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Effect of nonlinear perturbations on second order linear nonoscillatory differential equations

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Abstract. The aim of this paper is to show that any second order nonoscillatory linear differential equation can be converted into an oscillating system by applying a "sufficiently large" nonlinear perturbation. This can be achieved through a detailed analysis of possible nonoscillatory solutions of the perturbed differential equation which may exist when the perturbation is "sufficiently small". As a consequence the class of oscillation-generating perturbations is determined precisely with respect to the original nonoscillatory linear equation.

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1. Introduction

Consider the perturbed second order linear differential equation of the form

$$(p(t)x')' + q(t)x + Q(t)|x|^{\gamma}\operatorname{sgn} x = 0, \tag{A}$$

where γ is a positive constant with $\gamma \neq 1$, and $p(t)$, $q(t)$ and $Q(t)$ are continuous functions on $[a, \infty)$, $a \geqq 0$, such that $p(t) > 0$ and $Q(t) \geqq 0$ for $t \geqq a$. Equation (A) is called *superlinear* or *sublinear* according as $\gamma > 1$ or $\gamma < 1$.

Assume that the second order linear differential equation

$$(p(t)x')' + q(t)x = 0 \tag{B}$$

is nonoscillatory, that is, all of its nontrivial solutions are nonoscillatory. It is natural to expect that (A) inherits the nonoscillatory character from (B) as long as the perturbation $Q(t)|x|^{\gamma}\operatorname{sgn} x$ remains "small", and that application of a "sufficiently large" perturbation might turn (B) into an oscillating system (A), which means the oscillation of all of its

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solutions.

The objective of this paper is to verify the truth of this expectation by presenting the results on nonoscillation and oscillation of (A) implying, respectivley, preservation of nonoscillation and generation of oscillation of (B), which are determined by the convergence or divergence of the integrals

$$\int^{\infty} Q(t)u(t)^{\gamma}v(t)dt \quad \text{and} \quad \int^{\infty} Q(t)u(t)v(t)^{\gamma}dt,$$

where $\{u(t), v(t)\}$ is a fundamental system of solutions of (B) consisting of a (uniquely determined) principal solution $u(t)$ and a non-principal solution $v(t)$.

We recall that (Hartman [4]) $u(t)$ and $v(t)$ satisfy

$$\lim_{t\to\infty} \frac{u(t)}{v(t)} = 0 \tag{1.1}$$

and

$$\int_{t_0}^{\infty} \frac{dt}{p(t)u(t)^2} = \infty, \qquad \int_{t_0}^{\infty} \frac{dt}{p(t)v(t)^2} < \infty, \tag{1.2}$$

and it holds that

$$v(t) \sim cu(t) \int_{t_0}^{t} \frac{ds}{p(s)u(s)^2} \qquad \text{for some constant } c > 0, \tag{1.3}$$

where the symbol $\sim$ denotes the asymptotic equivalence:

$$f(t) \sim g(t) \qquad \Longleftrightarrow \qquad \lim_{t\to\infty} \frac{f(t)}{g(t)} = 1. \tag{1.4}$$

In Section 2, in order to gain useful information about the structure of nonoscillatory solutions of (A), we classify the set of nonoscillatory solutions of (A) into three types according to their asymptotic behavior at infinity. Next, in Section 3 we give sharp conditions for the existence of solutions belonging to these types. Some examples illustrating the nonoscllation theorems obtained are also presented. Finally, in Section 4 we consider the case where the function $Q(t)$ in equation (A) is of alternating sign and show that the results of Section 3 can be extended to this case. Moreover, we establish sharp oscillation criteria for (A) on the basis of oscillation and nonoscillation results due to Belohorec [2] and Kiguradze [6].

2. Classification of positive solutions

In this section we classify the set of all nonoscillatory solutions of (A) according to their asymptotic behavior as $t \to \infty$. It suffices to restrict our attention to the totality of eventually positive solutions of (A) since if $x(t)$ is a solution of (A), then so is $-x(t)$.

The subsequent development is based on the fact that the linear differential operator in equation (B) can be represented as

$$(p(t)x')' + q(t)x = \frac{1}{u(t)} \left(p(t)u(t)^2 \left(\frac{x}{u(t)} \right)' \right)', \tag{2.1}$$

in terms of the principal solution $u(t)$ of (B) which may be assumed to be positive on $[t_0, \infty)$, $t_0 \geqq a$, so that the analysis of the equation (A) is reduced to that of the equation

$$(p(t)u(t)^2y')' + Q(t)u(t)^{\gamma+1}|y|^{\gamma}\mathrm{sgn}\, y = 0. \tag{2.2}$$

In dealing with (2.2) a crucial role is played by the function

$$P(t) = \int_{t_0}^{t} \frac{ds}{p(s)u(s)^2}, \tag{2.3}$$

which satisfies $\lim_{t\to\infty} P(t) = \infty$ (cf. (1.2)).

