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PICONE TYPE FORMULA FOR NON-SELFADJOINT IMPULSIVE DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS SOLUTIONS

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Abstract. A Picone type formula for second order linear non-selfadjoint impulsive differential equations with discontinuous solutions having fixed moments of impulse actions is derived. Applying the formula, Leighton and Sturm-Picone type comparison theorems as well as several oscillation criteria for impulsive differential equations are obtained.

1. Introduction

As the impulsive differential equations are useful in modelling many real processes observed in physics, chemistry, biology, engineering, etc., see [1, 11, 13, 20, 21, 22, 25, 26, 27], there has been an increasing interest in studying such equations from the point of view of stability, asymptotic behavior, existence of periodic solutions, and oscillation of solutions. The classical theory can be found in the monographs [9, 18]. Recently, the oscillation theory of impulsive differential equations has also received considerable attention, see [2, 14] for the Sturmian theory of impulsive differential equations, and [15] for a Picone type formula and its applications. Due to difficulties caused by the impulsive perturbations the solutions are usually assumed to be continuous in most works in the literature. In this paper, we consider second order non-selfadjoint linear impulsive differential equations with discontinuous solutions. Our aim is to derive a Picone type identity for such impulsive differential equations, and hence extend and generalize several results in the literature.

We consider second order linear impulsive differential equations of the form

$$\begin{aligned} l_0[x] &= (k(t)x')' + r(t)x' + p(t)x = 0, && t \neq \theta_i \\ l_1[x] &= \Delta x + (1+p_i)x - k(t)x' = 0, && t = \theta_i \\ l_2[x] &= \Delta(k(t)x') + (1-\tilde{p}_i)k(t)x' + (1+p_i\tilde{p}_i)x = 0, && t = \theta_i \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} L_0[y] &= (m(t)y')' + s(t)y' + q(t)y = 0, && t \neq \theta_i \\ L_1[y] &= \Delta y + (1+q_i)y - m(t)y' = 0, && t = \theta_i \\ L_2[y] &= \Delta(m(t)y') + (1-\tilde{q}_i)m(t)y' + (1+q_i\tilde{q}_i)y = 0, && t = \theta_i \end{aligned} \tag{1.2}$$

where $\Delta z(t) = z(t^+) - z(t^-)$ and $z(t^\pm) = \lim_{\tau\to t^\pm} z(\tau)$. For our purpose, we fix $t_0 \in \mathbb{R}$ and let I_0 be an interval contained in $[t_0,\infty)$. We assume without further mention that

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(i) $\{p_i\}$, $\{\tilde{p}_i\}$, $\{q_i\}$ and $\{\tilde{q}_i\}$ are real sequences and $\{\theta_i\}$ is a strictly increasing unbounded sequence of real numbers;

(ii) $k, r, p, m, s, q \in \mathrm{PLC(I_0)} := \big\{\mathrm{h} : \mathrm{I_0} \to \mathbb{R}$ is continuous on each interval (θ_i, θ_{i+1}), $h(\theta_i^{\pm})$ exist, $h(\theta_i) = h(\theta_i^-)$ for $i \in \mathbb{N}\big\}$ with $k(t) > 0$, $m(t) > 0$ for all $t \in I_0$.

Note that if $z \in \mathrm{PLC}(I_0)$ and $\Delta z(\theta_i) = 0$ for all $i \in \mathbb{N}$, then z becomes continuous and conversely. If $\tau \in \mathbb{R}$ is a jump point of the function $z(t)$ i.e. $\Delta z(\tau) \neq 0$, then there exists a $j \in \mathbb{N}$ such that $\theta_j = \tau$. Throughout this work, we denote by j_τ, the index j satisfying $\theta_j = \tau$.

By a solution of the impulsive system (1.1) on an interval $I_0 \subset [t_0, \infty)$, we mean a nontrivial function x which is defined on I_0 such that $x, x', (kx')' \in \mathrm{PLC}(I_0)$ and that x satisfies (1.1) for all $t \in I_0$.

