

A PRIORI ESTIMATE FOR DISCONTINUOUS SOLUTIONS OF A SECOND ORDER LINEAR HYPERBOLIC PROBLEM

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Abstract. *In the paper we investigate a non-local contact-boundary value problem for a system of second order hyperbolic equations with discontinuous solutions. Under some conditions on input data a priori estimate is obtained for the solution of this problem.*

In the paper we consider the following hyperbolic system:

$$\begin{aligned} (Lz)(t, x) &\equiv z_{tx}(t, x) + z(t, x) A_{0,0}(t, x) + \\ &+ z_t(t, x) A_{1,0}(t, x) + z_x(t, x) A_{0,1}(t, x) = \varphi(t, x), \end{aligned} \quad (1)$$

$$(t, x) \in G = G_0 \cup G_1, \quad G_0 = (0, T) \times (0, \alpha), \quad G_1 = (0, T) \times (\alpha, l),$$

where $z(t, x) = (z_1(t, x), \dots, z_n(t, x))$ is the desired vector-function; $A_{i,j}(t, x)$, $i, j = 0, 1$ are the given $n \times n$ -matrices on G ; $\varphi(t, x)$ is the given n -dimensional vector-function on G ; α is a fixed point from $(0, l)$.

For the system (1) we give the following non-local contact boundary conditions

$$\begin{aligned} (L_k z)(t) &\equiv z(t, 0) \beta_{0,k}(t) + z(t, \alpha - 0) \beta_{1,k}(t) + z(t, \alpha + 0) \beta_{2,k}(t) + \\ &+ z(t, l) \beta_{3,k}(t) + z_t(t, 0) g_{0,k}(t) + z_t(t, \alpha - 0) g_{1,k}(t) + z_t(t, \alpha + 0) g_{2,k}(t) + \\ &+ z_t(t, l) g_{3,k}(t) = \varphi_k(t), \quad t \in (0, T), \quad k = 1, 2; \end{aligned} \quad (2)$$

$$(L_3 z)(x) \equiv z_x(0, x) = \varphi_3(x), \quad x \in (0, l); \quad (3)$$

$$L_0 z \equiv z(0, 0) = \varphi_0. \quad (4)$$

Here: $\beta_{i,k}(t)$, $g_{i,k}(t)$, $i = 0, 1, 2, 3$; $k = 1, 2$ are the given $n \times n$ matrices on $(0, T)$; $\varphi_k(t)$, $k = 1, 2$ are the given n -dimensional vector-functions on $(0, T)$; $\varphi_3(x)$ is the given n -dimensional vector-function on $(0, l)$; φ_0 is the given constant n -dimensional vector.

We assume that the following conditions are satisfied:

1) The matrices $A_{i,j}(t,x)$ are measurable on G , $A_{0,0} \in \mathcal{L}_{p,n \times n}(G)$; there exist the functions $A_{1,0}^0 \in \mathcal{L}_p(0,l)$ and $A_{0,1}^0 \in \mathcal{L}_p(0,T)$, such that $\|A_{1,0}(t,x)\| \leq A_{1,0}^0(x)$, $\|A_{0,1}(t,x)\| \leq A_{0,1}^0(t)$ almost everywhere on G , where $\mathcal{L}_{p,n \times n}(G)$, $1 \leq p \leq \infty$ is a Banach space of $n \times n$ matrices $g = (g_{ij})$ with elements $g_{ij} \in \mathcal{L}_p(G)$, wherein the norm is determined by the equality $\|g\|_{\mathcal{L}_{p,n \times n}(G)} = \|g^0\|_{\mathcal{L}_p(G)}$, moreover $g^0 = \|g\| \equiv \sum_{i,j=1}^n |g_{ij}|$ is the norm of the matrix g ;

2) $\beta_{i,k} \in \mathcal{L}_{p,n \times n}(0,T)$ and $g_{i,k} \in \mathcal{L}_{\infty,n \times n}(0,T)$;

3) $\varphi \in \mathcal{L}_{p,n}(G)$, $\varphi_k \in \mathcal{L}_{p,n}(0,T)$, $\varphi_3 \in \mathcal{L}_{p,n}(0,l)$, where $\mathcal{L}_{p,n}(G)$, $1 \leq p \leq \infty$, is a space of n -dimensional vector-functions $\varphi = (\varphi_1, \dots, \varphi_n)$ with elements from $\mathcal{L}_p(G)$; norm of $\varphi \in \mathcal{L}_{p,n}(G)$ is defined as $\|\varphi\|_{\mathcal{L}_{p,n}(G)} = \|\varphi^0\|_{\mathcal{L}_p(G)}$ and $\varphi^0(t,x) = \|\varphi(t,x)\| = \sum_{i=1}^n |\varphi_i(t,x)|$ is norm of n -vectors $\varphi(t,x) \in R^n$ for fixed $(t,x) \in G$. R^n is the space of all vectors $\rho = (\rho_1, \dots, \rho_n)$ with norm $\|\rho\| = \sum_{i=1}^n |\rho_i|$.

