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EXISTENCE OF ALMOST AUTOMORPHIC SOLUTIONS TO SOME CLASSES OF NONAUTONOMOUS HIGHER-ORDER DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we obtain the existence of almost automorphic solutions to some classes of nonautonomous higher order abstract differential equations with Stepanov almost automorphic forcing terms. A few illustrative examples are discussed at the very end of the paper.

1. Introduction

The main motivation of this paper comes from the work of Andres, Bersani, and Radová [8], in which the existence (and uniqueness) of almost periodic solutions was established for the class of n-order autonomous differential equations

$$u^{(n)}(t) + \sum_{k=1}^{n} a_k u^{(n-k)}(t) = f(u) + p(t), \quad t \in \mathbb{R}, \tag{1.1}$$

where $f, p : \mathbb{R} \mapsto \mathbb{R}$ are (Stepanov) almost periodic, f is Lipschitz, and $a_k \in \mathbb{R}$ for $k = 1, ..., n$ are given real constants such that the real part of each root of the characteristic polynomial associated with the (linear) differential operator on the left-hand side of Eq. (1.1), that is,

$$Q(\lambda) := \lambda^n + \sum_{k=1}^{n} a_k \lambda^{n-k}$$

is at least nonzero.

The method utilized in [8] makes extensive use of a very complicated representation formula for solutions to Eq. (1.1). For details on that representation formula, we refer the reader to [9] and [10] and the references therein.

Let $\mathbb{H}$ be a Hilbert space. In this paper, we study a more general equation than Eq. (1.1). Namely, using similar techniques as in [14, 27], we study and obtain some reasonable sufficient conditions, which do guarantee the existence of *almost automorphic* solutions to the class of *nonautonomous* n-order differential equations

$$u^{(n)}(t) + \sum_{k=1}^{n-1} a_k(t) u^{(k)}(t) + a_0(t) A u(t) = f(t, u), \quad t \in \mathbb{R}, \tag{1.2}$$

where $A : D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ is a (possibly unbounded) self-adjoint linear operator on $\mathbb{H}$ whose spectrum consists of isolated eigenvalues

$$0 < \lambda_1 < \lambda_2 < ... < \lambda_l \to \infty \ \text{ as } \ l \to \infty$$

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with each eigenvalue having a finite multiplicity γ_j equals to the multiplicity of the corresponding eigenspace, the functions $a_k : \mathbb{R} \mapsto \mathbb{R}$ $(k = 0, 1, ..., n-1)$ are almost automorphic with

$$\inf_{t\in\mathbb{R}} \Big| a_0(t) \Big| = \gamma_0 > 0,$$

and the function $f : \mathbb{R}\times\mathbb{H} \mapsto \mathbb{H}$ is Stepanov almost automorphic in the first variable uniformly in the second variable.

Consider the time-dependent polynomial defined by

$$Q_t^l(\rho) := \rho^n + \sum_{k=1}^{n-1} a_k(t)\rho^k + \lambda_l a_0(t)$$

and denote its roots by

$$\rho_k^l(t) = \mu_k^l(t) + i\nu_k^l(t), \quad k = 1, 2, ..., n, \;\; l \geq 1, \;\; \text{and} \;\; t \in \mathbb{R}.$$

In the rest of this paper, we suppose that there exists $\delta_0 > 0$ such that

$$\text{(1.3)} \qquad \sup_{l\geq 1, t\in\mathbb{R}} \left[\max\Big(\mu_1^l(t), \mu_2^l(t), ..., \mu_n^l(t)\Big) \right] \leq -\delta_0 < 0.$$

To deal with Eq. (1.2), the main idea consists of rewriting it as a nonautonomous first-order differential equation on $\mathbb{X}^n = \mathbb{H} \times \mathbb{H} \times \mathbb{H}.... \times \mathbb{H}$ (n-times) involving the family of $n \times n$-operator matrices $\{A(t)\}_{t\in\mathbb{R}}$.

Indeed, assuming that u is differentiable n times and setting

$$z := \begin{pmatrix} u \\ u' \\ u'' \\ u^{(3)} \\ . \\ u^{(n-1)} \end{pmatrix},$$

then Eq. (1.2) can be rewritten in the Hilbert space $\mathbb{X}^n$ in the following form

$$\text{(1.4)} \qquad z'(t) = A(t)z(t) + F(t, z(t)), \; t \in \mathbb{R},$$

where $A(t)$ is the family of $n \times n$-operator matrices defined by

$$\text{(1.5)} \qquad A(t) = \begin{pmatrix} 0 & I_{\mathbb{H}} & 0 & 0 & .... & 0 & \\ 0 & 0 & I_{\mathbb{H}} & .... & 0 & & \\ . & . & . & I_{\mathbb{H}} & . & . & . \\ . & . & . & . & . & . & . \\ -a_0(t)A & -a_1(t)I_{\mathbb{H}} & . & . & . & . & -a_{n-1}(t)I_{\mathbb{H}} \end{pmatrix}$$

whose domains $D(A(t))$ are constant in $t \in \mathbb{R}$ and are precisely given by

$$D = D(A) \times \mathbb{H} \times \mathbb{H}... \times \mathbb{H} := D(A) \times \mathbb{X}^{n-1}$$

for all $t \in \mathbb{R}$.

Moreover, the semilinear term F appearing in Eq. (1.4) is defined on $\mathbb{R} \times \mathbb{X}^n_\alpha$ for some $\alpha \in (0,1)$ by

$$F(t,z) := \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ . \\ . \\ f(t,u) \end{pmatrix},$$

where $\mathbb{X}^n_\alpha$ is the real interpolation space between $\mathbb{X}^n$ and $D(A(t))$ given by $\mathbb{X}^n_\alpha = \mathbb{H}_\alpha \times \mathbb{X}^{n-1}$, with

$$\mathbb{H}_\alpha := (\mathbb{H}, D(A))_{\alpha,\infty}.$$

Under some reasonable assumptions, it will be shown that the linear operator matrices $A(t)$ satisfy the well-known Acquistapace-Terreni conditions [3], which do guarantee the existence of an evolution family $U(t,s)$ associated with it. Moreover, it will be shown that $U(t,s)$ is exponentially stable under those assumptions.

The existence of almost automorphic solutions to higher-order differential equations is important due to their (possible) applications. For instance when $n = 2$, we have thermoelastic plate equations [14, 27] or telegraph equation [31] or Sine-Gordon equations [26]. Let us also mention that when $n = 2$, some contributions on the maximal regularity, bounded, almost periodic, asymptotically almost periodic solutions to abstract second-order differential and partial differential equations have recently been made, among them are [11], [12], [44], [45], [46], and [47]. In [8], the existence of almost periodic solutions to higher-order differential equations with constant coefficients in the form Eq. (1.1) was obtained in particular in the case when the forcing term is almost periodic. However, to the best of our knowledge, the existence of almost automorphic solutions to higher-order nonautonomous equations in the form Eq. (1.2) in the case when the forcing term is Stepanov almost automorphic is an untreated original question, which in fact constitutes the main motivation of the present paper.

The paper is organized as follows: Section 2 is devoted to preliminaries facts needed in the sequel. In particular, facts related to the existence of evolution families as well as preliminary results on intermediate spaces will be stated there. In addition, basic definitions and classical results on (Stepanov) almost automorphic functions are also given. In Sections 3 and 4, we prove the main result. In Section 5, we provide the reader with an example to illustrate our main result.

2. Preliminaries

Let $\mathbb{H}$ be a Hilbert space equipped with the norm $\|\cdot\|$ and the inner product $\langle\cdot,\cdot\rangle$. In this paper, $A : D(A) \subset \mathbb{H} \mapsto \mathbb{H}$ stands for a self-adjoint (possibly unbounded) linear operator on $\mathbb{H}$ whose spectrum consists of isolated eigenvalues

$$0 < \lambda_1 < \lambda_2 < ... < \lambda_l \to \infty \quad \text{as} \quad l \to \infty$$

with each eigenvalue having a finite multiplicity γ_j equals to the multiplicity of the corresponding eigenspace.