Definition 1.1. *A function $z \in \mathrm{PLC}(I_0)$ is said to have a generalized zero at $t = t_*$ if $z(t_*^+)z(t_*) \leq 0$ for $t_* \in I_0$. A solution is called oscillatory if it has arbitrarily large generalized zeros, and a differential equation is oscillatory if every solution of the equation is oscillatory.*

2. The Main Results

Let I be a nondegenerate subinterval of I_0. In what follows we shall make use of the following condition:

$$k(t) \neq m(t) \text{ whenever } r(t) \neq s(t) \text{ for all } t \in I. \tag{C}$$

We will see that condition (C) is quite crucial in obtaining a Picone type formula as in the case of nonimpulsive differential equations. If (C) fails to hold then a device of Picard is helpful.

The Picone type formula is obtained by making use of the following Picone type identity, consisting of a pair of identities.

Lemma 2.1 (Picone type identity)**.** *Let $u', v', (ku')', (mv')' \in \mathrm{PLC}(I)$, and (C) hold. If $v(t) \neq 0$ for any $t \in I$, then*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{u}{v}\,(vku' - umv')\right\} \\
&\quad = \left\{q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m}\right\}u^2 + (k-m)\left\{u' + \frac{(s-r)}{2(k-m)}\,u\right\}^2 \\
&\quad + \frac{m}{v^2}\left\{u'v - uv' - \frac{s}{2m}\,uv\right\}^2 + \frac{u}{v}\,\Big\{vl_0[u] - uL_0[v]\Big\}, \qquad t \neq \theta_i; \quad (2.1)
\end{aligned}$$

and

$$\Delta\Big\{\frac{u}{v}\,(vku'-umv')\Big\} = (\tilde{p}_i-\tilde{q}_i)u_+^2(q_i-p_i)u^2+\frac{1}{vv_+}\Big\{v\Delta u-u\Delta v\Big\}^2$$
$$+\frac{u}{v}\,\Big\{vl_1[u]-uL_1[v]\Big\}+\frac{u_+}{v_+}\,\Big\{v_+l_2[u]-u_+L_2[v]\Big\}$$
$$+\frac{u_+}{v_+}\,\Big\{\tilde{q}_iu_+L_1[v]-\tilde{p}_iv_+l_1[u]\Big\},\quad t=\theta_i, \tag{2.2}$$

where the notations $z_+ = z(t^+)$ and $z = z(t)$ are used.

Proof. Let $t \in I$. If $t \neq \theta_i$, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{u}{v}\,(vku'-umv')\Big\}$$
$$=(k-m)(u')^2+(q-p)u^2+m(u'-\frac{u}{v}\,v')^2+u^2\,\frac{sv'}{v}-ruu'$$
$$+\frac{u}{v}\,\Big\{vl_0[u]-uL_0[v]\Big\}$$
$$=(k-m)\Big\{(u')^2+\frac{s-r}{k-m}\,uu'+\frac{(s-r)^2}{4(k-m)^2}\,u^2\Big\}+(q-p)u^2$$
$$+m\Big\{(u'-\frac{uv'}{v})^2-\frac{su}{m}(u'-\frac{uv'}{v})+\frac{s^2u^2}{4m^2}\Big\}-\frac{s^2u^2}{4m}-\frac{(s-r)^2}{4(k-m)}\,u^2$$
$$+\frac{u}{v}\,\Big\{vl_0[u]-uL_0[v]\Big\}.$$

Rearranging we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big\{\frac{u}{v}\,(vku'-umv')\Big\}$$
$$=\Big\{q-p-\frac{(s-r)^2}{4(k-m)}-\frac{s^2}{4m}\Big\}u^2+(k-m)\Big\{u'+\frac{(s-r)}{2(k-m)}\,u\Big\}^2$$
$$+\frac{m}{v^2}\Big\{u'v-uv'-\frac{s}{2m}\,uv\Big\}^2+\frac{u}{v}\,\Big\{vl_0[u]-uL_0[v]\Big\}.$$