Non-local boundary value problems for integro-differential equations with continuous coefficients were studied in the paper [2].

We'll consider the solution of problem (1)-(4) in the space $\widehat{W}_{p,n}(G)$, $1 \leq p \leq \infty$, [4] (p. 52) of all n -dimensional vector-functions $z(t,x)$, which on each domain G_k ($k = 0, 1$) belong to $W_{p,n}(G_k)$ and are continuous at the point $(0, \alpha)$. Here $W_{p,n}(G_k)$ is a space of all n -dimensional vector-functions $z \in \mathcal{L}_{p,n}(G_k)$, possessing generalized in S.L.Sobolev's sense derivatives z_t, z_x and z_{tx} from $\mathcal{L}_{p,n}(G_k)$, $k = 0, 1$. We'll define the norm in the space $\widehat{W}_{p,n}(G)$ by the equality [4] (p. 54)

$$\|z\|_{\widehat{W}_{p,n}(G)} = \sum_{k=0}^1 \|z\|_{W_{p,n}(G_k)},$$

where

$$\|z\|_{W_{p,n}(G_k)} = \|z\|_{\mathcal{L}_{p,n}(G_k)} + \|z_t\|_{\mathcal{L}_{p,n}(G_k)} + \|z_x\|_{\mathcal{L}_{p,n}(G_k)} + \|z_{tx}\|_{\mathcal{L}_{p,n}(G_k)}.$$

Since the operator $Nz = (z(0,0), z_t(t,0), z_t(t,\alpha+0), z_x(0,x), z_{tx}(t,x))$, brings about isomorphism from $\widehat{W}_{p,n}(G)$ to $\widehat{Q}_{p,n} = R^n \times \mathcal{L}_{p,n}(0,T) \times \mathcal{L}_{p,n}(0,T) \times \mathcal{L}_{p,n}(0,l) \times \mathcal{L}_{p,n}(G)$, [1], we can reduce problem (1) - (4) to the following operator equation

$$\widehat{L}z = \widehat{\varphi},$$

where $\widehat{L} = (L_0, L_1, L_2, L_3, L)$, $z \in \widehat{W}_{p,n}(G)$ is desired solution and $\widehat{\varphi} = (\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi) \in \widehat{Q}_{p,n}$ is the given element. This equation is equiva-

lent to the system of integro-algebraic equations with respect to elements of five components

$$\begin{aligned} \widehat{b} &= (b_0, b_1(t), b_2(t), b_3(x), b(t, x)) \equiv \\ &\equiv (z(0, 0), z_t(t, 0), z_t(t, a + 0), z_x(0, x), z_{tx}(t, x)) \end{aligned}$$

of the space $\widehat{Q}_{p,n}$:

$$\begin{aligned} &b(t, x) + \int_0^t \int_0^x b(\tau, \zeta) q_1(\zeta, x) A_{0,0}(t, x) d\tau d\zeta + \\ &+ \int_0^x b(t, \zeta) q_1(\zeta, x) A_{1,0}(t, x) d\zeta + \int_0^t b(\tau, x) A_{0,1}(t, x) d\tau + \\ &+ \left(\int_0^t b_1(\tau) \theta(\alpha - x) d\tau + \int_0^t b_2(\tau) \theta(x - \alpha) d\tau \right) A_{0,0}(t, x) + \\ &+ (b_1(t) \theta(\alpha - x) + b_2(t) \theta(x - \alpha)) A_{1,0}(t, x) + b_3(x) A_{0,1}(t, x) + \\ &+ \int_0^x b_3(\zeta) A_{0,0}(t, x) d\zeta + b_0 A_{0,0}(t, x) = \varphi(t, x), \quad (t, x) \in G; \end{aligned} \quad (5)$$

$$\begin{aligned} &b_1(t) (g_{0,k}(t) + g_{1,k}(t)) + b_2(t) (g_{2,k}(t) + g_{3,k}(t)) + \\ &+ \int_0^t b_1(\tau) (\beta_{0,k}(t) + \beta_{1,k}(t)) d\tau + \\ &+ \int_0^t b_2(\tau) (\beta_{2,k}(t) + \beta_{3,k}(t)) d\tau = \varphi_k^0(t), \quad t \in (0, T), \quad k = 1, 2; \end{aligned} \quad (6)$$