Let $x(t)$ be a positive solution of (A) on $[t_0, \infty)$, and put $x(t) = u(t)y(t)$. Then, $y(t)$ is a positive solution of (2.2) on $[t_0, \infty)$. It follows from (2.2) that $(p(t)u(t)^2y'(t))' \leqq 0$, $t \geqq t_0$, which shows that $p(t)u(t)^2y'(t)$ is nonincreasing for $t \geqq t_0$. We claim that $y'(t) > 0$, $t \geqq t_0$. In fact, if $y'(t_1) < 0$ for some $t_1 > t_0$, then we have

$$p(t)u(t)^2y'(t) \leqq -p(t_1)u(t_1)^2y'(t_1) = -k < 0 \quad \text{for} \quad t \geqq t_1.$$

Dividing the above inequality by $p(t)u(t)^2$ and integrating it from t_1 to t, we obtain

$$y(t) - y(t_1) \leqq -k\int_{t_1}^{t} \frac{ds}{p(s)u(s)^2} = -k[P(t) - P(t_1)],$$

from which, letting $t \to \infty$, we find that $y(t) \to -\infty$ as $t \to \infty$, a contradiction to the assumed positivity of $y(t)$. Therefore, we must have $y'(t) > 0$ for $t \geqq t_0$, so that $y(t)$ is increasing on $[t_0, \infty)$. Integrating the inequality

$$0 < p(t)u(t)^2y'(t) \leqq p(t_0)u(t_0)y'(t_0) = c, \qquad t \geqq t_0,$$

we get

$$y(t) - y(t_0) \leqq c\int_{t_0}^{t} \frac{ds}{p(s)u(s)^2} = c[P(t) - P(t_0)], \qquad t \geqq t_0,$$

from which it follows that there exist positive constants c_1 and c_2 such that

$$c_1 \leqq y(t) \leqq c_2P(t) \quad \text{for all sufficiently large } t. \tag{2.4}$$

Multiplying (2.4) by $u(t)$ and taking (1.3) into account, we obtain the following result.

Lemma 2.1. *Let $x(t)$ be a positive solution of* (A) *on* $[t_0, \infty)$. *Then there exist positive constants c_1 and c_2 such that*

$$c_1u(t) \leqq x(t) \leqq c_2v(t) \quad \textit{for all sufficiently large } t. \tag{2.5}$$

On the basis of this lemma and in view of (1.1) it is convenient to classify the set of all positive solutions $x(t)$ of (A) into the following three types according to their asymptotic behavior as $t \to \infty$:

(I) $x(t) \sim cu(t)$ for some $c > 0$;

(II) $u(t) \prec x(t) \prec v(t)$;

(III) $x(t) \sim cv(t)$ for some $c > 0$,

where the symbol $\prec$ is used to mean

$$f(t) \prec g(t) \quad \Longleftrightarrow \quad \lim_{t\to\infty} \frac{f(t)}{g(t)} = 0. \tag{2.6}$$

3. Existence of positive solutions

A natural question arises: Is it possible to obtain conditions under which equation (A) possesses a positive solution of the types (I), (II) or (III) defined at the end of Section 2? Our purpose here is to show that an almost complete answer to this question can be provided.

Theorem 3.1. (i) *Equation* (A) *possesses a positive solution* $x(t)$ *such that* $x(t) \sim cu(t)$ *for some* $c > 0$ *if and only if*

$$\int_{t_0}^{\infty} Q(t)u(t)^{\gamma}v(t)dt < \infty. \tag{3.1}$$

(ii) *Equation* (A) *possesses a positive solution* $x(t)$ *such that* $x(t) \sim cv(t)$ *for some* $c > 0$ *if and only if*

$$\int_{t_0}^{\infty} Q(t)u(t)v(t)^{\gamma}dt < \infty. \tag{3.2}$$

Proof: Let $x(t)$ be a positive solution of (A) on $[t_0, \infty)$ and put $y(t) = x(t)/u(t)$. Then, $y(t) > 0$ is a solution of (2.2) for $t \geqq t_0$, and in view of (2.4) $y(t)$ falls into one of the following three types:

(I') $\quad y(t) \sim c \quad$ for some $c > 0$;

(II') $\quad 1 \prec y(t) \prec P(t)$;

(III') $\quad y(t) \sim cP(t) \quad$ for some $c > 0$,

which naturally correspond, respectively, to the types (I), (II) and (III) for the solutions $x(t)$ of (A). Integrating (2.2) on $[t, \infty)$, we have

$$p(t)u(t)^2y'(t) = c' + \int_t^{\infty} Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds, \quad t \geqq t_0, \tag{3.3}$$

where $c' = \lim\limits_{t\to\infty} p(t)u(t)^2y'(t) \in [0, \infty)$. Assume that $c' > 0$. Then, since $y(t)$ is of the type (III'), integrating (3.3) over $[t_0, t]$ gives

$$y(t) = y(t_0) + c'P(t) + \int_{t_0}^{t} \frac{1}{p(s)u(s)^2} \int_s^{\infty} Q(r)u(r)^{\gamma+1}y(r)^{\gamma}drds, \quad t \geqq t_0. \tag{3.4}$$

Assume next that $c' = 0$. Then, (3.4) reduces to

$$y(t) = y(t_0) + \int_{t_0}^{t} \frac{1}{p(s)u(s)^2} \int_s^{\infty} Q(r)u(r)^{\gamma+1}y(r)^{\gamma}drds, \quad t \geqq t_0. \tag{3.5}$$

Put $c = \lim\limits_{t\to\infty} y(t)$. Then, there are two possibilities: either $c = \infty$ or $c \in (0, \infty)$. If $c = \infty$, then (3.5) will be used as an integral equation for $y(t)$ of the type (II'). If $c \in (0, \infty)$, then integration of (3.3) from t to ∞ yields

$$\begin{aligned} y(t) &= c - \int_t^{\infty} \frac{1}{p(s)u(s)^2} \int_s^{\infty} Q(r)u(r)^{\gamma+1}y(r)^{\gamma}drds \\ &= c - \int_t^{\infty} [P(s) - P(t_0)]Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds, \quad t \geqq t_0, \end{aligned} \tag{3.6}$$

which will be used as an integral equation for $y(t)$ of the type (I').