If $t = \theta_i$, then the computation becomes more involved. We see that

$$\Delta\left\{\frac{u}{v}\left(vku' - umv'\right)\right\}$$
$$= u_+\left\{l_2[u] + \tilde{p}_i ku' - (1 + p_i\tilde{p}_i)u\right\} - uku' + \frac{u^2}{v}mv'$$
$$-\frac{u_+^2}{v_+}\left\{L_2[v] + \tilde{q}_i mv' - (1 + q_i\tilde{q}_i)v\right\}$$
$$= \left\{\tilde{p}_i u_+ - u\right\}ku' - (1 + p_i\tilde{p}_i)uu_+$$
$$+\left\{\frac{u^2}{v} - \tilde{q}_i\frac{u_+^2}{v_+}\right\} mv' + (1 + q_i\tilde{q}_i)\frac{u_+^2 v}{v_+} + \frac{u_+}{v_+}\left\{v_+ l_2[u] - u_+ L_2[v]\right\}$$
$$= \left\{\tilde{p}_i u_+ - u\right\}\left\{u_+ + p_i u - l_1[u]\right\} + \left\{\frac{u^2}{v} - \tilde{q}_i\frac{u_+^2}{v_+}\right\}\left\{v_+ + q_i v - L_1[v]\right\}$$
$$-(1 + p_i\tilde{p}_i)uu_+ + (1 + q_i\tilde{q}_i)\frac{u_+^2 v}{v_+} + \frac{u_+}{v_+}\left\{v_+ l_2[u] - u_+ L_2[v]\right\}$$

$$= (\tilde{p}_i - \tilde{q}_i)u_+^2 + (q_i - p_i)u^2 + \frac{1}{vv_+}\left\{uv_+ - u_+ v\right\}^2$$
$$+ul_1[u] - \frac{u^2}{v}L_1[v] + \tilde{q}_i\frac{u_+^2}{v_+}L_1[v] - \tilde{p}_i u_+ l_1[u] + \frac{u_+}{v_+}\left\{v_+ l_2[u] - u_+ L_2[v]\right\}$$
$$= (\tilde{p}_i - \tilde{q}_i)u_+^2 + (q_i - p_i)u^2 + \frac{1}{vv_+}\left\{v\Delta u - u\Delta v\right\}^2 + \frac{u}{v}\left\{vl_1[u] - uL_1[v]\right\}$$
$$+\frac{u_+}{v_+}\left\{v_+ l_2[u] - u_+ L_2[v]\right\} + \frac{u_+}{v_+}\left\{\tilde{q}_i u_+ L_1[v] - \tilde{p}_i v_+ l_1[u]\right\}$$

□

Theorem 2.1 (Picone type formula). *Let* (C) *be satisfied. Suppose that* u', v', $(ku')'$, $(mv')' \in \mathrm{PLC}(I)$. *If* $v(t) \neq 0$ *for any* $t \in I$, *and* $[\alpha, \beta] \subseteq I$ *then*

$$\begin{aligned}
&\frac{u}{v}\,(vku' - umv')\Big|_{t=\alpha}^{t=\beta} \\
&\quad = \int_\alpha^\beta \Bigg\{ \left[q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m} \right] u^2 + (k-m) \left[u' + \frac{(s-r)}{2(k-m)}\, u \right]^2 \\
&\quad + \frac{m}{v^2} \left[u'v - uv' - \frac{s}{2m}\, uv \right]^2 + \frac{u}{v} \Big[v l_0[u] - u L_0[v] \Big] \Bigg\} \mathrm{d}t \\
&\quad + \sum_{\alpha \le \theta_i < \beta} \Bigg\{ (\tilde{p}_i - \tilde{q}_i) u^2(\theta_i^+) + (q_i - p_i) u^2(\theta_i) \\
&\quad + \frac{1}{v(\theta_i) v(\theta_i^+)} \Big[v(\theta_i) \Delta u(\theta_i) - u(\theta_i) \Delta v(\theta_i) \Big]^2 \\
&\quad + \frac{u(\theta_i)}{v(\theta_i)} \Big[v(\theta_i) l_1[u] - u(\theta_i) L_1[v] \Big] + \frac{u(\theta_i^+)}{v(\theta_i^+)} \Big[v(\theta_i^+) l_2[u] - u(\theta_i^+) L_2[v] \Big] \\
&\quad + \frac{u(\theta_i^+)}{v(\theta_i^+)} \Big[\tilde{q}_i u(\theta_i^+) L_1[v] - \tilde{p}_i v(\theta_i^+) l_1[u] \Big] \Bigg\}. \qquad (2.3)
\end{aligned}$$