Let $\{e_j^k\}$ be a (complete) orthonormal sequence of eigenvectors associated with the eigenvalues $\{\lambda_j\}_{j\geq 1}$.

Clearly, for each $u \in D(A) := \left\{ u \in \mathbb{H} : \quad \sum_{j=1}^{\infty} \lambda_j^2 \left\| E_j u \right\|^2 < \infty \right\}$, we have

$$Au = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k = \sum_{j=1}^{\infty} \lambda_j E_j u$$

where $E_j u = \sum_{k=1}^{\gamma_j} \langle u, e_j^k \rangle e_j^k$.

Note that $\{E_j\}_{j\geq 1}$ is a sequence of orthogonal projections on $\mathbb{H}$. Moreover, each $u \in \mathbb{H}$ can written as follows:

$$u = \sum_{j=1}^{\infty} E_j u.$$

It should also be mentioned that the operator $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$, which is explicitly expressed in terms of those orthogonal projections E_j by, for all $u \in \mathbb{H}$,

$$T(t)u = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j u.$$

In addition, the fractional powers A^r $(r \geq 0)$ of A exist and are given by

$$D(A^r) = \left\{ u \in \mathbb{H} : \sum_{j=1}^{\infty} \lambda_j^{2r} \left\| E_j u \right\|^2 < \infty \right\}$$

and

$$A^r u = \sum_{j=1}^{\infty} \lambda_j^{2r} E_j u, \quad \forall u \in D(A^r).$$

Let $(\mathbb{X}, \left\| \cdot \right\|)$ be a Banach space. If L is a linear operator on the Banach space $\mathbb{X}$, then:

- $D(L)$ stands for its domain;
- $\rho(L)$ stands for its resolvent;
- $\sigma(L)$ stands for its spectrum;
- $N(L)$ stands for its null-space or kernel; and
- $R(L)$ stands for its range.

If $L : D = D(L) \subset \mathbb{X} \mapsto \mathbb{X}$ is a closed linear operator, one denotes its graph norm by $\left\| \cdot \right\|_D$. Clearly, $(D, \left\| \cdot \right\|_D)$ is a Banach space. Moreover, one sets

$$R(\lambda, L) := (\lambda I - L)^{-1}$$

for all $\lambda \in \rho(A)$.

We set $Q = I - P$ for a projection P. If $Y, \mathbb{Z}$ are Banach spaces, then the space $B(\mathbb{Y}, \mathbb{Z})$ denotes the collection of all bounded linear operators from $\mathbb{Y}$ into $\mathbb{Z}$ equipped with its natural topology. This is simply denoted by $B(\mathbb{Y})$ when $\mathbb{Y} = \mathbb{Z}$.

2.1. **Evolution Families. Hypothesis** (H.1). The family of closed linear operators $A(t)$ for $t \in \mathbb{R}$ on $\mathbb{X}$ with domain $D(A(t))$ (possibly not densely defined) satisfy the so-called Acquistapace-Terreni conditions, that is, there exist constants $\omega \in \mathbb{R}$, $\theta \in \Big(\frac{\pi}{2}, \pi\Big)$, $K, L \geq 0$ and $\mu, \nu \in (0, 1]$ with $\mu + \nu > 1$ such that

$$S_\theta \cup \Big\{0\Big\} \subset \rho\Big(A(t) - \omega\Big) \ni \lambda, \qquad \Big\|R\Big(\lambda, A(t) - \omega\Big)\Big\| \leq \frac{K}{1 + \Big|\lambda\Big|} \tag{2.1}$$

and

$$\Big\|\Big(A(t) - \omega\Big)R\Big(\lambda, A(t) - \omega\Big)\Big[R\Big(\omega, A(t)\Big) - R\Big(\omega, A(s)\Big)\Big]\Big\| \leq L\,\Big|t - s\Big|^{\mu}\,\Big|\lambda\Big|^{-\nu} \tag{2.2}$$

for $t, s \in \mathbb{R}$, $\lambda \in S_\theta := \Big\{\lambda \in \mathbb{C} \setminus \{0\} : \Big|\arg\lambda\Big| \leq \theta\Big\}$.

Note that in the particular case when $A(t)$ has a constant domain $\mathbb{D} = D(A(t))$, it is well-known [6, 38] that Eq. (2.2) can be replaced with the following: There exist constants L and $0 < \mu \leq 1$ such that

$$\Big\|\Big(A(t) - A(s)\Big)R\Big(\omega, A(r)\Big)\Big\| \leq L\Big|t - s\Big|^{\mu}, \; s, t, r \in \mathbb{R}. \tag{2.3}$$

It should mentioned that (H.1) was introduced in the literature by Acquistapace and Terreni in [2, 3] for $\omega = 0$. Among other things, it ensures that there exists a unique evolution family $\mathcal{U} = U(t, s)$ on $\mathbb{X}$ associated with $A(t)$ satisfying

(a) $U(t, s)U(s, r) = U(t, r)$;
(b) $U(t, t) = I$ for $t \geq s \geq r$ in $\mathbb{R}$;
(c) $(t, s) \mapsto U(t, s) \in B(\mathbb{X})$ is continuous for $t > s$;
(d) $U(\cdot, s) \in C^1((s, \infty), B(\mathbb{X}))$, $\dfrac{\partial U}{\partial t}(t, s) = A(t)U(t, s)$ and

$$\Big\|A(t)^k U(t, s)\Big\| \leq K\,(t - s)^{-k}$$

for $0 < t - s \leq 1$, $k = 0, 1$, $0 \leq \alpha < \mu$, $x \in D((\omega - A(s))^\alpha)$, and a constant C depending only on the constants appearing in (H.1); and
(e) $\partial_s^+ U(t, s)x = -U(t, s)A(s)x$ for $t > s$ and $x \in D(A(s))$ with $A(s)x \in \overline{D(A(s))}$.

It should also be mentioned that the above-mentioned proprieties were mainly established in [1, Theorem 2.3] and [49, Theorem 2.1], see also [3, 48]. In that case we say that $A(\cdot)$ generates the evolution family $U(\cdot, \cdot)$.

One says that an evolution family $\mathcal{U}$ has an *exponential dichotomy* (or is *hyperbolic*) if there are projections $P(t)$ ($t \in \mathbb{R}$) that are uniformly bounded and strongly continuous in t and constants $\delta > 0$ and $N \geq 1$ such that

(f) $U(t, s)P(s) = P(t)U(t, s)$;
(g) the restriction $U_Q(t, s) : Q(s)\mathbb{X} \to Q(t)\mathbb{X}$ of $U(t, s)$ is invertible (we then set $\widetilde{U}_Q(s, t) := U_Q(t, s)^{-1}$); and

(h) $\left\|U(t,s)P(s)\right\| \leq Ne^{-\delta(t-s)}$ and $\left\|\widetilde{U}_Q(s,t)Q(t)\right\| \leq Ne^{-\delta(t-s)}$ for $t \geq s$ and $t,s \in \mathbb{R}$.

According to [40], the following sufficient conditions are required for $A(t)$ to have exponential dichotomy.

(i) Let $(A(t), D(t))_{t\in\mathbb{R}}$ be generators of analytic semigroups on $\mathbb{X}$ of the same type. Suppose that $D(A(t)) \equiv D(A(0))$, $A(t)$ is invertible,

$$\sup_{t,s\in\mathbb{R}} \left\|A(t)A(s)^{-1}\right\|$$

is finite, and

$$\left\|A(t)A(s)^{-1} - I\right\| \leq L_0\left|t-s\right|^{\mu}$$

for $t,s \in \mathbb{R}$ and constants $L_0 \geq 0$ and $0 < \mu \leq 1$.

(j) The semigroups $(e^{\tau A(t)})_{\tau\geq 0}$, $t \in \mathbb{R}$, are hyperbolic with projection P_t and constants $N, \delta > 0$. Moreover, let

$$\left\|A(t)e^{\tau A(t)}P_t\right\| \leq \psi(\tau)$$

and

$$\left\|A(t)e^{\tau A_Q(t)}Q_t\right\| \leq \psi(-\tau)$$

for $\tau > 0$ and a function ψ such that $\mathbb{R} \ni s \mapsto \varphi(s) := \left|s\right|^{\mu}\psi(s)$ is integrable with $L_0\left\|\varphi\right\|_{L^1(\mathbb{R})} < 1$.