where

$$\begin{aligned} \varphi_k^0(t) &= \varphi_k(t) - b_0(\beta_{0,k}(t) + \beta_{1,k}(t) + \beta_{2,k}(t) + \beta_{3,k}(t)) - \\ &- \int_0^\alpha b_3(\zeta) (\beta_{1,k}(t) + \beta_{2,k}(t)) d\zeta - \int_0^l b_3(\zeta) \beta_{3,k}(t) d\zeta - \varphi_{k,b}(t); \\ \varphi_{k,b}(t) &= \int_0^t \int_0^\alpha b(\tau, \zeta) \beta_{1,k}(t) d\tau d\zeta + \int_0^t \int_\alpha^l b(\tau, \zeta) \beta_{3,k}(t) d\tau d\zeta + \\ &+ \int_0^\alpha b(t, \zeta) g_{1,k}(t) d\zeta + \int_\alpha^l b(t, \zeta) g_{3,k}(t) d\zeta, \quad t \in (0, T); \end{aligned} \quad (6^*)$$

$$b_3(x) = \varphi_3(x), \quad x \in (0, l); \quad (7)$$

$$b_0 = \varphi_0, \quad (8)$$

where $\theta(y)$ is one-dimensional Heaviside function on $R = R^1$ and $q_1(\zeta, x) = \theta(\zeta - \alpha)\theta(x - \alpha) + \theta(\alpha - x)$.

If we succeed to estimate the components $b_0, b_1(t), b_2(t), b_3(x), b(t, x)$ of the vector \widehat{b} , on the basis of [1] we get a priori estimate for the solution $z \in \widehat{W}_{p,n}(G)$ of problem (1)-(4)

$$\begin{aligned} z(t, x) = & b_0 + \theta(\alpha - x) \int_0^T b_1(\tau) \theta(t - \tau) d\tau + \\ & + \theta(x - \alpha) \int_0^T b_2(\tau) \theta(t - \tau) d\tau + \int_0^l b_3(\zeta) \theta(x - \zeta) d\zeta + \\ & + \int_G \int \theta(t - \tau) \theta(x - \zeta) q_1(\zeta, x) b(\tau, \zeta) d\tau d\zeta, \quad (t, x) \in G. \end{aligned} \quad (9)$$

The components $b_1(t), b_2(t), b(t, x)$ are determined from the system of equations (5),(6), since the components $b_0, b_3(x)$ are explicitly given by conditions (7), (8), therefore, it remains to estimate only $b_1(t), b_2(t), b(t, x)$.

It is obvious that by means of the matrix

$$\Delta(t) = \begin{pmatrix} g_{0,1}(t) + g_{1,1}(t) & g_{0,2}(t) + g_{1,2}(t) \\ g_{2,1}(t) + g_{3,1}(t) & g_{2,2}(t) + g_{3,2}(t) \end{pmatrix}$$

we can write the equality (6) in the compact form

$$(b_1(t), b_2(t)) \Delta(t) + \int_0^t (b_1(\tau), b_2(\tau)) B(\tau) d\tau = (\varphi_1^0(t), \varphi_2^0(t)), t \in (0, T), \quad (10)$$

where

$$B(t) = \begin{pmatrix} \beta_{0,1}(t) + \beta_{1,1}(t) & \beta_{0,2}(t) + \beta_{1,2}(t) \\ \beta_{2,1}(t) + \beta_{3,1}(t) & \beta_{2,2}(t) + \beta_{3,2}(t) \end{pmatrix}$$

Assume that almost for all $t \in (0, T)$ the matrix $\Delta(t)$ is invertible and it holds

$$\|\Delta(t)\| \leq M_1, \|\Delta^{-1}(t)\| \leq M_1 \quad (11)$$

in the sense of almost everywhere on $(0, T)$. Then, from (10) we have

$$(b_1(t), b_2(t)) + \int_0^t (b_1(\tau), b_2(\tau)) B_1(\tau) d\tau = (\varphi_1^0(t), \varphi_2^0(t)) \Delta^{-1}(t), t \in (0, T), \quad (12)$$

where

$$B_1(t) = B(t) \Delta^{-1}(t).$$

Passing in (12) to the vector norm we have

$$\alpha(t) \leq \int_0^t \alpha(\tau) l(\tau) d\tau + S^0(t), t \in (0, T), \quad (13)$$

where

$$\alpha(t) = \|b_1(t)\| + \|b_2(t)\|,$$

$$l(t) = \|B_1(t)\| \leq M_1 \|B(t)\|,$$

$$S^0(t) = \|(\varphi_1^0(t), \varphi_2^0(t))\Delta^{-1}(t)\| \leq M_1(\|\varphi_1^0(t)\| + \|\varphi_2^0(t)\|);$$

here and below M_i are constants independent on $\hat{\varphi} = (\varphi_0, \varphi_1, \varphi_2, \varphi_3, \varphi)$.