Proof. Using (2.1) and (2.2), and employing Lemma 2.1 with

$$\nu(\beta) - \nu(\alpha) = \int_\alpha^\beta \nu'(t)\mathrm{d}t + \sum_{\alpha \le \theta_i < \beta} \Delta \nu(\theta_i)$$

where

$$\nu(t) = \frac{u(t)}{v(t)} \Big[v(t)k(t)u'(t) - u(t)m(t)v'(t) \Big], \qquad t \in I,$$

we easily see that (2.3) holds. □

The following corollary is an extension of the classical comparison theorem of Leighton [10, Corollary 1].

Theorem 2.2 (Leighton type comparison). *Let* x *be a solution of (1.1) having two generalized zeros* $a, b \in I$. *Suppose that* (C) *holds, and that*

$$\int_a^b \left\{ \left[q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m} \right] x^2 + (k-m) \left[x' + \frac{s-r}{2(k-m)}\, x \right]^2 \right\} \mathrm{d}t > 0, \qquad (2.4)$$

and

$$\tilde{p}_i \ge \tilde{q}_i, \quad q_i \ge p_i \qquad (2.5)$$

for all i *for which* $\theta_i \in [a, b]$, *then every solution* y *of (1.2) must have at least one generalized zero on* $[a, b]$.

Proof. Assume that y has no generalized zero in $[a,b]$. Since x and y are solutions of (1.1) and (1.2) respectively, $l_k[x] = L_k[y] = 0$, $k = 0,1,2$. Let $\epsilon > 0$ be sufficiently small. From Theorem 2.1 we obtain

$$\begin{aligned}\frac{x}{y}\,(ykx' - xmy')\Big|_{t=a+\epsilon}^{t=b-\epsilon} &= \int_{a+\epsilon}^{b-\epsilon}\Bigg\{\left[q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m}\right]x^2 \\ &+(k-m)\left\{x' + \frac{(s-r)}{2(k-m)}\,x\right\}^2 + \frac{m}{y^2}\left(x'y - xy' - \frac{s}{2m}\,xy\right)^2\Bigg\}\mathrm{d}t \\ &+ \sum_{a+\epsilon\le\theta_i<b-\epsilon}\Bigg\{(\tilde{p}_i - \tilde{q}_i)x^2(\theta_i^+) + (q_i - p_i)x^2(\theta_i) \\ &+ \frac{1}{y(\theta_i)y(\theta_i^+)}\Big[y(\theta_i)\Delta x(\theta_i) - x(\theta_i)\Delta y(\theta_i)\Big]^2\Bigg\}. \end{aligned} \tag{2.6}$$

As $\epsilon \to 0^+$ the left-hand side of (2.6) tends to

$$\begin{aligned}&\frac{x(b^-)}{y(b^-)}\Big[y(b^-)k(b^-)x'(b^-) - x(b^-)m(b^-)y'(b^-)\Big] \\ &-\frac{x(a^+)}{y(a^+)}\Big[y(a^+)k(a^+)x'(a^+) - x(a^+)m(a^+)y'(a^+)\Big] \\ &= x(b^-)x(b^+) + x(a^-)x(a^+) + (p_{i_b} - q_{i_b})x^2(b^-) - (\tilde{p}_{i_a} - \tilde{q}_{i_a})x^2(a^+) \\ &-x^2(b^-)\frac{y(b^+)}{y(b^-)} - x^2(a^+)\frac{y(a^-)}{y(a^+)} \le 0. \end{aligned} \tag{2.7}$$