Proof of Statement (i). Let $x(t)$ be a positive solution of (A) on $[t_0, \infty)$ such that $x(t) \sim cu(t)$ for some $c > 0$. Then, $y(t) = x(t)/u(t)$ is a solution of (2.2) such that $y(t) \sim c$, so that $y(t)$ satisfies (3.6) for $t \geqq t_0$. This means in particular that

$$\int_{t_0}^{\infty} [P(t) - P(t_0)]Q(t)u(t)^{\gamma+1}y(t)dt < \infty,$$

which, combined with $y(t) \sim c$, shows that

$$\int_{t_0}^{\infty} [P(t) - P(t_0)]Q(t)u(t)^{\gamma+1}dt < \infty,$$

implying the validity of (3.1). Therefore, (3.1) is a necessary condition for (A) to possess a solution of the type (I).

Assume now that (3.1) is satisfied. Let $c > 0$ be any fixed constant and choose $T \geqq t_0$ sufficiently large so that

$$\int_{T}^{\infty} P(t)Q(t)u(t)^{\gamma+1}dt \leqq \frac{1}{2}c^{1-\gamma}. \tag{3.7}$$

Define the set Y by

$$Y = \left\{ y \in C[T,\infty) : \ \frac{c}{2} \leqq y(t) \leqq c, \quad t \geqq T \right\}, \tag{3.8}$$

which is a closed convex subset of the locally convex space $C[T, \infty)$ of continuous functions on $[T, \infty)$ equipped with the topology of uniform convergence on compact subintervals of $[T, \infty)$.

Consider the integral operator $\mathcal{F}$ defined by

$$\mathcal{F}y(t) = c - \int_{t}^{\infty} [P(s) - P(T)]Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds, \qquad t \geqq T. \tag{3.9}$$

Using (3.7), we have for $y \in Y$

$$\frac{c}{2} \leqq c - c^{\gamma} \int_{T}^{\infty} P(s)Q(s)u(s)^{\gamma+1}ds \leqq \mathcal{F}y(t) \leqq c, \qquad t \geqq T.$$

This shows that $y \in Y$ implies $\mathcal{F}y \in Y$, and hence $\mathcal{F}$ maps Y into itself. Let $\{y_n\}$ be a sequence in Y converging to $y_0 \in Y$ as $n \to \infty$ in $C[T, \infty)$, that is, $y_n(t) \to y_0(t)$ uniformly on compact subintervals of $[T, \infty)$. Then, it can be shown with the help of the Lebesgue dominated convergence theorem that $\mathcal{F}y_n(t) \to \mathcal{F}y_0(t)$ on any compact subinterval of $[T, \infty)$, which means that $\mathcal{F}y_n \to \mathcal{F}y_0$ in $C[T, \infty)$. This implies that $\mathcal{F}$ is a continuous mapping. It is clear that the set $\mathcal{F}(Y)$ is uniformly bounded on $[T, \infty)$. Differentiating (3.9), we see that

$$0 < (\mathcal{F}y)'(t) = [P(t) - P(T)]Q(t)u(t)^{\gamma+1}y(t)^{\gamma} \leqq c^{\gamma}P(t)Q(t)u(t)^{\gamma+1}, \tag{3.10}$$

for $t \geqq T$, which means that $\mathcal{F}(Y)$ is locally equicontinuous on $[T, \infty)$. From the Ascoli-Arzela lemma it then follows that $\mathcal{F}(Y)$ is relatively compact in $C[T, \infty)$. Thus all the hypotheses of the Schauder-Tychonoff fixed point theorem are fulfilled for $\mathcal{F}$, and we conclude that there exists $y \in Y$ such that $y = \mathcal{F}y$, which is the integral equation (3.6) with t_0 replaced by T. Differentiating this integral equation, we see that $y = y(t)$ is a solution of equation (B) of the type (I') such that $y(t) \sim c$ as $t \to \infty$. The function $x(t) = u(t)y(t)$ provides a solution of (A) on $[T, \infty)$ such that $x(t) \sim cu(t)$ as $t \to \infty$. It follows that (3.1) is a sufficient condition for the existence of a type (I)-solution of equation (A). This completes the proof of the first statement of Theorem 3.1.

Proof of Statement (ii). Let $x(t)$ be a positive solution of (A) such that $x(t) \sim cv(t)$ for some $c > 0$. Put $y(t) = x(t)/u(t)$. Then, it is a solution of (2.2) such that $y(t) \sim cP(t)$, and so $y(t)$ satisfies (3.4), which means in particular that

$$\int_{t_0}^{\infty} Q(t)u(t)^{\gamma+1}y(t)^{\gamma}dt < \infty.$$

Using $y(t) \sim cv(t)$ in the above inequality, we obtain

$$\int_{t_0}^{\infty} P(t)^{\gamma}Q(t)u(t)^{\gamma+1}dt < \infty,$$

which is equivalent to (3.2).