Let the point $\tau \in (0, T)$ be fixed and $t \in (0, \tau)$. Then integrating (13) with respect to t on $(0, \tau)$ we get

$$R(\tau) \leq \int_0^\tau R(t)l(t)dt + S^1(\tau), \tau \in (0, T), \quad (14)$$

where

$$S^1(\tau) = \int_0^\tau S^0(t)dt,$$

$$R(\tau) = \int_0^\tau \alpha(t)dt.$$

We write the inequality (14) in the form

$$R(\tau) \leq R^1(\tau) + S^1(\tau) + \varepsilon, \quad (15)$$

where $\varepsilon > 0$ is an arbitrary number and

$$R^1(\tau) = \int_0^\tau R(t)l(t)dt.$$

Hence

$$\frac{R(\tau)}{R^1(\tau) + S^1(\tau) + \varepsilon} \leq 1, \tau \in (0, T).$$

Therefore

$$\frac{R(\tau)l(\tau)}{R^1(\tau) + S^1(\tau) + \varepsilon} \leq l(\tau), \tau \in (0, T),$$

or

$$\frac{\dot{R}^1(\tau)}{R^1(\tau) + S^1(\tau) + \varepsilon} \leq l(\tau), \tau \in (0, T), \quad (16)$$

where the sign of point over some function of one argument means its first derivative.

The function $S^1(\tau)$ is a monotonically increasing function. Therefore, if t is fixed and $\tau \in (0, t)$, then $S^1(\tau) \leq S^1(t)$. Therefore from (16) we have

$$\frac{\dot{R}^1(\tau)}{R^1(\tau) + S^1(t) + \varepsilon} \leq \frac{\dot{R}^1(\tau)}{R^1(\tau) + S^1(\tau) + \varepsilon} \leq l(\tau), \tau \in (0, t),$$

integrating it with respect to τ on $(0, t)$ we get

$$\ln \frac{R^1(t) + S^1(t) + \varepsilon}{R^1(0) + S^1(t) + \varepsilon} \leq \int_0^t l(\tau) d\tau$$

or

$$R^1(t) + S^1(t) + \varepsilon \leq (S^1(t) + \varepsilon)e^{\int_0^t l(\tau) d\tau}.$$

Taking this into account in (15) we get

$$R(t) \leq S^1(t)e^{\int_0^t l(\tau) d\tau}. \quad (17)$$

Writing (13) in the form

$$\alpha(t) \leq R(t)l(t) + S^0(t)$$

and using (17) we get

$$\alpha(t) \leq l(t)e^{\int_0^t l(\tau) d\tau} \int_0^t S^0(\tau) d\tau + S^0(t) \quad (18)$$

Thus, we have proved

Lemma 1. *If, for some non-negative functions $\alpha, l, S^0 \in \mathcal{L}_p(0, T)$, the inequality (13) holds, then the function $\alpha(t)$ also satisfies the condition (18).*

Taking into account the expression of the function $l(t)$ in (18) we get

$$\alpha(t) \leq M_2 \|B(t)\| \int_0^t S^0(\tau) d\tau + S^0(t), t \in (0, T), \quad (19)$$

where

$$M_2 = M_1 \exp \left(M_1 \int_0^T \|B(\tau)\| d\tau \right).$$

Notice that from the conditions imposed on the matrix functions $\beta_{i,k}(t)$ it follows that $\|B(\cdot)\| \in \mathcal{L}_p(0, T)$. Therefore $M_2 < +\infty$.

Now by means of this lemma for the sum $\alpha(t) = \|b_1(t)\| + \|b_2(t)\|$ we have the estimate

$$\alpha(t) = \|b_1(t)\| + \|b_2(t)\| \leq M_2 \int_0^t S^0(\tau) d\tau \|B(t)\| + S^0(t), t \in (0, T).$$

Therefore, using the Hölder inequality, we obtain

$$\|b_k(t)\| \leq S^0(t) + M_2 T^{\frac{1}{q}} \|S^0\|_{\mathcal{L}_p(0, T)} \|B(t)\|, t \in (0, T), k = 1, 2,$$

here and below $q = p/(p - 1)$ denotes the number conjugate to p .