Using (2.5) and (2.7) in (2.6) we get

$$\mathcal{L}[x] := \int_a^b \left\{\left[q - p - \frac{(s-r)^2}{4(k-m)} - \frac{s^2}{4m}\right]x^2 + (k-m)\left[x' + \frac{s-r}{2(k-m)}\,x\right]^2\right\}\mathrm{d}t \le 0,$$

a contradiction to (2.4). □

Remark 2.1. It is clear that, if the solution $x(t)$ of equation (1.1) is continuous at both $t = a$ and $t = b$ (i.e. $x(a^\pm) = x(b^\pm) = 0$) then (2.6) implies $L[x] \le 0$ which contradicts (2.4). If $x(t)$ is continuous at $t = b$ (i.e. $x(b^\pm) = 0$) but not at $t = a$, then it follows from (2.6) that

$$\mathcal{L}[x] \le -(\tilde{p}_{i_a} - \tilde{q}_{i_a})x^2(a^+) + x(a)x(a^+) - x^2(a^+)\frac{y(a)}{y(a^+)} \le 0 \tag{2.8}$$

and similarly, if $x(t)$ is continuous at $t = a$ (i.e. $x(a^\pm) = 0$) but not at $t = b$, then

$$\mathcal{L}[x] \le x(b)x(b^+) - x^2(b)\frac{y(b^+)}{y(b)} + (p_{i_b} - q_{i_b})x^2(b) \le 0. \tag{2.9}$$

Both inequalities (2.8) and (2.9) contradict (2.4). Hence $y(t)$ must have a generalized zero on $[a,b]$.

From Theorem 2.2 we obtain the following corollary. The result gives an extension and improvement for [2, Corollary 1].

Corollary 2.1 (Sturm-Picone type comparison)**.** *Let x be a solution of (1.1) having two consecutive generalized zeros $a, b \in I$. Suppose that (C) holds, and that*

$$k \geq m, \tag{2.10}$$

$$q \geq p + \frac{(s-r)^2}{4(k-m)} + \frac{s^2}{4m} \tag{2.11}$$

for all $t \in [a, b]$, and that (2.5) holds for all i for which $\theta_i \in [a, b]$.

If either (2.10) or (2.11) is strict in a subinterval of $[a, b]$, or one of the inequalities in (2.5) is strict for some $i \in \mathbb{N}$, then every solution y of (1.2) must have at least one generalized zero on $[a, b]$.

Corollary 2.2. *Suppose that the conditions (2.10)-(2.11) are satisfied for all $t \in [t_*, \infty)$ for some integer $t_* \geq t_0$, and that (2.5) is satisfied for all i for which $\theta_i \geq t_*$. If one of the inequalities in (2.5) or in (2.10)-(2.11) is strict, then every solution y of (1.2) is oscillatory whenever a solution x of (1.1) is oscillatory.*

As a consequence of Theorem 2.2 and Corollary 2.1, we have the following oscillation criterion.

Corollary 2.3. *Suppose for any given $T \geq t_0$ there exists an interval $(a, b) \subset [T, \infty)$ for which either the conditions of Theorem 2.2 or Corollary 2.1 are satisfied, then every solution y of (1.2) is oscillatory.*

3. Device of Picard

If the condition (C) fails to hold, then we introduce the so called device of Picard [16] (see also [7, p. 12]), and thereby obtain different versions of Corollary 2.1.

Clearly, for any $h \in \mathrm{PLC}(I)$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(x^2 h) = 2xx'h + x^2 h', \qquad t \neq \theta_i.$$

Let

$$u := \frac{x}{y}\,(ykx' - xmy') + x^2 h, \qquad t \in I.$$

It is not difficult to see that

$$\begin{aligned} u' &= \left\{q - p + h' - \frac{(s-r+2h)^2}{4(k-m)} - \frac{s^2}{4m}\right\}x^2 + (k-m)\left\{x' + \frac{s-r+2h}{2(k-m)}\,x\right\}^2 \\ &+ \frac{m}{y^2}\left\{x'y - xy' - \frac{s}{2m}\,xy\right\}^2, \qquad t \neq \theta_i \end{aligned}$$

and

$$\Delta u = [\tilde{p}_i - \tilde{q}_i + h_+]\,x_+^2 + \Big[q_i - p_i - h(\theta_i)\Big]x^2 + \frac{1}{yy_+}\Big[y\Delta x - x\Delta y\Big]^2, \quad t = \theta_i.$$