Assume that (3.2) is satisfied. Let $c > 0$ be any fixed constant and choose $T \geqq t_0$ so that

$$\int_{T}^{\infty} P(t)^{\gamma}Q(t)u(t)^{\gamma+1}dt \leqq 2^{-\gamma}c^{1-\gamma}. \tag{3.11}$$

Define the integral operator $\mathcal{G}$ and the set Z by

$$\mathcal{G}z(t) = c\int_{T}^{t} \frac{ds}{p(s)u(s)^2} + \int_{T}^{t} \frac{1}{p(s)u(s)^2}\int_{s}^{\infty} Q(r)u(r)^{\gamma+1}z(r)^{\gamma}drds, \quad t \geqq T, \tag{3.12}$$

and

$$Z = \{z \in C[T, \infty) : c[P(t) - P(T)] \leqq z(t) \leqq 2c[P(t) - P(T)],\ t \geqq T\}. \tag{3.13}$$

Then it can be verified without difficulty that $\mathcal{G}$ is a self-map on Z and sends Z into a relatively compact subset of $C[T, \infty)$. Therefore, by the Schauder-Tychonoff fixed point theorem there exists a fixed point $z \in Z$ of $\mathcal{G}$, which satisfies

$$z(t) = c\int_{T}^{t} \frac{ds}{p(s)u(s)^2} + \int_{T}^{t} \frac{1}{p(s)u(s)^2}\int_{s}^{\infty} Q(r)u(r)^{\gamma+1}z(r)^{\gamma}drds, \quad t \geqq T,$$

from which it readily follows that $z(t)$ is a solution of the differential equation (2.2) such that $z(t) \sim c'P(t)$ for some $c' \in [c, 2c]$. The function $x(t) = u(t)z(t)$ then gives a solution of equation (A) such that $x(t) \sim c''v(t)$ for some $c'' > 0$. This completes the proof of Theorem 3.1.

Remark 3.1. The conclusion of Theorem 3.1 holds true for any value of γ including $\gamma = 1$. Note that if $\gamma = 1$, the two conditions (3.1) and (3.2) degenerate into the single condition

$$\int_{t_0}^{\infty} Q(t)u(t)v(t)dt < \infty.$$

It remains to study the existence or nonexistence of type (II)-solutions of equation (A). In the case where (A) is sublinear a necessary and sufficient condition for the existence of such solutions can be given as the following theorem shows.

Theorem 3.2. *Let* $0 < \gamma < 1$. *Equation* (A) *possesses a positive solution* $x(t)$ *such that* $u(t) \prec x(t) \prec v(t)$ *if and only if* (3.2) *holds and*

$$\int_{t_0}^{\infty} Q(t)u(t)^{\gamma}v(t)dt = \infty. \tag{3.14}$$

Proof: (The "only if"part) Suppose that (A) has a solution $x(t)$ such that $u(t) \prec x(t) \prec v(t)$ as $t \to \infty$. Put $y(t) = x(t)/u(t)$. Then, $y(t)$ is a solution of (2.2) such that $1 \prec y(t) \prec P(t)$, and it satisfies (3.5) which can be expressed as

$$y(t) = y(t_0) + \int_{t_0}^{t} [P(s) - P(t_0)]Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds + [P(t) - P(t_0)] \int_{t}^{\infty} Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds. \tag{3.15}$$

Note that the following two inequalities follow from (3.15):

$$y(t) \geqq \int_{t_0}^{t} [P(s) - P(t_0)]Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds, \quad t \geqq t_0, \tag{3.16}$$

$$y(t) \geqq [P(t) - P(t_0)] \int_{t}^{\infty} Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds, \quad t \geqq t_0. \tag{3.17}$$

Using (3.3) with $c' = 0$, we transform (3.15) into

$$y(t) = y(t_0) + [P(t) - P(t_0)]p(t)u(t)^2y'(t) + \int_{t_0}^{t} [P(s) - P(t_0)]Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds. \tag{3.18}$$

Put $\Phi(t) = \int_t^{\infty} Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds$. Then (3.17) is reduced to

$$-\Phi(t)^{-\gamma}\Phi'(t) \geqq [P(t) - P(t_0)]^{\gamma}Q(t)u(t)^{\gamma+1}, \quad t \geqq t_0.$$

Integrating the above from t_0 to t, we have

$$\int_{t_0}^{t} [P(s) - P(t_0)]^{\gamma} Q(s)u(s)^{\gamma+1}ds \leqq \frac{1}{1-\gamma}\Phi(t_0)^{1-\gamma}, \quad t \geqq t_0,$$

which clearly implies (3.2). Thus, (3.2) is a necessary condition for the sublinear equation (A) to possess a solution of the type (II). To prove the necessity of (3.14) we proceed as follows. Suppose to the contrary that

$$\int_{t_0}^{\infty} Q(t)u(t)^{\gamma}v(t)dt < \infty, \text{ or equivalently } \int_{t_0}^{\infty} P(t)Q(t)u(t)^{\gamma+1}dt < \infty.$$

Choose t_0 so large that $y(t) \geqq 1$ for $t \geqq t_0$ and

$$\int_{t_0}^{\infty} P(t)Q(t)u(t)^{\gamma+1}dt \leqq \frac{1}{2}. \tag{3.19}$$

Using (3.18) and (3.19), we see that

$$y(t) = y(t_0) + [P(t) - P(t_0)]p(t)u(t)^2y'(t) + \int_{t_0}^{t}[P(s) - P(t_0)]Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds$$

$$\leqq y(t_0) + P(t)p(t)u(t)^2y'(t) + y(t)\int_{t_0}^{t} P(s)Q(s)u(s)^{\gamma+1}ds$$

$$\leqq y(t_0) + P(t)p(t)u(t)^2y'(t) + \frac{1}{2}y(t), \quad t \geqq t_0.$$

This implies that $\liminf_{t\to\infty} P(t)p(t)u(t)^2y'(t)/y(t) \geqq 1/2$, so that there exist a $t_1 \geqq t_0$ and a positive constant k such that

$$y(t) \leqq kP(t)p(t)u(t)^2y'(t) \quad \text{for} \ \ t \geqq t_1. \tag{3.20}$$

In view of (3.3) (with $c' = 0$) and the decreasing nature of $p(t)u(t)^2y'(t)$, we obtain