Here, passing to the norm, by Minkowsky inequality we get

$$\begin{aligned} \|b_k\|_{\mathcal{L}_{p,n}(0,T)} &\leq \|S^0\|_{\mathcal{L}_{p,(0,T)}} (1 + M_2 T^{\frac{1}{q}} \|B\|_{\mathcal{L}_{p,2n \times 2n}(0,T)}) = \\ &= M_3 \|S^0\|_{\mathcal{L}_{p,(0,T)}}, \quad k = 1, 2; \\ M_3 &= 1 + M_2 T^{\frac{1}{q}} \|B\|_{\mathcal{L}_{p,2n \times 2n}(0,T)}. \end{aligned}$$

Obviously

$$\|S^0\|_{\mathcal{L}_{p,(0,T)}} \leq M_1 (\|\varphi_1^0\|_{\mathcal{L}_{p,n}(0,T)} + \|\varphi_2^0\|_{\mathcal{L}_{p,n}(0,T)}).$$

Therefore

$$\|b_k\|_{\mathcal{L}_{p,n}(0,T)} \leq M_4 (\|\varphi_1^0\|_{\mathcal{L}_{p,n}(0,T)} + \|\varphi_2^0\|_{\mathcal{L}_{p,n}(0,T)}), \quad k = 1, 2, \quad (20)$$

where $M_4 = M_3 M_1$.

Now, let's estimate the norms $\|\varphi_k^0\|_{\mathcal{L}_{p,n}(0,T)}$, $k = 1, 2$. Obviously if the vector $z \in \widehat{W}_{p,n}(G)$ satisfies the conditions (1)-(4), then its independent elements $\hat{b} = (z(0, 0), z_t(t, 0), z_t(t, \alpha + 0), z_x(0, x), z_{tx}(t, x)) = (b_0, b_1(t), b_2(t), b_3(x), b(t, x))$ satisfy the equalities (5)-(8) and therewith

$$\|b_0\| \leq \|\hat{\varphi}\|_{\hat{Q}_{p,n}}$$

$$\|b_3\|_{\mathcal{L}_{p,n}(0,l)} \leq \|\hat{\varphi}\|_{\hat{Q}_{p,n}} \quad (21)$$

where

$$\begin{aligned} \|\hat{\varphi}\|_{\hat{Q}_{p,n}} &= \|\hat{L}z\|_{\widehat{W}_{p,n}(G)} = \|\hat{L}(N^{-1}(\hat{b}))\|_{\hat{Q}_{p,n}} = \\ &= \|\varphi_0\| + \|\varphi_1\|_{\mathcal{L}_{p,n}(0,T)} + \|\varphi_2\|_{\mathcal{L}_{p,n}(0,T)} + \\ &+ \|\varphi_3\|_{\mathcal{L}_{p,n}(0,l)} + \|\varphi\|_{\mathcal{L}_{p,n}(G)}, \quad z = N^{-1}\hat{b}. \end{aligned}$$

Therefore, from the expressions (6*) of the vectors $\varphi_k^0(t)$ we have

$$\begin{aligned} \|\varphi_k^0(t)\| &\leq \|\varphi_k(t)\| + \|b_0\| (\|\beta_{0,k}(t)\| + \|\beta_{1,k}(t)\| + \\ &+ \|\beta_{2,k}(t)\| + \|\beta_{3,k}(t)\|) + \|b_3\|_{\mathcal{L}_{p,n}(0,l)} \alpha^{\frac{1}{q}} \times \\ &\times (\|\beta_{1,k}(t)\| + \|\beta_{2,k}(t)\|) + l^{\frac{1}{q}} \|b_3\|_{\mathcal{L}_{p,n}(0,l)} \|\beta_{3,k}(t)\| + \|\varphi_{k,b}(t)\|, \quad k = 1, 2. \end{aligned}$$

Hence by means of the Minkowsky inequality allowing for (20) we have

$$\begin{aligned}
\|\varphi_k^0\|_{\mathcal{L}_{p,n}(0,T)} &\leq \|\varphi_k\|_{\mathcal{L}_{p,n}(0,T)} + \|b_0\| \sum_{i=0}^3 \|\beta_{i,k}\|_{\mathcal{L}_{p,n \times n}(0,T)} + \\
&\quad + \|b_3\|_{\mathcal{L}_{p,n}(0,l)} \alpha^{\frac{1}{q}} \sum_{i=1}^2 \|\beta_{i,k}\|_{\mathcal{L}_{p,n \times n}(0,T)} + \\
&\quad + \|b_3\|_{\mathcal{L}_{p,n}(0,l)} l^{\frac{1}{q}} \|\beta_{3,k}\|_{\mathcal{L}_{p,n \times n}(0,T)} + \\
&\quad + \|\varphi_{k,b}\|_{\mathcal{L}_{p,n}(0,T)} \leq \|\hat{\varphi}\|_{\hat{Q}_{p,n}} \left(1 + \sum_{i=0}^3 \|\beta_{i,k}\|_{\mathcal{L}_{p,n \times n}(0,T)}\right) + \\
&\quad + \alpha^{\frac{1}{q}} \sum_{i=1}^2 \|\beta_{i,k}\|_{\mathcal{L}_{p,n \times n}(0,T)} + l^{\frac{1}{q}} \|\beta_{3,k}\|_{\mathcal{L}_{p,n \times n}(0,T)} + \|\varphi_{k,b}\|_{\mathcal{L}_{p,n}(0,T)}
\end{aligned}$$