Assuming $r', s' \in \mathrm{PLC}(I)$, and taking $h = (r - s)/2$ we get

$$u' = \left\{q - p - \frac{s' - r'}{2} - \frac{s^2}{4m}\right\}x^2 + (k - m)(x')^2 + \frac{m}{y^2}\left\{x'y - xy' - \frac{s}{2m}\, xy\right\}^2, \quad t \neq \theta_i$$

and

$$\Delta u = \left\{\tilde{p}_i - \tilde{q}_i + \frac{r_+ - s_+}{2}\right\}x_+^2 + \left\{q_i - p_i - \frac{r - s}{2}\right\}x^2 + \frac{1}{yy_+}\left[y\Delta x - x\Delta y\right]^2, \quad t \neq \theta_i.$$

Thus we obtain the following results in a similar manner as in the previous section.

Theorem 3.1. *Let $r', s' \in \mathrm{PLC}(I)$ and x be a solution of (1.1) having two consecutive generalized zeros a and b in I. Suppose that*

$$k \geq m, \tag{3.1}$$

$$q \geq p + \frac{s' - r'}{2} + \frac{s^2}{4m} \tag{3.2}$$

are satisfied for all $t \in [a, b]$, and that

$$\tilde{p}_i \geq \tilde{q}_i - \big[r(\theta_i^+) - s(\theta_i^+)\big]/2, \quad q_i \geq p_i + \big[r(\theta_i) - s(\theta_i)\big]/2 \tag{3.3}$$

for all i for which $\theta_i \in [a, b]$.

If either (3.1) or (3.2) is strict in a subinterval of $[a, b]$, or one of the inequalities in (3.3) is strict for some i, then every solution y of (1.2) must have at least one generalized zero in $[a, b]$.

Corollary 3.1. *Suppose that the conditions (3.1)-(3.2) are satisfied for all $t \in [t_*, \infty)$ for some integer $t_* \geq t_0$, and that the conditions in (3.3) are satisfied for all i for which $\theta_i \geq t_*$. If $r', s' \in \mathrm{PLC}[t^*, \infty)$ and one of the inequalities (3.1)-(3.3) is strict, then (1.2) is oscillatory whenever a solution x of (1.1) is oscillatory.*

Theorem 3.2 (Leighton type comparison)**.** *Let $r', s' \in \mathrm{PLC}[a, b]$ and x be a solution of (1.1) having two generalized zeros $a, b \in I$ such that*

$$\int_a^b \left[\left(q - p - \frac{s' - r'}{2} - \frac{s^2}{4m}\right)x^2 + \big(k - m\big)(x')^2\right]\mathrm{d}t > 0,$$

and that the inequalities in (3.3) hold for all i for which $\theta_i \in [a, b]$. Then every solution y of (1.2) must have at least one generalized zero on $[a, b]$.

From Theorem 3.1 and Theorem 3.2, we have the following oscillation criterion.

Corollary 3.2. *Suppose for any given $t_1 \geq t_0$ there exists an interval $(a, b) \subset [t_1, \infty)$ for which either the conditions of Theorem 3.1 or Theorem 3.2 are satisfied, then (1.2) is oscillatory.*

4. Further results

The lemma below, cf. [2, Lemma 1.] and [14, Lemma 3.1.], is used for comparison purposes. The proof is a straightforward verification.