$$p(t)u(t)^2y'(t) = \int_t^{\infty} Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds \leqq \int_t^{\infty} Q(s)u(s)^{\gamma+1}y(s)ds$$

$$\leqq k\int_t^{\infty} Q(s)u(s)^{\gamma+1}P(s)p(s)u(s)^2y'(s)ds$$

$$\leqq kp(t)u(t)^2y'(t)\int_t^{\infty} P(s)Q(s)u(s)^{\gamma+1}ds, \quad t \geqq t_1,$$

from which we conclude that

$$1 \leqq k\int_t^{\infty} P(s)Q(s)u(s)^{\gamma+1}ds, \quad t \geqq t_1.$$

This is a contradiction, and so (3.14) must be satisfied. Hence (3.14) is also necessary for (A) to have a positive solution of the type (II) (We note that the above argument is an extended adaptation of the one given in [8]).

(The "if" part) Assume that (3.2) and (3.14) hold. Let $c > 0$ be any fixed constant and choose $T \geqq t_0$ such that

$$\int_T^{\infty} [1 + P(t)]^{\gamma}Q(t)u(t)^{\gamma+1}dt \leqq c^{1-\gamma}. \tag{3.21}$$

Let W denote the set

$$W = \{w \in C[T, \infty) : \ c \leqq w(t) \leqq c[1 + P(t)], \ \ t \geqq T\} \tag{3.22}$$

and define the integral operator $\mathcal{H}$ by

$$\mathcal{H}w(t) = c + \int_T^{t} \frac{1}{p(s)u(s)^2}\int_s^{\infty} Q(r)u(r)^{\gamma+1}w(r)^{\gamma}drds, \quad t \geqq T. \tag{3.23}$$

It can be verified routinely that $\mathcal{H}$ maps W continuously into a relatively compact subset of W. By the Schauder-Tychonoff theorem $\mathcal{H}$ has a fixed point w in W. From the integral equation $w = \mathcal{H}w$ we see that $w(t)$ is a positive solution of (2.2) on $[T, \infty)$ and satisfies

$\lim_{t\to\infty} p(t)u(t)^2 w'(t) = 0$, which implies that $\lim_{t\to\infty} w(t)/P(t) = 0$. On the other hand, using (3.16), we have

$$w(t) \geqq \int_T^t [P(s) - P(T)]Q(s)u(s)^{\gamma+1}w(s)^\gamma ds \geqq c^\gamma \int_T^t [P(s) - P(T)]Q(s)u(s)^{\gamma+1}ds, \quad t \geqq T,$$

which, combined with (3.14), implies that $\lim_{t\to\infty} w(t) = \infty$. Accordingly, the solution $w(t)$ of equation (2.2) satisfies $1 \prec w(t) \prec P(t)$, and so the function $x(t) = u(t)w(t)$ is a solution of (A) satisfying $u(t) \prec x(t) \prec v(t)$. This completes the proof of Theorem 3.2.

Remark 3.2. The problem of characterizing the existence of a solution of type (II) for the superlinear case ($\gamma > 1$) of (A) seems to be difficult. In this case we conjecture that (A) has such a solution if and only if (3.1) holds and

$$\int_{t_0}^{\infty} Q(t)u(t)v(t)^\gamma dt = \infty, \tag{3.24}$$

but we have so far been able to prove the “only if”part of the conjecture.

Proposition 3.1. *Let $\gamma > 1$. If equation* (A) *has a positive solution $x(t)$ such that $u(t) \prec x(t) \prec v(t)$ as $t \to \infty$, then* (3.1) *and* (3.24) *are satisfied.*

Proof: Let $x(t)$ be a solution of (A) such that $u(t) \prec x(t) \prec v(t)$. Then, $y(t) = x(t)/u(t)$ is a solution of (2.2) such that $1 \prec y(t) \prec P(t)$, and it satisfies (3.16). Let $\Psi(t)$ denote the right hand side of (3.16):

$$\Psi(t) = \int_{t_0}^t [P(s) - P(t_0)]Q(s)u(s)^{\gamma+1}y(s)^\gamma ds, \quad t \geqq t_0.$$

Then, it satisfies

$$\Psi(t)^{-\gamma}\Psi'(t) \geqq [P(t) - P(t_0)]Q(t)u(t)^{\gamma+1}, \quad t \geqq t_0,$$

from which it readily follows that

$$\int_{t_0}^t [P(s) - P(t_0)]Q(s)u(s)^{\gamma+1}ds \leqq \frac{1}{\gamma-1}\Psi(t_0)^{1-\gamma}, \quad t \geqq t_0.$$

This implies that $\int_{t_0}^\infty P(t)Q(t)u(t)^{\gamma+1}dt < \infty$, which is equivalent to (3.1).

