}(t) < b.$$

**Proof.** At first we show that  $A : P \rightarrow P$ . Let  $x \in P$  then  $(-1)^{n-1}Ax(t) \geq 0$ . Moreover,

$$(-1)^{n-1}(Ax)''(t) = (-1)^{n-1}f(t, A_{n-1}x(t), A_{n-2}x(t), \dots, A_1x(t), x(t)) < 0.$$

By lemma 3.2,  $\min_{t \in [0,1]} (-1)^{n-1}Ax(t) \geq \gamma \|Ax\|$ , this implies that  $A : P \rightarrow P$ . Also, it is easy to see that the operator  $A$  is completely continuous.

Choose  $x \in \overline{P}_c$ , then  $\|x\| \leq c$ . Note that

$$\|A_j x\| = \max_{t \in [0,1]} \left| \int_0^1 G_j(t,s)x(s)ds \right| \leq \prod_{i=2}^{j+1} M_{n-i} \|x\| \leq \prod_{i=2}^{j+1} M_{n-i} c.$$

Thus, according to assumption  $(H_4)$  we have

$$\begin{aligned} \|Ax\| &= \max_{0 \leq t \leq 1} |Ax(t)| \\ &= \max_{0 \leq t \leq 1} \left\{ \int_0^1 |g_{n-1}(t,s)f(s, A_{n-1}x(s), A_{n-2}x(s), \dots, A_1x(s), x(s))| ds \right\} \\ &\leq \frac{c}{M_{n-1}} \max_{0 \leq t \leq 1} \left\{ \int_0^1 |g_{n-1}(t,s)| ds \right\} \\ &= c. \end{aligned}$$

Therefore,  $A : \overline{P}_c \rightarrow \overline{P}_c$ .

In a completely analogous argument, assumption  $(H_3)$  implies that Condition (C2) of the Leggett-Williams Fixed Point Theorem is satisfied.

We now show that condition (C1) is satisfied. Note that for  $0 \leq t \leq 1$ .

$$x(t) = (-1)^{n-1} \frac{b}{\gamma} \in P \left( \alpha, b, \frac{b}{\gamma} \right) \quad \text{and} \quad \alpha(x) = \frac{b}{\gamma} > b.$$

Thus,

$$\left\{ x \in P \left( \alpha, b, \frac{b}{\gamma} \right) \mid \alpha(x) > b \right\} \neq \emptyset.$$

Also, if  $x \in P(\alpha, b, \frac{b}{\gamma})$ , then  $\alpha(x) = \min_{t \in [0,1]} (-1)^{n-1} x(t) \geq b$  for each  $0 \leq t \leq 1$ , so  $(-1)^{n-1} x(t) \geq b$ ,  $0 \leq t \leq 1$ , this implies

$$\begin{aligned} (-1)^{n-2} A_1 x(t) &= \int_0^1 -G_1(t, s) (-1)^{n-1} x(s) ds \\ &\geq b \int_0^1 |G_1(t, s)| ds \geq b m_{n-2}. \end{aligned}$$

Inductively, we have

$$(-1)^{n-1-j} A_j x(t) \geq \prod_{i=2}^{j+1} m_{n-i} b, \quad 0 \leq t \leq 1, \quad 1 \leq j \leq n-1.$$

and it is easy to see that

$$|A_j x(t)| \leq \prod_{i=2}^{j+1} M_{n-i} \frac{b}{\gamma}.$$

Applying condition  $(H_5)$  we get

$$(-1)^n f(t, A_{n-1} x(t), A_{n-2} x(t), \dots, A_1 x(t), x(t)) \geq \frac{b}{m_{n-1}}, \quad 0 \leq t \leq 1.$$

So,

$$\begin{aligned} \alpha(Ax) &= \min_{0 \leq t \leq 1} (-1)^{n-1} Ax(t) \\ &= \min_{0 \leq t \leq 1} \left\{ \int_0^1 -g_{n-1}(t, s) (-1)^n f(s, A_{n-1} x(s), A_{n-2} x(s), \dots, A_1 x(s), x(s)) ds \right\} \\ &\geq \frac{b}{m_{n-1}} \min_{0 \leq t \leq 1} \int_0^1 |g_{n-1}(t, s)| ds \\ &= b. \end{aligned}$$

Therefore, condition (C1) is satisfied.

Finally, we show that condition (C3) is also satisfied. That is, we show that if  $x \in P(\alpha, b, c)$  and  $\|Ax\| > d = b/\gamma$ , then  $\alpha(Ax) > b$ . This follows since  $A : P \rightarrow P$ , then

$$\alpha(Ax) = \min_{0 \leq t \leq 1} (-1)^{n-1} Ax(t) \geq \gamma \|Ax\| > b.$$

Therefore, condition (C3) is also satisfied. So we complete the proof.