or

$$\|\varphi_k^0\|_{\mathcal{L}_{p,n}(0,T)} \leq M_5 \|\hat{\varphi}\|_{\hat{Q}_{p,n}} + \|\varphi_{k,b}\|_{\mathcal{L}_{p,n}(0,T)}, \quad (22)$$

where

$$\begin{aligned}
M_5 &= 1 + \sum_{i=0}^3 \|\beta_{i,k}\|_{\mathcal{L}_{p,n \times n}(0,T)} + \alpha^{\frac{1}{q}} \sum_{i=1}^2 \|\beta_{i,k}\|_{\mathcal{L}_{p,n \times n}(0,T)} + \\
&\quad + l^{\frac{1}{q}} \|\beta_{3,k}\|_{\mathcal{L}_{p,n \times n}(0,T)}.
\end{aligned}$$

Now let's estimate the norm of the vector $\varphi_{k,b}(t)$. Obviously

$$\begin{aligned}
\|\varphi_{k,b}(t)\| &\leq (T\alpha)^{\frac{1}{q}} \|\beta_{1,k}\|_{\mathcal{L}_{p,n \times n}(0,T)} \|b\|_{\mathcal{L}_{p,n}(G)} + \\
&\quad + (T(l-\alpha))^{\frac{1}{q}} \|\beta_{3,k}\|_{\mathcal{L}_{p,n \times n}(0,T)} \|b\|_{\mathcal{L}_{p,n}(G)} + \\
&\quad + \alpha^{\frac{1}{q}} \|b(t, \cdot)\|_{\mathcal{L}_{p,n}(0,l)} \|g_{1,k}\|_{\mathcal{L}_{\infty, n \times n}(0,T)} + \\
&\quad + (l-\alpha)^{\frac{1}{q}} \|b(t, \cdot)\|_{\mathcal{L}_{p,n}(0,l)} \|g_{3,k}\|_{\mathcal{L}_{\infty, n \times n}(0,T)}.
\end{aligned}$$

Therefore

$$\|\varphi_{k,b}\|_{\mathcal{L}_{p,n}(0,T)} \leq c^k \|b\|_{\mathcal{L}_{p,n}(G)}, \quad k = 1, 2, \quad (23)$$

$$\begin{aligned}
c^k &= (T\alpha)^{\frac{1}{q}} \|\beta_{1,k}\|_{\mathcal{L}_{p,n \times n}(0,T)} + (T(l-\alpha))^{\frac{1}{q}} \|\beta_{3,k}\|_{\mathcal{L}_{p,n \times n}(0,T)} + \\
&\quad + \alpha^{\frac{1}{q}} \|g_{1,k}\|_{\mathcal{L}_{\infty, n \times n}(0,T)} + (l-\alpha)^{\frac{1}{q}} \|g_{3,k}\|_{\mathcal{L}_{\infty, n \times n}(0,T)}. \quad (24)
\end{aligned}$$

Then using (23) we get from (22)

$$\|\varphi_k^0\|_{\mathcal{L}_{p,n}(0,T)} \leq M_5 \|\hat{\varphi}\|_{\hat{Q}_{p,n}} + c^k \|b\|_{\mathcal{L}_{p,n}(G)}, k = 1, 2. \quad (25)$$