This setting requires some estimates related to $U(t,s)$. For that, we introduce the interpolation spaces for $A(t)$. We refer the reader to the following excellent books [6], [23], and [29] for proofs and further information on theses interpolation spaces.

Let A be a sectorial operator on $\mathbb{X}$ (for that, in assumption (H.1), replace $A(t)$ with A) and let $\alpha \in (0,1)$. Define the real interpolation space

$$\mathbb{X}_\alpha^A := \left\{x \in \mathbb{X} : \left\|x\right\|_\alpha^A := \sup_{r>0}\left\|r^\alpha\Big(A-\omega\Big)R\Big(r, A-\omega\Big)x\right\| < \infty\right\},$$

which, by the way, is a Banach space when endowed with the norm $\left\|\cdot\right\|_\alpha^A$. For convenience we further write

$$\mathbb{X}_0^A := \mathbb{X},\ \left\|x\right\|_0^A := \left\|x\right\|,\ \mathbb{X}_1^A := D(A)$$

and

$$\left\|x\right\|_1^A := \left\|(\omega - A)x\right\|.$$

Moreover, let $\hat{\mathbb{X}}^A := \overline{D(A)}$ of $\mathbb{X}$. In particular, we have the following continuous embedding

$$D(A) \hookrightarrow \mathbb{X}_\beta^A \hookrightarrow D((\omega - A)^\alpha) \hookrightarrow \mathbb{X}_\alpha^A \hookrightarrow \hat{\mathbb{X}}^A \hookrightarrow \mathbb{X}, \tag{2.4}$$

for all $0 < \alpha < \beta < 1$, where the fractional powers are defined in the usual way.

In general, $D(A)$ is not dense in the spaces $\mathbb{X}_\alpha^A$ and $\mathbb{X}$. However, we have the following continuous injection

$$\mathbb{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \tag{2.5}$$

for $0 < \alpha < \beta < 1$.

Given the family of linear operators $A(t)$ for $t \in \mathbb{R}$, satisfying (H.1), we set

$$\mathbb{X}_\alpha^t := \mathbb{X}_\alpha^{A(t)}, \quad \hat{\mathbb{X}}^t := \hat{\mathbb{X}}^{A(t)}$$

for $0 \leq \alpha \leq 1$ and $t \in \mathbb{R}$, with the corresponding norms. Then the embedding in Eq. (2.4) holds with constants independent of $t \in \mathbb{R}$. These interpolation spaces are of class $\mathcal{J}_\alpha$ ([29, Definition 1.1.1]) and hence there is a constant $c(\alpha)$ such that

$$\Big\|y\Big\|_\alpha^t \leq c(\alpha)\Big\|y\Big\|^{1-\alpha}\Big\|A(t)y\Big\|^\alpha, \quad y \in D(A(t)). \tag{2.6}$$

We have the following fundamental estimates for the evolution family $\mathcal{U}$.

Proposition 2.1. [14] *For $x \in \mathbb{X}$, $0 \leq \alpha \leq 1$ and $t > s$, the following hold:*

(i) *There is a constant $c(\alpha)$, such that*

$$\Big\|U(t,s)P(s)x\Big\|_\alpha^t \leq c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\Big\|x\Big\|. \tag{2.7}$$

(ii) *There is a constant $m(\alpha)$, such that*

$$\Big\|\widetilde{U}_Q(s,t)Q(t)x\Big\|_\alpha^s \leq m(\alpha)e^{-\delta(t-s)}\Big\|x\Big\|. \tag{2.8}$$

In addition to above, we also need the following assumptions:

Hypothesis (H.2). The evolution family $\mathcal{U}$ generated by $A(\cdot)$ has an exponential dichotomy with constants $N, \delta > 0$ and dichotomy projections $P(t)$ for $t \in \mathbb{R}$.

Hypothesis (H.3). There exist α, β with $0 \leq \alpha < \beta < 1$ and such that

$$\mathbb{X}_\alpha^t = \mathbb{X}_\alpha \quad \text{and} \quad \mathbb{X}_\beta^t = \mathbb{X}_\beta$$

for all $t \in \mathbb{R}$, with uniform equivalent norms.

2.2. **Stepanov Almost Automorphic Functions.** Let $(\mathbb{X}, \|\cdot\|), (\mathbb{Y}, \|\cdot\|_{\mathbb{Y}})$ be two Banach spaces. Let $BC(\mathbb{R}, \mathbb{X})$ (respectively, $BC(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denote the collection of all $\mathbb{X}$-valued bounded continuous functions (respectively, the class of jointly bounded continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$). The space $BC(\mathbb{R}, \mathbb{X})$ equipped with the sup norm $\|\cdot\|_\infty$ is a Banach space. Furthermore, $C(\mathbb{R}, \mathbb{Y})$ (respectively, $C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$) denotes the class of continuous functions from $\mathbb{R}$ into $\mathbb{Y}$ (respectively, the class of jointly continuous functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$).

Definition 2.2. [37] The Bochner transform $f^b(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$ of a function $f : \mathbb{R} \mapsto \mathbb{X}$ is defined by $f^b(t,s) := f(t+s)$.

Remark 2.3. (i) A function $\varphi(t,s)$, $t \in \mathbb{R}$, $s \in [0,1]$, is the Bochner transform of a certain function f, $\varphi(t,s) = f^b(t,s)$, if and only if $\varphi(t+\tau, s-\tau) = \varphi(s,t)$ for all $t \in \mathbb{R}$, $s \in [0,1]$ and $\tau \in [s-1, s]$.

(ii) Note that if $f = h + \varphi$, then $f^b = h^b + \varphi^b$. Moreover, $(\lambda f)^b = \lambda f^b$ for each scalar λ.

Definition 2.4. The Bochner transform $F^b(t,s,u)$, $t \in \mathbb{R}$, $s \in [0,1]$, $u \in \mathbb{X}$ of a function $F(t,u)$ on $\mathbb{R} \times \mathbb{X}$, with values in $\mathbb{X}$, is defined by

$$F^b(t,s,u) := F(t+s,u)$$

for each $u \in \mathbb{X}$.

Definition 2.5. Let $p \in [1,\infty)$. The space $BS^p(\mathbb{X})$ of all Stepanov bounded functions, with the exponent p, consists of all measurable functions $f : \mathbb{R} \mapsto \mathbb{X}$ such that f^b belongs to $L^\infty\big(\mathbb{R}; L^p((0,1),\mathbb{X})\big)$. This is a Banach space with the norm

$$\|f\|_{S^p} := \|f^b\|_{L^\infty(\mathbb{R},L^p)} = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} \|f(\tau)\|^p \, d\tau \right)^{1/p}.$$

2.3. **S^p-Almost Automorphy.**

Definition 2.6. (Bochner) A function $f \in C(\mathbb{R},\mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$g(t) := \lim_{n\to\infty} f(t+s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} g(t-s_n) = f(t)$$

for each $t \in \mathbb{R}$.

The collection of all almost automorphic functions from $\mathbb{R}$ to $\mathbb{X}$ will be denoted $AA(\mathbb{X})$.

Similarly

Definition 2.7. (Bochner) A function $F \in C(\mathbb{R} \times \mathbb{Y}, \mathbb{X})$ is said to be almost automorphic if for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ such that

$$G(t,u) := \lim_{n\to\infty} F(t+s_n,u)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} G(t-s_n,u) = F(t,u)$$

for each $t \in \mathbb{R}$ uniformly in $u \in \mathbb{Y}$.

The collection of all almost automorphic functions from $\mathbb{R} \times \mathbb{Y}$ to $\mathbb{X}$ will be denoted $AA(\mathbb{R} \times \mathbb{Y})$.