Lemma 4.1. *Let ψ be a positive function for $t \geq \alpha$ with $\psi', \psi'' \in \mathrm{PLC}[\alpha, \infty)$, where α is a fixed real number. Then the function*

$$x(t) = \frac{1}{\sqrt{\psi(t)}} \sin\left(\int_\alpha^t \psi(s)\mathrm{d}s\right), \qquad t \geq \alpha \tag{4.1}$$

is a solution of equation

$$\begin{array}{ll} (a_2(t)x')' + a_1(t)x' + a_0(t)x = 0, & t \neq \theta_i, \\ \Delta x + (1+e_i)x - a_2(t)x' = 0, & t = \theta_i, \\ \Delta(a_2(t)x') + (1-\tilde{e}_i)a_2(t)x' + (1+e_i\tilde{e}_i)x = 0, & t = \theta_i \quad (i \in \mathbb{N}) \end{array} \tag{4.2}$$

where $a_j^j \in \mathrm{PLC}[\alpha, \infty)$, $j = 0, 1, 2$, $\{e_i\}$ and $\{\tilde{e}_i\}$ are real sequences, with

$$a_0 = \frac{1}{2}\left(a_2'' + a_1' + a_1\frac{a_2'}{a_2} + \frac{a_1^2}{a_2}\right) - \frac{(a_2'+a_1)^2}{4a_2} + a_2\left[\psi^2 + \frac{\psi''}{2\psi} - \frac{3}{4}\left(\frac{\psi'}{\psi}\right)^2\right], \tag{4.3}$$

$$e_i = \psi(\theta_i)\cot\left(\int_\alpha^{\theta_i}\psi(s)\mathrm{d}s\right) - \left(\frac{a_2(\theta_i)\psi(\theta_i)}{a_2(\theta_i^+)\psi(\theta_i^+)}\right)^{1/2} - \frac{(a_2\psi)'(\theta_i)}{2\psi(\theta_i)} - \frac{a_1(\theta_i)}{2}, \tag{4.4}$$

$$\begin{aligned} \tilde{e}_i = {} & \left\{\frac{a_2(\theta_i)}{a_2(\theta_i^+)}\right\}^{1/2}\left\{\psi(\theta_i)\psi(\theta_i^+)\right\}^{1/2}\cot\left(\int_\alpha^{\theta_i}\psi(s)\mathrm{d}s\right) + \left(\frac{a_2(\theta_i^+)\psi(\theta_i^+)}{a_2(\theta_i)\psi(\theta_i)}\right)^{1/2} \\ & - \frac{(a_2\psi)'(\theta_i^+)}{2\psi(\theta_i^+)} - \frac{a_1(\theta_i^+)}{2}, \qquad \theta_i > \alpha. \end{aligned} \tag{4.5}$$

In view of Lemma 4.1 and Corollary 2.2, we can state the next theorem.

Theorem 4.1. *Let the function $\psi(t)$ satisfy the conditions of Lemma 4.1 and let the functions $a_j^j \in \mathrm{PLC}[\alpha, \infty)$, $j = 0, 1, 2$, and the sequences $\{e_i\}$ and $\{\tilde{e}_i\}$ be defined by the equalities (4.3)-(4.5), respectively. Suppose that $a_2(t) \neq k(t)$ whenever $a_1(t) \neq r(t)$,*

$$\int_\alpha^\infty \psi(t)\mathrm{d}t = \infty, \tag{4.6}$$

$$a_2 \geq k, \quad p \geq a_0 + \frac{(r-a_1)^2}{4(a_2-k)} + \frac{r^2}{4k}, \quad \textit{for all } t \geq \alpha \tag{4.7}$$

and that

$$\tilde{e}_i \geq \tilde{p}_i, \quad p_i \geq e_i, \quad \textit{for all } i \textit{ for which } \theta_i > \alpha. \tag{4.8}$$

If one of the inequalities in (4.7) and (4.8) is strict, then every solution x of (1.1) is oscillatory.

If $r' \in \mathrm{PLC}[\alpha, \infty)$, then from Lemma 4.1 and Theorem 3.1 we have the following theorem.