Now, equation (5) is written in the form

$$(\Omega b)(t, x) = \varphi^0(t, x), (t, x) \in G, \quad (26)$$

where

$$\begin{aligned} (\Omega b)(t, x) &= b(t, x) + \int_0^t \int_0^x b(\tau, \zeta) q_1(\zeta, x) A_{0,0}(t, x) d\tau d\zeta + \\ &+ \int_0^x b(t, \zeta) q_1(\zeta, x) A_{1,0}(t, x) d\zeta + \int_0^t b(\tau, x) A_{0,1}(t, x) d\tau, \quad (27) \\ \varphi^0(t, x) &= \varphi^{0,0}(t, x) + \varphi^{0,1}(t, x); \\ \varphi^{0,0}(t, x) &= \varphi(t, x) - b_3(x) A_{0,1}(t, x) - \\ &- b_0 A_{0,0}(t, x) - \int_0^x b_3(\zeta) A_{0,0}(t, x) d\zeta; \\ \varphi^{0,1}(t, x) &= - \left(\int_0^t b_1(\tau) \theta(\alpha - x) d\tau + \int_0^t b_2(\tau) \theta(x - \alpha) d\tau \right) A_{0,0}(t, x) - \\ &- (b_1(t) \theta(\alpha - x) + b_2(t) \theta(x - \alpha)) A_{1,0}(t, x). \end{aligned}$$

In the expression of the vector $\varphi^{0,0}(t, x)$ there are the vectors $\varphi(t, x)$, $b_3(x)$, b_0 and the given matrices $A_{0,1}(t, x)$, $A_{0,0}(t, x)$. Above we have estimated the norms of the vectors $\varphi(t, x)$, $b_3(x)$, b_0 by $\|\hat{\varphi}\|_{\hat{Q}_{p,n}}$. Therefore, from the expression of the vector $\varphi^{0,0}(t, x)$ by means of the Holder and Minkowsky inequalities we can easily get the estimate

$$\|\varphi^{0,0}\|_{\mathcal{L}_{p,n}(G)} \leq M_6 \|\hat{\varphi}\|_{\hat{Q}_{p,n}}, \quad (28)$$

where $M_6 > 0$ is a suitable constant.

Further, from the expression of the vector $\varphi^{0,1}(t, x)$ it is seen that

$$\begin{aligned} \|\varphi^{0,1}(t, x)\| &\leq (T^{\frac{1}{q}} \|b_1\|_{\mathcal{L}_{p,n}(0,T)} \theta(\alpha - x) + \\ &+ T^{\frac{1}{q}} \|b_2\|_{\mathcal{L}_{p,n}(0,T)} \theta(x - \alpha)) \|A_{0,0}(t, x)\| + \\ &+ (\|b_1\|_{\mathcal{L}_{p,n}(0,T)} \theta(\alpha - x) + \|b_2\|_{\mathcal{L}_{p,n}(0,T)} \theta(x - \alpha)) A_{1,0}^0(x). \end{aligned}$$

Hence we get

$$\|\varphi^{0,1}\|_{\mathcal{L}_{p,n}(G)} \leq (T^{\frac{1}{q}} \|A_{0,0}\|_{\mathcal{L}_{p,n \times n}(G)} +$$

$$\begin{aligned}
& + \|A_{1,0}^0\|_{\mathcal{L}_p(0,t)}(\|b_1\|_{\mathcal{L}_{p,n}(0,T)} + \|b_2\|_{\mathcal{L}_{p,n}(0,T)}) = \\
& = M_7(\|b_1\|_{\mathcal{L}_{p,n}(0,T)} + \|b_2\|_{\mathcal{L}_{p,n}(0,T)}), \tag{29}
\end{aligned}$$

where

$$M_7 = T^{\frac{1}{q}} \|A_{0,0}\|_{\mathcal{L}_{p,n \times n}(G)} + \|A_{1,0}^0\|_{\mathcal{L}_p(0,t)}.$$

The operator Ω , defined by the equality (27) acts in $\mathcal{L}_{p,n}(G)$, is bounded and has a bounded inverse in it [3]. Therefore from (26) we have

$$\|b\|_{\mathcal{L}_{p,n}(G)} \leq \|\Omega^{-1}\| \|\varphi^0\|_{\mathcal{L}_{p,n}(G)}.$$

Hence by (28) and (29) we get

$$\|b\|_{\mathcal{L}_{p,n}(G)} \leq \|\Omega^{-1}\| (M_6 \|\hat{\varphi}\|_{\hat{Q}_{p,n}} + M_7(\|b_1\|_{\mathcal{L}_{p,n}(0,T)} + \|b_2\|_{\mathcal{L}_{p,n}(0,T)})). \tag{30}$$

Take into account, (25) in (20) and get

$$\|b_1\|_{\mathcal{L}_{p,n}(0,T)} + \|b_2\|_{\mathcal{L}_{p,n}(0,T)} \leq 2M_4(2M_5 \|\hat{\varphi}\|_{\hat{Q}_{p,n}} + (c^1 + c^2) \|b\|_{\mathcal{L}_{p,n}(G)}). \tag{31}$$

Hence substituting (31) into (30) we get

$$\|b\|_{\mathcal{L}_{p,n}(G)} \leq \|\Omega^{-1}\| \left(M_8 \|\hat{\varphi}\|_{\hat{Q}_{p,n}} + 2M_4M_7(c^1 + c^2) \|b\|_{\mathcal{L}_{p,n}(G)} \right),$$

where $M_8 = M_6 + 4M_4M_5M_7 > 0$.