We have the following composition result:

Theorem 2.8. [34] *Suppose $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ belongs to $AA(\mathbb{R} \times \mathbb{Y})$ and that the mapping $x \mapsto f(t,x)$ is Lipschitz in the sense that there exists $L \geq 0$ such that*

$$\Big\|F(t,x) - F(t,y)\Big\| \leq L\Big\|x-y\Big\|$$

for all $x,y \in \mathbb{Y}$ uniformly in $t \in \mathbb{R}$.

Then, then the function defined by $G(t) = F(t,\varphi(t))$ belongs to $AA(\mathbb{X})$ provided $\varphi \in AA(\mathbb{Y})$.

We also have the following composition result, which is a straightforward consequence of the composition of pseudo almost automorphic functions obtained in [43].

Theorem 2.9. [43] *If $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ belongs to $AA(\mathbb{R} \times \mathbb{Y})$ and if $x \mapsto F(t,x)$ is uniformly continuous on any bounded subset K of $\mathbb{Y}$ for each $t \in \mathbb{R}$, then the function defined by $h(t) = F(t, \varphi(t))$ belongs to $AA(\mathbb{X})$ provided $\varphi \in AA(\mathbb{Y})$.*

We will denote by $AA_u(\mathbb{X})$ the closed subspace of all functions $f \in AA(\mathbb{X})$ with $g \in C(\mathbb{R}, \mathbb{X})$. Equivalently, $f \in AA_u(\mathbb{X})$ if and only if f is almost automorphic and the convergence in Definition 2.7 are uniform on compact intervals, i.e. in the Fréchet space $C(\mathbb{R}, \mathbb{X})$. Indeed, if f is almost automorphic, then, its range is relatively compact. Obviously, the following inclusions hold:

$$AP(\mathbb{X}) \subset AA_u(\mathbb{X}) \subset AA(\mathbb{X}) \subset BC(\mathbb{X}),$$

where $AP(\mathbb{X})$ is the Banach space of almost periodic functions from $\mathbb{R}$ to $\mathbb{X}$.

Definition 2.10. [36] The space $AS^p(\mathbb{X})$ of Stepanov almost automorphic functions (or S^p-almost automorphic) consists of all $f \in BS^p(\mathbb{X})$ such that $f^b \in AA\big(L^p(0,1;\mathbb{X})\big)$. That is, a function $f \in L^p_{loc}(\mathbb{R};\mathbb{X})$ is said to be S^p-almost automorphic if its Bochner transform $f^b : \mathbb{R} \to L^p(0,1;\mathbb{X})$ is almost automorphic in the sense that for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $g \in L^p_{loc}(\mathbb{R};\mathbb{X})$ such that

$$\left[\int_t^{t+1} \|f(s_n+s) - g(s)\|^p ds\right]^{1/p} \to 0, \text{ and}$$

$$\left[\int_t^{t+1} \|g(s-s_n) - f(s)\|^p ds\right]^{1/p} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$.

Remark 2.11. It is clear that if $1 \le p < q < \infty$ and $f \in L^q_{loc}(\mathbb{R};\mathbb{X})$ is S^q-almost automorphic, then f is S^p-almost automorphic. Also if $f \in AA(\mathbb{X})$, then f is S^p-almost automorphic for any $1 \le p < \infty$. Moreover, it is clear that $f \in AA_u(\mathbb{X})$ if and only if $f^b \in AA(L^\infty(0,1;\mathbb{X}))$. Thus, $AA_u(\mathbb{X})$ can be considered as $AS^\infty(\mathbb{X})$.

Definition 2.12. A function $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}, (t,u) \mapsto F(t,u)$ with $F(\cdot,u) \in L^p_{loc}(\mathbb{R};\mathbb{X})$ for each $u \in \mathbb{Y}$, is said to be S^p-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{Y}$ if $t \mapsto F(t,u)$ is S^p-almost automorphic for each $u \in \mathbb{Y}$, that is, for every sequence of real numbers $(s'_n)_{n\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a function $G(\cdot,u) \in L^p_{loc}(\mathbb{R};\mathbb{X})$ such that

$$\left[\int_t^{t+1} \|F(s_n+s,u) - G(s,u)\|^p ds\right]^{1/p} \to 0, \text{ and}$$

$$\left[\int_t^{t+1} \|G(s-s_n,u) - F(s,u)\|^p ds\right]^{1/p} \to 0$$

as $n \to \infty$ pointwise on $\mathbb{R}$ for each $u \in \mathbb{Y}$.

The collection of those S^p-almost automorphic functions $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ will be denoted by $AS^p(\mathbb{R} \times \mathbb{Y})$.

We have the following straightforward composition theorems, which generalize Theorem 2.8 and Theorem 2.9, respectively:

Theorem 2.13. *Let $F : \mathbb{R}\times\mathbb{Y} \mapsto \mathbb{X}$ be a S^p-almost automorphic function. Suppose that $u \mapsto F(t, u)$ is Lipschz in the sense that there exists $L \geq 0$ such*

$$\Big\|F(t,u) - F(t,v)\Big\| \leq L\Big\|u - v\Big\|_{\mathbb{Y}}$$

for all $t \in \mathbb{R}, (u, v) \in \mathbb{Y} \times \mathbb{Y}$.

If $\phi \in AS^p(\mathbb{Y})$, then $\Gamma : \mathbb{R} \to \mathbb{X}$ defined by $\Gamma(\cdot) := F(\cdot, \phi(\cdot))$ belongs to $AS^p(\mathbb{X})$.

Theorem 2.14. *Let $F : \mathbb{R} \times \mathbb{Y} \mapsto \mathbb{X}$ be a S^p-almost automorphic function, where. Suppose that $F(t, u)$ is uniformly continuous in every bounded subset $K \subset \mathbb{X}$ uniformly for $t \in \mathbb{R}$. If $g \in AS^p(\mathbb{Y})$, then $\Gamma : \mathbb{R} \to \mathbb{X}$ defined by $\Gamma(\cdot) := F(\cdot, g(\cdot))$ belongs to $AS^p(\mathbb{X})$.*

3. Main results

Consider the nonautonomous differential equation

$$u'(t) = A(t)u(t) + F(t, u(t)), \quad t \in \mathbb{R}, \tag{3.1}$$

where $F : \mathbb{R} \times \mathbb{X}_\alpha \mapsto \mathbb{X}$ is S^p-almost automorphic.

Definition 3.1. A function $u : \mathbb{R} \mapsto \mathbb{X}_\alpha$ is said to be a bounded solution to Eq. (3.1) provided that

$$u(t) = \int_{-\infty}^{t} U(t,s)P(s)F(s,u(s))ds - \int_{t}^{\infty} U_Q(t,s)Q(s)F(s,u(s))ds. \tag{3.2}$$

for all $t \in \mathbb{R}$.

Throughout the rest of the paper, we set $S_1u(t) := S_{11}u(t) - S_{12}u(t)$, where

$$S_{11}u(t) := \int_{-\infty}^{t} U(t,s)P(s)F(s,u(s))ds, \quad S_{12}u(t) := \int_{t}^{\infty} U_Q(t,s)Q(s)F(s,u(s))ds.$$

for all $t \in \mathbb{R}$.

To study Eq. (3.1), in addition to the previous assumptions, we require that $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and that the following assumptions hold:

- (H.4) $R(\omega, A(\cdot)) \in B(\mathbb{R}, AA(\mathbb{X}_\alpha))$.
- (H.5) The function $F : \mathbb{R} \times \mathbb{X} \mapsto \mathbb{X}_\beta$ is S^p-almost automorphic in $t \in \mathbb{R}$ uniformly in $u \in \mathbb{X}$. Moreover, F is Lipschitz in the following sense: there exists $L > 0$ for which

$$\Big\|F(t,u) - F(t,v)\Big\|_\beta \leq L\Big\|u - v\Big\|$$

for all $u, v \in \mathbb{X}$ and $t \in \mathbb{R}$.

Lemma 3.2. *Under assumptions* (H.1)-(H.2)-(H.3)-(H.4)-(H.5) *and if*

$$N(\alpha, q, \delta) := \sum_{n=1}^{\infty} \left[\int_{n-1}^{n} e^{-q\frac{\delta}{2}s} s^{-q\alpha} ds \right]^{1/q} < \infty, \tag{3.3}$$

then the integral operator S_1 defined above maps $AA(\mathbb{X}_\alpha)$ into itself.