The truth of (3.24) is proved by reductio ad absurdum. Assume that (3.24) fails to holds, that is, $\int_{t_0}^\infty P(t)^\gamma Q(t)u(t)^{\gamma+1}dt < \infty$. Choose t_0 large enough so that

$$\frac{y(t)}{P(t)} \leqq 1 \quad \text{for } t \geqq t_0 \quad \text{and} \quad \int_{t_0}^\infty P(t)^\gamma Q(t)u(t)^{\gamma+1}dt \leqq \frac{1}{2}. \tag{3.25}$$

From (3.18) and (3.25) we see that

$$
\begin{aligned}
y(t) &\leqq y(t_0) + P(t)p(t)u(t)^2y'(t) + \int_{t_0}^{t} P(s)Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds \\
&\leqq y(t_0) + P(t)p(t)u(t)^2y'(t) + \int_{t_0}^{t} P(s)^{\gamma}Q(s)u(s)^{\gamma+1}\left(\frac{y(s)}{P(s)}\right)^{\gamma-1} y(s)ds \\
&\leqq y(t_0) + P(t)p(t)u(t)^2y'(t) + y(t)\int_{t_0}^{t} P(s)^{\gamma}Q(s)u(s)^{\gamma+1}ds \\
&\leqq y(t_0) + P(t)p(t)u(t)^2y'(t) + \frac{1}{2}y(t), \quad t \geqq t_0,
\end{aligned}
$$

whence we obtain $\liminf_{t\to\infty} P(t)p(t)u(t)^2y'(t)/y(t) \geqq 1/2$, so that there exist $t_1 \geqq t_0$ and $l > 0$ such that

$$
y(t) \leqq lP(t)p(t)u(t)^2y'(t) \quad \text{for } t \geqq t_1. \tag{3.26}
$$

Since $\lim_{t\to\infty} p(t)u(t)^2y'(t) = 0$, we may assume that

$$
p(t)u(t)^2y'(t) \leqq 1 \quad \text{for } t \geqq t_1. \tag{3.27}
$$

Then, from (3.3) with $c' = 0$ combined with (3.26) and (3.27) we see that

$$
\begin{aligned}
p(t)u(t)^2y'(t) &= \int_{t}^{\infty} Q(s)u(s)^{\gamma+1}y(s)^{\gamma}ds \\
&\leqq l^{\gamma}\int_{t}^{\infty} P(s)^{\gamma}Q(s)u(s)^{\gamma+1}(p(s)u(s)^2y'(s))^{\gamma}ds \\
&\leqq l^{\gamma}\int_{t}^{\infty} P(s)^{\gamma}Q(s)u(s)^{\gamma+1}(p(s)u(s)^2y'(s))ds \\
&\leqq l^{\gamma}p(t)u(t)^2y'(t)\int_{t}^{\infty} P(s)^{\gamma}Q(s)u(s)^{\gamma+1}ds, \quad t \geqq t_1.
\end{aligned}
$$

Accordingly, we have

$$
1 \leqq l^{\gamma}\int_{t}^{\infty} P(s)^{\gamma}Q(s)u(s)^{\gamma+1}ds, \quad t \geqq t_1,
$$

which is a contradiction. This shows that (3.24) must be satisfied.

We present examples illustrating Theorems 3.1 and 3.2.

Example 3.1. Consider the generalized Airy equation with nonlinear perturbation

$$
x'' - k^2t^{2\rho}x + Q(t)|x|^{\gamma}\operatorname{sgn} x = 0, \tag{3.28}
$$

where $k > 0$ is a constant and 2ρ is a positive integer, and $Q(t)$ is a nonnegative continuous function on $[0, \infty)$. It is known that the unperturbed Airy equation

$$
x'' - k^2t^{2\rho}x = 0, \tag{3.29}
$$

is nonoscillatory (Sirovich [9]) and has a fundamental set of solutions $\{u(t), v(t)\}$ such that

$$u(t) \sim t^{-\frac{\rho}{2}} \exp\left(-\frac{kt^{\rho+1}}{\rho+1}\right), \quad v(t) \sim t^{-\frac{\rho}{2}} \exp\left(\frac{kt^{\rho+1}}{\rho+1}\right). \tag{3.30}$$

From Theorem 3.1 it follows that (3.28) has a solution $x(t)$ such that

$$x(t) \sim ct^{-\frac{\rho}{2}} \exp\left(-\frac{kt^{\rho+1}}{\rho+1}\right) \quad \text{for some } c > 0 \tag{3.31}$$

if and only if

$$\int_1^\infty t^{-\frac{1}{2}(\gamma+1)\rho} \exp\left(-\frac{(\gamma-1)kt^{\rho+1}}{\rho+1}\right) Q(t)dt < \infty, \tag{3.32}$$

and that (3.28) has a solution $x(t)$ such that

$$x(t) \sim ct^{-\frac{\rho}{2}} \exp\left(\frac{kt^{\rho+1}}{\rho+1}\right) \quad \text{for some } c > 0 \tag{3.33}$$

if and only if

$$\int_1^\infty t^{-\frac{1}{2}(\gamma+1)\rho} \exp\left(\frac{(\gamma-1)kt^{\rho+1}}{\rho+1}\right) Q(t)dt < \infty. \tag{3.34}$$

Applying Theorem 3.2, we conclude that the sublinear equation (3.28) ($0 < \gamma < 1$) possesses a solution $x(t)$ such that

$$t^{-\frac{\rho}{2}} \exp\left(\frac{-kt^{\rho+1}}{\rho+1}\right) \prec x(t) \prec t^{-\frac{\rho}{2}} \exp\left(\frac{kt^{\rho+1}}{\rho+1}\right), \tag{3.35}$$

if and only if (3.34) holds and

$$\int_1^\infty t^{-\frac{1}{2}(\gamma+1)\rho} \exp\left(-\frac{(\gamma-1)kt^{\rho+1}}{\rho+1}\right) Q(t)dt = \infty. \tag{3.36}$$