#### 4. Example

In this section, we present an example to demonstrate the application of Theorem 3.1. Consider

the boundary value problem

$$\begin{cases} x^{(4)}(t) = f(t, x(t), x''(t)), & 0 \leq t \leq 1, \\ x'(0) = \frac{1}{2}x' \left( \frac{1}{2} \right), & x(1) = \frac{1}{2}x \left( \frac{1}{2} \right), \\ x^{(3)}(0) = \frac{1}{4}x^{(3)} \left( \frac{1}{2} \right), & x''(1) = \frac{3}{4}x'' \left( \frac{1}{2} \right). \end{cases} \quad (12)$$

where

$$f(t, x, y) = \begin{cases} \frac{1}{1000} \sin t + 4x + \frac{1}{1000}y^3, & x \in (-\infty, 1/32], \\ \frac{1}{1000} \sin t - \frac{15584}{25} \left( x - \frac{3}{32} \right)^2 + \frac{64}{25} + \frac{1}{1000}y^3, & x \in [1/32, 3/32], \\ \frac{1}{1000} \sin t + \frac{32768}{16875} \left( x - \frac{13}{32} \right)^2 + \frac{64}{27} + \frac{1}{1000}y^3, & x \in [3/32, 13/32], \\ \frac{1}{1000} \sin t + \frac{64}{27} + \frac{1}{1000}y^3, & x \in [13/32, +\infty). \end{cases}$$

By Lemma 3.3, we have

$$|g_0(t, s)| = \begin{cases} \frac{3}{4} - \frac{1}{2}t, & 0 \leq t \leq 1, 0 \leq s \leq \frac{1}{2}, s \leq t; \\ \frac{3}{4} - \frac{1}{4}t - \frac{1}{4}s, & 0 \leq t \leq 1, 0 \leq s \leq \frac{1}{2}, t \leq s; \\ \frac{1}{2} - \frac{1}{4}t - \frac{1}{4}s, & 0 \leq t \leq 1, \frac{1}{2} \leq s \leq 1, s \leq t; \\ \frac{1}{2} - \frac{1}{2}s, & 0 \leq t \leq 1, \frac{1}{2} \leq s \leq 1, t \leq s. \end{cases}$$

$$|g_1(t, s)| = \begin{cases} \frac{5}{8} - \frac{1}{4}t, & 0 \leq t \leq 1, 0 \leq s \leq \frac{1}{2}, s \leq t; \\ \frac{5}{8} - \frac{1}{16}t - \frac{3}{16}s, & 0 \leq t \leq 1, 0 \leq s \leq \frac{1}{2}, t \leq s; \\ \frac{3}{4} - \frac{3}{16}t - \frac{9}{16}s, & 0 \leq t \leq 1, \frac{1}{2} \leq s \leq 1, s \leq t; \\ \frac{3}{4} - \frac{3}{4}s, & 0 \leq t \leq 1, \frac{1}{2} \leq s \leq 1, t \leq s. \end{cases}$$

We first consider the condition  $i = 0$ .

1) For  $0 \leq t \leq \frac{1}{2}$ , we have

$$\begin{aligned} \int_0^1 |g_0(t, s)| ds &= \int_0^t |g_0(t, s)| ds + \int_t^{\frac{1}{2}} |g_0(t, s)| ds + \int_{\frac{1}{2}}^1 |g_0(t, s)| ds \\ &= \int_0^t \left( \frac{3}{4} - \frac{1}{2}t \right) ds + \int_t^{\frac{1}{2}} \left( \frac{3}{4} - \frac{1}{4}t - \frac{1}{4}s \right) ds + \int_{\frac{1}{2}}^1 \left( \frac{1}{2} - \frac{1}{2}s \right) ds \\ &= \frac{13}{32} - \frac{1}{8}t - \frac{1}{8}t^2. \end{aligned}$$

2) For  $\frac{1}{2} \leq t \leq 1$ , we have

$$\begin{aligned} \int_0^1 |g_0(t, s)| ds &= \int_0^{\frac{1}{2}} |g_0(t, s)| ds + \int_{\frac{1}{2}}^t |g_0(t, s)| ds + \int_t^1 |g_0(t, s)| ds \\ &= \int_0^{\frac{1}{2}} \left( \frac{3}{4} - \frac{1}{2}t \right) ds + \int_{\frac{1}{2}}^t \left( \frac{1}{2} - \frac{1}{4}t - \frac{1}{4}s \right) ds + \int_t^1 \left( \frac{1}{2} - \frac{1}{2}s \right) ds \\ &= \frac{13}{32} - \frac{1}{8}t - \frac{1}{8}t^2. \end{aligned}$$

So,

$$M_0 = \max_{0 \leq t \leq 1} \int_0^1 |g_0(t, s)| ds = \frac{13}{32}, \quad m_0 = \min_{0 \leq t \leq 1} \int_0^1 |g_0(t, s)| ds = \frac{5}{32}.$$

Next, we consider the condition  $i = 1$ .