Proof. Let $u \in AA(\mathbb{X}_\alpha)$. Setting $\varphi(t) := F(t, u(t))$ and using Theorem 2.13 it follows that $\varphi \in AS^p(\mathbb{X}_\beta)$. The next step consists of showing that $S_1 \in AA(\mathbb{X}_\alpha)$.

Define for all $n = 1, 2, ...,$ the sequence of integral operators

$$\Phi_n(t) = \int_{n-1}^{n} U(t, t-s)P(t-s)\varphi(t-s)ds$$

for each $t \in \mathbb{R}$.

Letting $r = t - s$ it follows that

$$\Phi_n(t) = \int_{t-n}^{t-n+1} U(t, r)P(r)\varphi(r)dr,$$

and hence from the Hölder's inequality and the estimate Eq. (2.7) it follows that

$$\begin{aligned} \Big\|\Phi_n(t)\Big\|_\alpha &\leq \int_{t-n}^{t-n+1} c(\alpha)e^{-\frac{\delta}{2}(t-r)}(t-r)^{-\alpha}\Big\|\phi(r)\Big\|dr \\ &\leq c\int_{t-n}^{t-n+1} c(\alpha)e^{-\frac{\delta}{2}(t-r)}(t-r)^{-\alpha}\Big\|\phi(r)\Big\|_\alpha dr \\ &\leq cc'\int_{t-n}^{t-n+1} c(\alpha)e^{-\frac{\delta}{2}(t-r)}(t-r)^{-\alpha}\Big\|\varphi(r)\Big\|_\beta dr \\ &\leq q(\alpha)\left[\int_{n-1}^{n} e^{-q\frac{\delta}{2}s}s^{-q\alpha}ds\right]^{1/q}\Big\|\varphi\Big\|_{S^p}. \end{aligned}$$

Using Eq. (3.3), we then deduce from Weirstrass Theorem that the series defined by

$$D(t) := \sum_{n=1}^{\infty} \Phi_n(t)$$

is uniformly convergent on $\mathbb{R}$. Moreover, $D \in C(\mathbb{R}, \mathbb{X}_\alpha)$ and

$$\Big\|D(t)\Big\|_\alpha \leq \sum_{n=1}^{\infty}\Big\|\Phi_n(t)\Big\|_\alpha \leq q(\alpha)N(\alpha, q, \delta)\Big\|\phi\Big\|_{\mathbb{S}^p}$$

for all $t \in \mathbb{R}$.

Let us show that $\Phi_n \in AA(\mathbb{X}_\alpha)$ for each $n = 1, 2, 3, ...$ Indeed, since $\varphi \in AS^p(\mathbb{X}_\beta) \subset AS^p(\mathbb{X}_\alpha)$, for every sequence of real numbers $(\tau'_n)_{n\in\mathbb{N}}$ there exist a subsequence $(\tau_{n_k})_{k\in\mathbb{N}}$ and a function $\widehat{\varphi}$ such that

$$\int_{t}^{t+1}\Big\|\widehat{\varphi}(s) - \varphi(s+\tau_{n_k})\Big\|_\alpha^p ds \to 0$$

and

$$\int_t^{t+1} \Big\|\widehat{\varphi}(s-\tau_{n_k})-\varphi(s)\Big\|_\alpha^p ds \to 0$$

as $k \to \infty$ pointwise in $\mathbb{R}$.

Define for all $n = 1, 2, 3, ...$, the sequence of integral operators

$$\widehat{\Phi}_n(t) = \int_{n-1}^{n} U(t, t-s)P(t-s)\widehat{\varphi}(t-s)ds$$

for all $t \in \mathbb{R}$.

Now

$$\begin{aligned}
\Phi(t+\tau_{n_k}) - \widehat{\Phi}(t) &= \int_{n-1}^{n} U(t, t+\tau_{n_k}-s)P(t+\tau_{n_k}-s)\varphi(t+\tau_{n_k}-s)ds \\
&- \int_{n-1}^{n} U(t, t-s)P(t-s)\widehat{\varphi}(t-s)ds \\
&= \int_{n-1}^{n} U(t, t+\tau_{n_k}-s)P(t+\tau_{n_k}-s)\varphi(t+\tau_{n_k}-s)ds \\
&+ \int_{n-1}^{n} U(t, t+\tau_{n_k}-s)P(t+\tau_{n_k}-s)\widehat{\varphi}(t-s)ds \\
&- \int_{n-1}^{n} U(t, t+\tau_{n_k}-s)P(t+\tau_{n_k}-s)\widehat{\varphi}(t-s)ds \\
&- \int_{n-1}^{n} U(t, t-s)P(t-s)\widehat{\varphi}(t-s)ds \\
&= \int_{n-1}^{n} U(t, t+\tau_{n_k}-s)P(t+\tau_{n_k}-s)\Big[\varphi(t+\tau_{n_k}-s)-\widehat{\varphi}(t-s)\Big]ds \\
&+ \int_{n-1}^{n} \Big[U(t, t+\tau_{n_k}-s)P(t+\tau_{n_k}-s) - U(t, t-s)P(t-s)\Big]\widehat{\varphi}(t-s)ds.
\end{aligned}$$

Using Lebesgue Dominated Convergence Theorem, one can easily see that

$$\Big\|\int_{n-1}^{n} U(t, t+\tau_{n_k}-s)P(t+\tau_{n_k}-s)\Big[\varphi(t+\tau_{n_k}-s)-\widehat{\varphi}(t-s)\Big]ds\Big\|_\alpha \to 0 \text{ as } k \to \infty,\ t \in \mathbb{R}.$$

Similarly, using [15] it follows that

$$\Big\|\int_{n-1}^{n} \Big[U(t, t+\tau_{n_k}-s)P(t+\tau_{n_k}-s)-U(t, t-s)P(t-s)\Big]\widehat{\varphi}(t-s)ds\Big\|_\alpha \to 0 \text{ as } k \to \infty,\ t \in \mathbb{R}.$$

Thus

$$\widehat{\Phi}_n(t) = \lim_{k\to\infty} \Phi_n(t+\tau_{n_k}), \quad t \in \mathbb{R}.$$

Similarly, one can easily see that

$$\Phi_n(t) = \lim_{k\to\infty} \widehat{\Phi}_n(t-\tau_{n_k})$$

for all $t \in \mathbb{R}$ and $n = 1, 2, 3, ...$ Therefore the sequence $\Phi_n \in AA(\mathbb{X}_\alpha)$ for each $n = 1, 2, ...$ and hence $D \in AA(\mathbb{X}_\alpha)$. Consequently $t \mapsto S_{11}(t)$ belong to $AA(\mathbb{X}_\alpha)$. The proof for $t \mapsto S_{12}(t)$ is similar to that of $t \mapsto S_{11}(t)$ and hence omitted.