Theorem 4.2. *Let the function $\psi(t)$ satisfy the conditions of Lemma 4.1 and let the functions $a_j^j \in \mathrm{PLC}[\alpha,\infty)$, $j = 0, 1, 2.$, and the sequences $\{e_i\}$ and $\{\tilde{e}_i\}$ be defined by the equalities (4.3)-(4.5), respectively. Suppose that (4.6) holds*

$$a_2 \geq k, \quad p \geq a_0 + \frac{r' - a_1'}{2} + \frac{r^2}{4k}, \quad \text{for all } t \geq \alpha, \tag{4.9}$$

and that

$$\tilde{e}_i \geq \tilde{p}_i - \frac{a_1(\theta_i^+) - r(\theta_i^+)}{2}, \quad p_i \geq e_i + \frac{a_1(\theta_i) - r(\theta_i)}{2} \tag{4.10}$$

for all i for which $\theta_i > \alpha$.

If one of the inequalities in (4.9) and (4.10) is strict, then every solution x of (1.1) is oscillatory.

It is clear that an impulsive differential equation with a known solution can be used to obtain more concrete oscillation criteria. For instance, consider the impulsive differential equation

$$\begin{array}{ll} x'' - 2x' + x = 0, & t \neq i, \\ \Delta x + (1+p_i)\, x - x' = 0, & t = i, \\ \Delta x' + (1-\tilde{p}_i)\, x' + (1+p_i\tilde{p}_i)\, x = 0, & t = i, \quad (i \in \mathbb{N}). \end{array} \tag{4.11}$$

where

$$p_i = (e+1)i^{-1} + 2 \quad \text{and} \quad \tilde{p}_i = -i(1+e^{-1}) - e^{-1}, \quad (i \in \mathbb{N}.)$$

It is easy to verify that $x(t) = x_i(t)$, where

$$x_i(t) = \Big\{(e+i)(t-i) + i\Big\} e^{t-i}, \qquad t \in (i-1, i], \quad (i \in \mathbb{N}),$$

is an oscillatory solution with generalized zeros $\tau_i = i$ and $\xi_i = i(e+i-1)(e+i)^{-1} \in (i-1, i)$. Indeed, $x(\tau_i)x(\tau_i^+) < 0$ and $x(\xi_i) = 0$, $i \in \mathbb{N}$.

Applying Corollary 2.2, we easily see that equation (1.1) with $\theta_i = i$ is oscillatory if there exists an $n_0 \in \mathbb{N}$ such that, for each fixed $i \geq n_0$ and for all $t \in (i-1, i]$, one of the following conditions (a) or (b) is satisfied:

$$
\begin{aligned}
(a) \qquad & k(t) \le 1; \quad k(t) < 1 \quad \text{whenever} \quad r(t) \ne -2; \\
& p(t) \ge 1 + \frac{(r(t)+2)^2}{4(1-k(t))} + \frac{r^2(t)}{4k(t)}; \\
& \tilde{p}_i \le -\,(i+1)\,e^{-1} - i; \\
& p_i \ge (e+1)i^{-1} + 2.
\end{aligned}
$$

$$
\begin{aligned}
(b) \qquad & k(t) \le 1; \\
& p(t) \ge 1 + \frac{r'(t)}{2} + \frac{r^2(t)}{4k(t)}; \\
& \tilde{p}_i \le -\,(i+1)\,e^{-1} - i - 1 - \frac{r(i^+)}{2}; \\
& p_i \ge (e+1)i^{-1} + 1 - \frac{r(i)}{2}.
\end{aligned}
$$

We finally note that it is sometimes possible to exterminate the impulse effects from a differential equation. In our case, if the condition

$$|\tilde{p}_i - p_i| > 2$$

holds, then by substituting

$$x(t) = \xi_i\, z(t), \quad \xi_i = \frac{1}{2}\left\{\tilde{p}_i - p_i \pm \sqrt{(\tilde{p}_i - p_i)^2 - 4}\right\}$$

into equation (1.1), we find that z satisfies the non-selfadjoint differential equation

$$(k(t)z')' + r(t)z' + p(t)z = 0.$$

The oscillatory nature of x and z are the same. However, the restriction imposed is quite severe.

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