If we assume

$$\gamma = 2M_4M_7 \|\Omega^{-1}\| (c^1 + c^2) < 1, \tag{*}$$

then we can obtain

$$\|b\|_{\mathcal{L}_{p,n}(G)} \leq M_9 \|\hat{\varphi}\|_{\hat{Q}_{p,n}}, \tag{32}$$

with constant

$$M_9 = (1 - \gamma)^{-1} \|\Omega^{-1}\| M_8.$$

Taking into account (32) in (31) we have

$$\|b_1\|_{\mathcal{L}_{p,n}(0,T)} + \|b_2\|_{\mathcal{L}_{p,n}(0,T)} \leq 2M_4(2M_5 + M_9(c^1 + c^2)) \|\hat{\varphi}\|_{\hat{Q}_{p,n}}. \tag{33}$$

Now, summing up the inequalities (21), (32), (33) for the totality

$\hat{b} = (b_0, b_1(t), b_2(t), b_3(x), b(t, x))$ we have the estimate

$$\begin{aligned} \left\| \hat{b} \right\|_{\hat{Q}_{p,n}} &= \|b_0\| + \|b_1\|_{\mathcal{L}_{p,n}(0,T)} + \|b_2\|_{\mathcal{L}_{p,n}(0,T)} + \|b_3\|_{\mathcal{L}_{p,n}(0,l)} + \\ &+ \|b\|_{\mathcal{L}_{p,n}(G)} \leq M_{10} \|\hat{\varphi}\|_{\hat{Q}_{p,n}} = M_{10} \left\| \hat{L}z \right\|_{\hat{Q}_{p,n}}, \end{aligned}$$

where

$$M_{10} = 2M_4(2M_5 + M_9(c^1 + c^2)) > 0$$

and

$$\hat{L}z = \hat{\varphi}, \quad \hat{b} = Nz.$$

Using the last inequality we get

$$\|z\|_{\widehat{W}_{p,n}(G)} \leq M_{11} \|Nz\|_{\hat{Q}_{p,n}} \leq M_{11}M_{10} \left\| \hat{L}z \right\|_{\hat{Q}_{p,n}},$$

with suitable constant $M_{11} > 0$ independent on z . Hence the following theorem is true.

Theorem 1. *Let the matrix $\Delta(t)$ be invertible for almost all $t \in (0, T)$ and conditions (11) and (*) be fulfilled, where M_4 and M_7 are the constants defined above by using the number M_1 , the constants c^k ($k = 1, 2$) are given by the formula (24), and the operator Ω is given by the relation (27). Then, for every solution z of problem (1)-(4), the a priori estimate $\|z\|_{\widehat{W}_{p,n}(G)} \leq M \left\| \hat{L}z \right\|_{\hat{Q}_{p,n}}$ holds, where $M > 0$ is a positive constant independent on z .*

The operator \hat{L} is a linear and bounded operator from $\widehat{W}_{p,n}(G)$ to $\hat{Q}_{p,n}$. Therefore, there exists a bounded, conjugated operator $\hat{L}^* : (\hat{Q}_{p,n})^* \rightarrow (\widehat{W}_{p,n}(G))^*$. Using general forms of linear bounded functional determined on $\hat{Q}_{p,n}$ and $\widehat{W}_{p,n}(G)$ we can prove that \hat{L}^* is a bounded vector operator of the form $\hat{L}^* = (\omega_0, \omega_1, \omega_2, \omega_3, \omega)$ acting in the space $\hat{Q}_{q,n}$, where $1/p + 1/q = 1$. Therefore, we can consider the equation $\hat{L}^* \hat{f} = \hat{\psi}$ as a conjugated equation for problem(1)-(4), where \hat{f} is a desired solution, $\hat{\psi}$ is an element from $(\widehat{W}_{p,n}(G))^*$. It follows from Theorem 1 that the following theorem is true.

Theorem 2. *Let the conditions of Theorem 1 be satisfied. Then problem (1)-(4) may have at most one solution $z \in \widehat{W}_{p,n}(G)$, and the conjugated equation $\hat{L}^* \hat{f} = \hat{\psi}$ for any right hand side $\hat{\psi} \in (\widehat{W}_{p,n}(G))^*$ has at least one solution $\hat{f} \in \hat{Q}_{q,n}$.*

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