Example 3.2. Consider the perturbed Euler differential equation

$$x'' + \left(\frac{1}{4}\sum_{k=0}^{n-1} \frac{1}{L_k(t)^2}\right) x = 0, \tag{3.37}$$

where the sequence of functions $\{L_n(t)\}$ and the sequence of numbers $\{e_n\}$ are defined by

$$L_0(t) = t, \quad L_n(t) = L_{n-1}(t) \log_n t, \quad n = 1, 2, \cdots, \tag{3.38}$$

and

$$e_0 = 1, \quad e_n = \exp(e_{n-1}), \quad n = 1, 2, \cdots. \tag{3.39}$$

It is known (Hille [5], Swanson [10]) that equation (3.37) has a linearly independent positive solutions

$$u(t) = L_{n-1}(t)^{\frac{1}{2}}, \quad v(t) = L_{n-1}(t)^{\frac{1}{2}} \log_n t. \tag{3.40}$$

From Theorem 3.1 applied to the nonlinear perturbation of (3.37)

$$x'' + \left(\frac{1}{4} \sum_{k=0}^{n-1} \frac{1}{L_k(t)^2} \right) x + Q(t)|x|^\gamma \mathrm{sgn}\, x = 0, \tag{3.41}$$

it follows that (3.41) has a solution $x(t)$ such that $x(t) \sim cL_{n-1}(t)^{\frac{1}{2}}$ for some $c > 0$ if and only if

$$\int_{e_n}^{\infty} L_{n-1}(t)^{\frac{1}{2}(\gamma+1)} (\log_n t) Q(t) dt < \infty, \tag{3.42}$$

and that (3.41) has a solution $x(t)$ such that $x(t) \sim cL_{n-1}(t)^{\frac{1}{2}} \log_n t$ for some $c > 0$ if and only if

$$\int_{e_n}^{\infty} L_{n-1}(t)^{\frac{1}{2}(\gamma+1)} (\log_n t)^\gamma Q(t) dt < \infty. \tag{3.43}$$

Applying Theorem 3.2, we see that the sublinear equation (3.41) with $0 < \gamma < 1$ possesses a solution $x(t)$ such that $L_{n-1}(t)^{\frac{1}{2}} \prec x(t) \prec L_{n-1}(t)^{\frac{1}{2}} \log_n t$ if and only if (3.43) holds and

$$\int_{e_n}^{\infty} L_{n-1}(t)^{\frac{1}{2}(\gamma+1)} (\log_n t) Q(t) dt = \infty. \tag{3.44}$$

For the equation

$$x'' + \frac{x}{4t^2} + \frac{|x|^\gamma \mathrm{sgn}\, x}{t^{\frac{1}{2}(\gamma+3)} (\log t)^2 (\log_2 t)^\gamma} = 0, \quad 0 < \gamma < 1, \tag{3.45}$$

wihch is a special case of (3.41) with $n = 1$ and $Q(t) = 1/t^{\frac{1}{2}(\gamma+3)} (\log t)^2 (\log_2 t)^\gamma$, both (3.43) and (3.44) are fulfilled because $L_0(t) = t$ and

$$L_0(t)^{\frac{1}{2}(\gamma+1)} (\log t) Q(t) = \frac{1}{t \log t (\log_2 t)^\gamma}, \tag{3.46}$$

$$L_0(t)^{\frac{1}{2}(\gamma+1)} (\log t)^\gamma Q(t) = \frac{1}{t (\log t)^{2-\gamma} (\log_2 t)^\gamma}. \tag{3.47}$$

Therefore, (3.45) possesses a solution $x(t)$ such that $t^{\frac{1}{2}} \prec x(t) \prec t^{\frac{1}{2}} \log t$. In fact, (3.45) possesses one such solution $x(t) = t^{\frac{1}{2}} \log_2 t$.

Remark 3.3. We notice that $x(t) = t^{\frac{1}{2}} \log_2 t$ is also a solution of (3.45) in the case $\gamma > 1$. This example suggests that the conjecture stated in Remark 3.2 would be true because in this case

$$\int_e^{\infty} L_0(t)^{\frac{1}{2}(\gamma+1)} (\log t) Q(t) dt < \infty$$

and

$$\int_e^{\infty} L_0(t)^{\frac{1}{2}(\gamma+1)} (\log t)^\gamma Q(t) dt = \infty.$$

4. Oscillation criteria and remarks

We have assumed so far that the function $Q(t)$ in equation (A) is nonnegative. A question naturally arises: To what extent does a perturbation $Q(t)|x|^{\gamma}\operatorname{sgn} x$ with $Q(t)$ of alternating sign affect the oscillatory behavior of equation (B)? The aim of this section is to give an affirmative answer to this question by showing that necessary and sufficient conditions for all nontrivial solutions of (A) to be oscillatory can be established to the case of sign-changing $Q(t)$ provided the positive part $Q_+(t) = \frac{1}{2}(|Q(t)| + Q(t))$ of $Q(t)$ is much larger than its negative part $Q_-(t) = \frac{1}{2}(|Q(t)| - Q(t))$ in some sense. Our arguments are based on the oscillation and nonoscillation theorems (Propositions 4.1 – 4.3 below) for differential equations of the form

$$\ddot{y} + h(s)|y|^{\gamma}\operatorname{sgn} y = 0 \qquad \left(\cdot = \frac{d}{ds}\right), \tag{C}$$

where $h : [a, \infty) \to \mathbb{R}$ is a continuous function which may be of alternating or variable sign.