In view of the above, it follows that $S_1 \in AA(\mathbb{X}_\alpha)$. $\square$

Lemma 3.3. *The integral operator S_1 defined above is a contraction whenever L is small enough.*

Proof. Let $v, w \in AA(\mathbb{X}_\alpha)$. Now,

$$\begin{aligned}
\Big\|S_{11}v(t) - S_{11}w(t)\Big\|_\alpha &\leq \int_{-\infty}^t c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\Big\|F_1(s,v(s)) - F_1(s,w(s))\Big\|ds \\
&\leq c\int_{-\infty}^t c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\Big\|F_1(s,v(s)) - F_1(s,w(s))\Big\|_\beta ds \\
&\leq Lcc(\alpha)\int_{-\infty}^t (t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\Big\|v(s) - w(s)\Big\|ds \\
&\leq Lc'cc(\alpha)\int_{-\infty}^t (t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\Big\|v(s) - w(s)\Big\|_\alpha ds.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\Big\|S_{12}v(t) - S_{12}w(t)\Big\|_\alpha &\leq \int_t^\infty m(\alpha)e^{-\delta(t-s)}\Big\|F_1(s,v(s)) - F_1(s,w(s))\Big\|ds \\
&\leq cm(\alpha)\int_t^\infty e^{-\delta(t-s)}\Big\|F_1(s,v(s)) - F_1(s,w(s))\Big\|_\beta ds \\
&\leq Lcm(\alpha)\int_t^\infty e^{-\delta(t-s)}\Big\|v(s) - w(s)\Big\|ds \\
&\leq Lcc'm(\alpha)\int_t^\infty e^{-\delta(t-s)}\Big\|v(s) - w(s)\Big\|_\alpha ds.
\end{aligned}$$

Consequently,

$$\Big\|S_1v - S_1w\Big\|_{\infty,\alpha} \leq Lcc'\Big(c(\alpha)\Gamma(1-\alpha)(2\delta^{-1})^{1-\alpha} + m(\alpha)\delta^{-1}\Big)\Big\|v - w\Big\|_{\infty,\alpha}$$

and hence S_1 is a contraction whenever L is small enough. $\square$

Theorem 3.4. *Suppose assumptions* (H.1)-(H.2)-(H.3)-(H.4)-(H.5) *and Eq. (3.3) hold and that L is small enough, then the nonautonmous differential equation Eq.* (3.1) *has a unique almost automorphic solution u satisfying $u = S_1u$*

Proof. The proof makes use of Lemma 3.2, Lemma 3.3, and the Bananch fixed-point principle. $\square$

4. Almost Automorphic Solutions to Some Higher-Order Differential Equations

We have previously seen that each $u \in \mathbb{H}$ can be written in terms of the sequence of orthogonal projections E_n as follows: $u = \sum_{l=1}^\infty \sum_{k=1}^{\gamma_l} \langle u, e_l^k\rangle e_l^k = \sum_{l=1}^\infty E_l u$. Moreover, for each $u \in D(A)$, $Au = \sum_{l=1}^{\infty} \lambda_l \sum_{k=1}^{\gamma_l} \langle u, e_l^k \rangle e_l^k = \sum_{l=1}^{\infty} \lambda_l E_l u$. Therefore, for all $z := \begin{pmatrix} u_1 \\ \cdot \\ u_n \end{pmatrix} \in D = D(A(t)) = D(A) \times \mathbb{H}^{n-1}$, we obtain the following

$$A(t)z = \begin{pmatrix} 0 & I_{\mathbb{H}} & 0 & 0 & .... & 0 & \\ 0 & 0 & I_{\mathbb{H}} & .... & 0 & & \\ . & . & . & I_{\mathbb{H}} & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ -a_0(t)A & -a_1(t)I_{\mathbb{H}} & . & . & . & . & -a_{n-1}(t)I_{\mathbb{H}} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ . \\ . \\ . \\ . \\ u_n \end{pmatrix}$$

$$= \begin{pmatrix} u_2 \\ u_3 \\ . \\ . \\ . \\ . \\ . \\ -a_0(t)Au_1 - a_1(t)u_2 ... - a_{n-1}(t)u_n \end{pmatrix} = \begin{pmatrix} \sum_{l=1}^{\infty} E_l u_2 \\ \sum_{l=1}^{\infty} E_l u_3 \\ . \\ . \\ . \\ -a_0(t) \sum_{l=1}^{\infty} \lambda_l E_l u_0 - \sum_{k=1}^{n-1} a_k(t) \sum_{l=1}^{\infty} E_l u_{k+1} \end{pmatrix}$$

$$
= \sum_{l=1}^{\infty} \begin{pmatrix}
0 & 1 & 0 & 0 & .... & 0 & \\
0 & 0 & 1 & .... & 0 & & \\
. & . & . & 1 & . & . & . \\
. & . & . & . & . & 1 & . \\
. & . & . & 1 & . & . & . \\
. & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . \\
-a_0(t)\lambda_l & -a_1(t) & . & . & . & . & -a_{n-1}(t)
\end{pmatrix}
\begin{pmatrix}
E_l & 0 & 0 & 0 & .... & 0 & \\
0 & E_l & 0 & 0 & .... & 0 & \\
0 & 0 & E_l & 0 & .... & 0 & \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & E_l
\end{pmatrix}
\begin{pmatrix}
u_1 \\ u_2 \\ u_3 \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ . \\ u_n
\end{pmatrix}
$$

$$
= \sum_{l=1}^{\infty} A_l(t) P_l z, \text{where}
$$

$$
P_l := \begin{pmatrix}
E_l & 0 & 0 & 0 & .... & 0 & \\
0 & E_l & 0 & 0 & .... & 0 & \\
0 & 0 & E_l & 0 & .... & 0 & \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & . \\
. & . & . & . & . & . & E_l
\end{pmatrix}, \ l \geq 1,
$$

and

$$
(4.1) \qquad A_l(t) := \begin{pmatrix} 0 & 1 & 0 & 0 & .... & 0 & \\ 0 & 0 & 1 & .... & 0 & & \\ . & . & . & 1 & . & . & . \\ . & . & . & . & . & 1 & . \\ . & . & . & . & . & . & . \\ -a_0(t)\lambda_l & -a_1(t) & . & . & . & . & -a_{n-1}(t) \end{pmatrix}, \ l \geq 1.
$$

From Eq. (1.3) it easily follows that there exists $\omega \in \left(\dfrac{\pi}{2}, \pi\right)$ such that if we define

$$
S_\omega = \left\{ z \in \mathbb{C} \setminus \{0\} : \left| \arg z \right| \leq \omega \right\},
$$

then

$$
S_\omega \cup \{0\} \subset \rho(A(t)).
$$

On the other hand, one can show without difficulty that $A_l(t) = K_l^{-1}(t) J_l(t) K_l(t)$, where $J_l(t), K_l(t)$ are respectively given by

$$
J_l(t) = \begin{pmatrix} \rho_1^l(t) & 0 & 0 & 0 & .... & 0 \\ 0 & \rho_2^l(t) & 0 & 0 & .... & 0 \\ 0 & 0 & \rho_3^l(t) & 0 & .... & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & \rho_n^l(t) \end{pmatrix}, \quad l \geq 1
$$

and

$$K_l(t) = \begin{pmatrix} 1 & 1 & 1 & 1 & .... & 1 \\ \rho_1^l & \rho_2^l & \rho_3^l & . & .... & (\rho_n^l)^2 \\ (\rho_1^l)^2 & (\rho_2^l)^2 & (\rho_3^l)^2 & . & .... & (\rho_n^l)^2 \\ (\rho_1^l)^3 & (\rho_2^l)^3 & (\rho_3^l)^3 & . & . & (\rho_n^l)^3 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ (\rho_1^l)^{n-1} & (\rho_2^l)^{n-1} & (\rho_3^l)^{n-1} & . & . & (\rho_n^l)^{n-1} \end{pmatrix}.$$

For $\lambda \in S_\omega$ and $z \in \mathbb{X}$, one has

$$\begin{aligned} R(\lambda, A(t))z &= \sum_{l=1}^{\infty} (\lambda - A_l(t))^{-1} P_l z \\ &= \sum_{l=1}^{\infty} K_l(t)(\lambda - J_l(t)P_l)^{-1} K_l^{-1}(t) P_l z. \end{aligned}$$

Hence,

$$\begin{aligned} \left\| R(\lambda, A(t))z \right\|^2 &\leq \sum_{l=1}^{\infty} \left\| K_l(t)P_l(\lambda - J_l(t)P_l)^{-1} K_l^{-1}(t)P_l \right\|_{B(\mathbb{X})}^2 \left\| P_l z \right\|^2 \\ &\leq \sum_{l=1}^{\infty} \left\| K_l(t)P_l \right\|_{B(\mathbb{X})}^2 \left\| (\lambda - J_l(t)P_l)^{-1} \right\|_{B(\mathbb{X})}^2 \left\| K_l^{-1}(t)P_l \right\|_{B(\mathbb{X})}^2 \left\| P_l z \right\|^2. \end{aligned}$$