3) For  $0 \leq t \leq \frac{1}{2}$ , we have

$$\begin{aligned} \int_0^1 |g_1(t, s)| ds &= \int_0^t |g_1(t, s)| ds + \int_t^{\frac{1}{2}} |g_1(t, s)| ds + \int_{\frac{1}{2}}^1 |g_1(t, s)| ds \\ &= \int_0^t \left( \frac{5}{8} - \frac{1}{4}t \right) ds + \int_t^{\frac{1}{2}} \left( \frac{5}{8} - \frac{1}{16}t - \frac{3}{16}s \right) ds + \int_{\frac{1}{2}}^1 \left( \frac{3}{4} - \frac{3}{4}s \right) ds \\ &= \frac{49}{128} - \frac{1}{32}t - \frac{3}{32}t^2. \end{aligned}$$

4) For  $\frac{1}{2} \leq t \leq 1$ , we have

$$\begin{aligned} \int_0^1 |g_1(t, s)| ds &= \int_0^{\frac{1}{2}} |g_1(t, s)| ds + \int_{\frac{1}{2}}^t |g_1(t, s)| ds + \int_t^1 |g_1(t, s)| ds \\ &= \int_0^{\frac{1}{2}} \left( \frac{5}{8} - \frac{1}{4}t \right) ds + \int_{\frac{1}{2}}^t \left( \frac{3}{4} - \frac{3}{16}t - \frac{9}{16}s \right) ds + \int_t^1 \left( \frac{3}{4} - \frac{3}{4}s \right) ds \\ &= \frac{49}{128} - \frac{1}{32}t - \frac{3}{32}t^2. \end{aligned}$$

So,

$$M_1 = \max_{0 \leq t \leq 1} \int_0^1 |g_1(t, s)| ds = \frac{49}{128}, \quad m_1 = \min_{0 \leq t \leq 1} \int_0^1 |g_1(t, s)| ds = \frac{33}{128}.$$

As  $\gamma = \frac{3}{5}$ ,  $m_1/M_1 = \frac{33}{49}$ , so we can let  $a = \frac{1}{13}$ ,  $b = \frac{3}{5}$ ,  $c = 1$ , then

$$\begin{aligned} f(t, x, y) &< a/M_1 = \frac{128}{637} \quad \text{for } (t, |x|, |y|) \in [0, 1] \times [0, 1/32] \times [0, 1/13], \\ f(t, x, y) &< c/M_1 = \frac{128}{49} \quad \text{for } (t, |x|, |y|) \in [0, 1] \times [0, 13/32] \times [0, 1], \\ f(t, x, y) &\geq b/m_1 = \frac{128}{55} \quad \text{for } (t, |x|, |y|) \in [0, 1] \times [3/32, 13/32] \times [3/5, 1]. \end{aligned}$$

By Theorem 3.1, problem (12) has at least three positive solutions.

### Acknowledgement

The authors are grateful to the referee for his/her constructive and valuable comments and suggestions.

### REFERENCES

1. R. W. Leggett, L. R. Williams; Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.* **28** (1979) 673–688.

2. V. A. Il'in, E. I. Moiseev, Nonlocal boundary value problem of the second kind for a Sturm-Liouville operator, *Differential Equations* **23(8)** (1987) 979-987.
3. V. A. Il'in, E. I. Moiseev. Nonlocal boundary value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects, *Differential Equations*, **23(7)** (1987) 803-810.
4. W. Feng, J. R. L. Webb, Solvability of a m-point boundary value problems with nonlinear growth, *J. Math. Anal. Appl*, **212** (1997) 467-480.
5. W. Feng, On a m-point nonlinear boundary value problem, *Nonlinear Analysis TMA*, **30(6)** (1997) 5369-5374.
6. C. P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equation, *J. Math. Anal. Appl*, **168** (1992) 540-551.
7. C. P. Gupta, A sharper condition for the solvability of a three-point second order boundary value problem, *J. Math. Anal. Appl*, **205** (1997) 586-579.
8. C. P. Gupta, A generalized multi-point boundary value problem for second order ordinary differential equations, *Appl. Math. Comput*, **(89)** (1998) 133-146.
9. J. M. Davis, P. W. Elloe and J. Henderson. Triple positive solutions and dependence on higher order derivatives, *J. Math. Anal. Appl*, **237** (1999) 710-720.
10. J. M. Davis, J. Henderson and P. J. Y. Wong, General Lidstone problems: multiplicity and symmetry of solutions, *J. Math. Anal. Appl*, **251** (2000) 527-548.
11. Y. Guo, W. Ge and Y. Gao, Twin positive solutions for higher order m-point boundary value problems with sign changing nonlinearities, *Appl. Math. Comput*, **146** (2003) 299-311.
12. R. Ma, N. Castaneda, Existence of solution of nonlinear m-point boundary-value problems, *J. Math. Anal. Appl*, **256** (2001) 556-567.
13. Y. Guo, X. Liu and J. Qiu, Three positive solutions for higher order m-point boundary value problems, *J. Math. Anal. Appl*, **289** (2004) 545-553.
14. Y. Wang, W. Ge, Multiple positive solutions for a nonlinear 2nth order m-point boundary value problems (II), *Tamsui Oxford Journal of Mathematical Sciences*, **22(2)**(2006)143-158.
15. Y. Wang, W. Ge, Existence of multiple positive solutions for even order multi-point boundary value problems, *Georgian Mathematical Journal*, **14(4)**(2007)775-792.

(Received March 26, 2010)