Proposition 4.1. *Assume that $h : [a, \infty) \to \mathbb{R}$ is a continuous function.*
(i) *Let $\gamma > 1$. All nontrivial solutions of* (C) *are oscillatory if*

$$\int_a^{\infty} s h(s) ds = \infty. \tag{4.1}$$

(ii) *Let $0 < \gamma < 1$. All nontrivial solutions of* (C) *are oscillatory if*

$$\int_a^{\infty} s^{\gamma} h(s) ds = \infty. \tag{4.2}$$

Proposition 4.2. *Let $\gamma > 0$, $c \neq 0$, and $k \in \{0, 1\}$. Then the condition*

$$\int_a^{\infty} s^{1-k+\gamma k}|h(s)| ds < \infty \tag{4.3}$$

is sufficient, and in the case of constant sign for $h(s)$ is necessary for (C) *to have a solution $y(s)$ admitting the asymptotic representation*

$$y(s) = cs^k(1 + o(1)), \qquad t \to \infty. \tag{4.4}$$

The above results are due to Kiguradze [6] and Belohorec [2] (see also [7], Theorems 18.1 and 18.2) and give rise to the following proposition.

Proposition 4.3. *Let $\gamma > 1$ (respectively $0 < \gamma < 1$), and let $h : [a, \infty) \to \mathbb{R}$ be a continuous function such that*

$$\int_a^{\infty} s(|h(s)| - h(s)) ds < \infty \qquad \left(\text{respectively} \quad \int_a^{\infty} s^{\gamma}(|h(s)| - h(s)) ds < \infty\right). \tag{4.5}$$

Then, for all nontrivial solutions of (C) *to be oscillatory, it is necessary and sufficient that the condition* (4.1) (*respectively* (4.2)) *be fulfilled.*

Let $\{u(t), v(t)\}$ be a fundamental system of solutions of (B) consisting of a principal solution $u(t)$ and a non-principal solution $v(t)$ defined by

$$v(t) = u(t) \int_{t_0}^{t} \frac{ds}{p(s)u(s)^2}.$$

Then, it can be shown that the change of variables

$$x(t) = u(t)y(s), \qquad s = \frac{v(t)}{u(t)} \tag{4.6}$$

transforms equation (A) into the differential equation of the form (C) with $h(s)$ given by

$$h(s) = p(t)u(t)^{\gamma+3}Q(t). \tag{4.7}$$

Let us now apply Proposition 4.1 to equation (C) with $h(s)$ given by (4.7). Then, noting that (4.1) and (4.2) are equivalent to

$$\int_{t_0}^{\infty} Q(t)u(t)^{\gamma}v(t)dt = \infty \quad \text{and} \quad \int_{t_0}^{\infty} Q(t)u(t)v(t)^{\gamma}dt = \infty,$$

respectively, we obtain the following oscillation result for equation (A) with sign-changing $Q(t)$.

Theorem 4.1. (i) *Let* $\gamma > 1$. *If*

$$\int_{t_0}^{\infty} Q(t)u(t)^{\gamma}v(t)dt = \infty, \tag{4.8}$$

then, all nontrivial solutions of (A) *are oscillatory.*
(ii) *Let* $0 < \gamma < 1$. *If*

$$\int_{t_0}^{\infty} Q(t)u(t)v(t)^{\gamma}dt = \infty, \tag{4.9}$$

then, all nontrivial solutions of (A) *are oscillatory.*

Likewise, from Proposition 4.2 applied to equation (C) − (4.7) we obtain the following nonoscillation result.

Theorem 4.2. (i) *Let* $\gamma > 1$. *If*

$$\int_{t_0}^{\infty} |Q(t)|u(t)^{\gamma}v(t)dt < \infty, \tag{4.10}$$

then, equation (A) *has a solution* $x(t)$ *such that* $x(t) \sim cu(t)$ *for some* $c > 0$.
(ii) *Let* $0 < \gamma < 1$. *If*

$$\int_{t_0}^{\infty} |Q(t)|u(t)v(t)^{\gamma}dt < \infty, \tag{4.11}$$

then, equation (A) *has a solution* $x(t)$ *such that* $x(t) \sim cv(t)$ *for some* $c > 0$.

Finally, combining Proposition 4.3 with equation (C) − (4.7), we are able to indicate a class of equations of the form (A) with sign-changing $Q(t)$ for which the situation for oscillation of all nontrivial solutions is completely characterized.

Theorem 4.3. *Let $\{u(t), v(t)\}$ be a fundamental system of solutions of* (B) *consisting of a principal solution $u(t)$ and a non-principal solution $v(t)$ defined by*

$$v(t) = u(t) \int_{t_0}^{t} \frac{ds}{p(s)u(s)^2}.$$

(i) *Let $\gamma > 1$. Suppose that*

$$\int_{t_0}^{\infty} (|Q(t)| - Q(t))u(t)^{\gamma} v(t) dt < \infty. \tag{4.12}$$

Then, all nontrivial solutions of equation (A) *are oscillatory if and only if* (**??**) *is satisfied.*
(ii) *Let $0 < \gamma < 1$. Suppose that*

$$\int_{t_0}^{\infty} (|Q(t)| - Q(t))u(t)v(t)^{\gamma} dt < \infty. \tag{4.13}$$

Then, all nontrivial solutions of equation (A) *are oscillatory if and only if* (**??**) *is satisfied.*

In connection with part (i) of Theorem 4.3 we note that condition (4.12) is equivalent to

$$\int_{t_0}^{\infty} Q_-(t)u(t)^{\gamma} v(t) dt < \infty, \tag{4.14}$$

which means that $Q_-(t)$ is sufficiently small. On the other hand, the condition (**??**) can be rewritten as

$$\int_{t_0}^{\infty} (Q_+(t) - Q_-(t))u(t)^{\gamma} v(t) dt = \infty,$$

which, in view of (4.14), implies that

$$\int_{t_0}^{\infty} Q_+(t)u(t)^{\gamma} v(t) dt = \infty.$$

This can be the ground for asserting that $Q_+(t)$ is sufficiently large.

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