Moreover, for $z := \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} \in \mathbb{X}$, we obtain

$$\begin{aligned}\Big\|K_l(t)P_l z\Big\|^2 &= \Big\|E_l u_1\Big\|^2 + \sum_{k=2}^{n} \Big|\rho_k^l(t)\Big|^{2(k-1)} \Big\|E_l u_k\Big\|^2 \\ &\leq \left(1 + \sum_{k=2}^{n} \Big|\rho_k^l(t)\Big|^{2(k-1)}\right) \Big\|z\Big\|^2.\end{aligned}$$

Let $d_n^l(t) := \sum_{k=2}^{n} \Big|\rho_k^l(t)\Big|^{2(k-1)} > 0$. Thus, there exists $C_1 > 0$ such that

$$\Big\|K_l(t)P_l z\Big\| \leq C_1 d_n^l(t) \Big\|z\Big\| \quad \text{for all} \quad l \geq 1 \text{ and } t \in \mathbb{R}.$$

Using induction, one can compute $K_l^{-1}(t)$ and show that for $z := \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{pmatrix} \in \mathbb{X}$, there is $C_2 > 0$ such that

$$\Big\|K_l^{-1}(t)P_l z\Big\| \leq \frac{C_2}{d_n^l(t)} \Big\|z\Big\| \quad \text{for all} \quad l \geq 1 \text{ and } t \in \mathbb{R}.$$

Now, for $z \in \mathbb{X}$, we have

$$\left\|(\lambda - J_l P_l)^{-1} z\right\|^2 = \left\| \begin{pmatrix} \frac{1}{\lambda-\rho_1^l} & 0 & 0 & . & 0 & \\ 0 & \frac{1}{\lambda-\rho_2^l} & 0 & 0 & . & 0 \\ 0 & 0 & \frac{1}{\lambda-\rho_3^l} & 0 & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & \frac{1}{\lambda-\rho_n^l} & \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ . \\ . \\ u_n \end{pmatrix} \right\|^2$$

$$\leq \frac{1}{|\lambda-\rho_1^l|^2}\left\|u_1\right\|^2 + \frac{1}{|\lambda-\rho_2^l|^2}\left\|u_2\right\|^2 + ... + \frac{1}{|\lambda-\rho_n^l|^2}\left\|u_n\right\|^2.$$

Let $\lambda_0 > 0$. Define the function

$$\eta(\lambda) := \frac{1+\left|\lambda\right|}{\left|\lambda-\rho_k^l\right|}.$$

It is clear that η is continuous and bounded on the closed set

$$\Sigma := \left\{\lambda \in \mathbb{C} : \ \left|\lambda\right| \leq \lambda_0, \ \left|\arg\lambda\right| \leq \omega\right\}.$$

On the other hand, it is clear that η is bounded for $\left|\lambda\right| > \lambda_0$. Thus η is bounded on S_ω. If we take

$$N = \sup\left\{\frac{1+\left|\lambda\right|}{\left|\lambda-\rho_k^l\right|} \ : \ \lambda \in S_\omega, \quad l \geq 1\,; \ k = 1, 2, ..., n, \quad t \in \mathbb{R}\right\}.$$

Therefore,

$$\left\|(\lambda - J_l P_l)^{-1} z\right\| \leq \frac{N}{1+\left|\lambda\right|}\|z\|, \quad \lambda \in S_\omega.$$

Consequently,

$$\left\|R(\lambda, A(t))\right\| \leq \frac{K}{1+\left|\lambda\right|}$$

for all $\lambda \in S_\omega$ and $t \in \mathbb{R}$.

First of all, note that the domain $D = D(A(t))$ is independent of t. Thus to check that Eq. (2.2) is satisfied it is enough to check that Eq. (2.3) holds. For that, note that the operator $A(t)$ is invertible with

$$A(t)^{-1} = \begin{pmatrix} -\frac{a_1(t)}{a_0(t)}A^{-1} & -\frac{a_2(t)}{a_0(t)}A^{-1} & ... & -\frac{a_{n-1}(t)}{a_0(t)}A^{-1} & -\frac{1}{a_0(t)}A^{-1} & \\ I_{\mathbb{H}} & 0 & ... & 0 & 0 & \\ 0 & I_{\mathbb{H}} & ...0 & 0 & 0 & \\ . & . & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & I_{\mathbb{H}} & 0 \end{pmatrix}$$

for all $t \in \mathbb{R}$. Hence, for $t, s, r \in \mathbb{R}$, computing $\Big(A(t) - A(s)\Big)A(r)^{-1}$ and assuming that there exist $L_k \geq 0$ $(k = 0, 1, 2, ..., n-1)$ and $\mu \in (0, 1]$ such that

$$\Big|a_k(t) - a_k(s)\Big| \leq L_k\Big|t - s\Big|^{\mu}, \quad k = 0, 1, 2, ..., n-1 \tag{4.2}$$

it easily follows that there exists $C > 0$ such that

$$\Big\|(A(t) - A(s))A(r)^{-1}z\Big\| \leq C\Big|t - s\Big|^{\mu}\Big\|z\Big\|.$$

In summary, the family of operators $\Big\{A(t)\Big\}_{t\in\mathbb{R}}$ satisfy Acquistpace-Terreni conditions. Consequently, there exists an evolution family $U(t, s)$ associated with it. Let us now check that $U(t, s)$ has exponential dichotomy. For that, we will have to check that (i)-(j) hold. First of all note that For every $t \in \mathbb{R}$, the family of linear operators $A(t)$ generate an analytic semigroup $(e^{\tau A(t)})_{\tau\geq 0}$ on $\mathbb{X}$ given by

$$e^{\tau A(t)}z = \sum_{l=1}^{\infty} K_l(t)^{-1}P_l e^{\tau J_l}P_l K_l(t)P_l z, \; z \in \mathbb{X}.$$

On the other hand, we have

$$\Big\|e^{\tau A(t)}z\Big\| = \sum_{l=1}^{\infty}\Big\|K_l(t)^{-1}P_l\Big\|_{B(\mathbb{X})}\Big\|e^{\tau J_l}P_l\Big\|_{B(\mathbb{X})}\Big\|K_l(t)P_l\Big\|_{B(\mathbb{X})}\Big\|P_l z\Big\|,$$

with for each $z = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ u_n \end{pmatrix}$,

$$\begin{aligned}
\left\| e^{\tau J_l} P_l z \right\|^2 &= \left\| \begin{pmatrix} e^{\rho_1^l \tau} E_l & 0 & 0 & 0 & 0 & ... & 0 \\ 0 & e^{\rho_2^l \tau} E_l & 0 & 0 & 0 & ... & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & ... & 0 & 0 & 0 & e^{\lambda_n^l \tau} E_l \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ u_n \end{pmatrix} \right\|^2 \\
&\leq \left\| e^{\rho_1^l \tau} E_l u_1 \right\|^2 + \left\| e^{\rho_2^l \tau} E_l u_2 \right\|^2 + ... + \left\| e^{\rho_n^l \tau} E_l u_n \right\|^2 \\
&\leq e^{-2\delta_0 \tau} \left\| z \right\|^2.
\end{aligned}$$

Therefore

$$\left\| e^{\tau A(t)} \right\| \leq C e^{-\delta_0 \tau}, \quad \tau \geq 0. \tag{4.3}$$

Using the continuity of a_k $(k = 0, ..., n-1)$ and the equality

$$R(\lambda, A(t)) - R(\lambda, A(s)) = R(\lambda, A(t))(A(t) - A(s))R(\lambda, A(s)),$$

it follows that the mapping $J \ni t \mapsto R(\lambda, A(t))$ is strongly continuous for $\lambda \in S_\omega$ where $J \subset \mathbb{R}$ is an arbitrary compact interval. Therefore, $A(t)$ satisfies the assumptions of [40, Corollary 2.3], and thus the evolution family $(U(t,s))_{t \geq s}$ is exponentially stable.

It is clear that (H.2) holds. It remains to check assumption (H.4). For that we need to show that $A^{-1}(\cdot) \in AA(B(\mathbb{X}))$. Since $t \mapsto a_k(t)$ $(k = 0, 1, 2, ..., n-1)$, and $t \mapsto a_0(t)^{-1}$ are almost automorphic it follows that $t \mapsto d_k(t) = -\dfrac{a_k(t)}{a_0(t)}$ $(k = 1, 2, ..., n-1)$ is almost automorphic, too. Therefore, for every sequence of real numbers $(s'_n)_{n \in \mathbb{N}}$, there exists a subsequence $(s_n)_{n \in \mathbb{N}}$ such that for $m = 0, 1, ..., n-1$,

$$\tilde{a}_0^{-1}(t) := \lim_{n \to \infty} a_0^{-1}(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \to \infty} \tilde{a}_0^{-1}(t - s_n) = a_0^{-1}(t)$$

for each $t \in \mathbb{R}$, and

$$\tilde{d}_m(t) := \lim_{n \to \infty} d_m(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n \to \infty} \tilde{d}_m(t - s_n) = d_m(t)$$

for each $t \in \mathbb{R}$.

Setting

$$\tilde{A}(t) = \begin{pmatrix} \tilde{d}_1(t)A^{-1} & \tilde{d}_2(t)A^{-1} & \tilde{d}_3(t)A^{-1} & ... & \tilde{d}_{n-1}(t)A^{-1} & -\dfrac{1}{\tilde{a}_0(t)}A^{-1} \\ I_{\mathbb{H}} & 0 & 0 & ... & 0 & 0 \\ 0 & I_{\mathbb{H}} & 0 & 0 & 0 & 0 \\ . & . & . & . & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & I_{\mathbb{H}} & 0 \end{pmatrix}$$

for each $t \in \mathbb{R}$, one can easily see that, for the topology of $B(\mathbb{X})$, the following hold

$$\tilde{A}(t) := \lim_{n\to\infty} A^{-1}(t + s_n)$$

is well defined for each $t \in \mathbb{R}$, and

$$\lim_{n\to\infty} \tilde{A}(t - s_n) = A^{-1}(t)$$

for each $t \in \mathbb{R}$, and hence $t \mapsto A^{-1}(t)$ is almost automorphic with respect to operator-topology.

It is now clear that if f satisfies (H.5) and if L is small enough, then the higher-order differential equation Eq. (1.4) has an almost automorphic solution

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ . \\ u_n \end{pmatrix} \in \mathbb{X}_\alpha = \mathbb{H}_\alpha \times \mathbb{H}^{n-1}.$$

Therefore, If $f = f_1 + f_2$ satisfies (H.5) and if the Lipschitz constant of f_1 is small enough, then Eq. (1.2) has at least one almost automorphic solution $u \in \mathbb{H}_\alpha$.

5. Examples of Second-Order Boundary Value Problems

In this section, we provide with a few illustrative examples. Precisely, we study the existence of almost automorphic solutions to modified versions of the so-called (nonautonomous) Sine-Gordon equations (see [26]).

In this section, we take $n = 2$ and suppose a_0 and a_1, in addition of being almost automorphic, satisfy the other previous assumptions. Moreover, we let $\alpha = \dfrac{1}{2}$ and fix $\beta \in \left(\dfrac{1}{2}, 1\right)$.

5.1. **Nonautonomous Sine-Gordon Equations.** Let $L > 0$ and and let $J = (0, L)$. Let $\mathbb{H} = L^2(J)$ be equipped with its natural topology. Our main objective here is to study the existence of almost automorphic solutions to a slightly modified version of the so-called Sine-Gordon equation with Dirichlet boundary conditions, which had been studied in the literature especially by Leiva [26] in the following form

$$(5.1) \qquad \frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} - d\frac{\partial^2 u}{\partial x^2} + k \sin u = p(t, x), \qquad t \in \mathbb{R}, \quad x \in J$$

$$(5.2) \qquad u(t, 0) = u(t, L) = 0, \qquad t \in \mathbb{R}$$

where c, d, k are positive constants, $p : \mathbb{R} \times J \mapsto \mathbb{R}$ is continuous and bounded.

Precisely, we are interested in the following system of second-order partial differential equations

$$(5.3) \qquad \frac{\partial^2 u}{\partial t^2} + a_1(t, x)\frac{\partial u}{\partial t} - a_0(t, x)\frac{\partial^2 u}{\partial x^2} = Q(t, x, u), \qquad t \in \mathbb{R}, \quad x \in J$$

$$(5.4) \qquad u(t, 0) = u(t, L) = 0, \qquad t \in \mathbb{R}$$

where $a_1, a_0 : \mathbb{R} \times J \mapsto \mathbb{R}$ are almost automorphic positive functions and $Q : \mathbb{R} \times J \times L^2(J) \mapsto L^2(J)$ is S^p-almost automorphic for $p > 1$.

Let us take

$$Av = -v'' \text{ for all } v \in D(A) = \mathbb{H}_0^1(J) \cap \mathbb{H}^2(J)$$

and suppose that $Q : \mathbb{R} \times J \times L^2(J) \mapsto \mathbb{H}_0^\beta(J)$ is S^p-almost automorphic in $t \in \mathbb{R}$ uniformly in $x \in J$ and $u \in L^2(J)$ Moreover, Q is Lipschitz in the following sense: there exists $L'' > 0$ for which

$$\Big\|Q(t, x, u) - Q(t, x, v)\Big\|_{\mathbb{H}_0^\beta(J)} \le L''\Big\|u - v\Big\|_2$$

for all $u, v \in L^2(J)$, $x \in J$ and $t \in \mathbb{R}$.

Consequently, the system Eq. (5.3) - Eq. (5.4) has unique solution $u \in AA(\mathbb{H}_0^1(J))$ when K'' is small enough.

5.2. **A Slightly Modified Version of the Nonautonomous Sine-Gordon Equations.** Let $\Omega \subset \mathbb{R}^N$ $(N \ge 1)$ be a open bounded subset with C^2 boundary $\Gamma = \partial\Omega$ and let $\mathbb{H} = L^2(\Omega)$ equipped with its natural topology $\| \cdot \|_{L^2(\Omega)}$. Here, we are interested in a slightly modified version of the nonautonomous Sine-Gordon studied in the previous example, that is, the system of second-order partial differential equations given by

$$(5.5) \qquad \frac{\partial^2 u}{\partial t^2} + a_1(t, x)\frac{\partial u}{\partial t} - a_0(t, x)\Delta u = R(t, x, u), \qquad t \in \mathbb{R}, \quad x \in \Omega$$

$$(5.6) \qquad u(t, x) = 0, \qquad t \in \mathbb{R}, \quad x \in \partial\Omega$$

where $a_1, a_0 : \mathbb{R} \times \Omega \mapsto \mathbb{R}$ are almost automorphic positive functions, and $R : \mathbb{R} \times \Omega \times L^2(\Omega) \mapsto L^2(\Omega)$ is S^p-almost automorphic for $p > 1$.

Define the linear operator A as follows:

$$Au = -\mathbf{\Delta}u \quad \text{for all} \quad u \in D(A) = \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega).$$

For each $\mu \in (0,1)$, we take $\mathbb{H}_\mu = D((-\mathbf{\Delta})^\mu) = \mathbb{H}_0^\mu(\Omega) \cap \mathbb{H}^{2\mu}(\Omega)$ equipped with its μ-norm $\|\cdot\|_\mu$.

Suppose that $R : \mathbb{R} \times \Omega \times L^2(\Omega) \mapsto \mathbb{H}_0^\beta(\Omega)$ is S^p-almost automorphic in $t \in \mathbb{R}$ uniformly in $x \in \Omega$ and $u \in L^2(\Omega)$. Moreover, R is Lipschitz in the following sense: there exists $L''' > 0$ for which

$$\Big\|R(t,x,u) - R(t,x,v)\Big\|_\beta \leq L'''\Big\|u - v\Big\|_2$$

for all $u, v \in L^2(\Omega)$, $x \in \Omega$ and $t \in \mathbb{R}$.

Therefore, the system Eq. (5.5) - Eq. (5.6) has a unique solution $u \in AA(\mathbb{H}_0^1(\Omega))$ when L''' is small enough.

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