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GENERAL EXISTENCE RESULTS FOR NONCONVEX THIRD ORDER DIFFERENTIAL INCLUSIONS

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Abstract. In this paper we prove the existence of solutions to the following third order differential inclusion:

$$\begin{cases} x^{(3)}(t) \in F(t, x(t), \dot{x}(t), \ddot{x}(t)) + G(x(t), \dot{x}(t), \ddot{x}(t)), \text{ a.e. on } [0,T] \\ x(0) = x_0, \dot{x}(0) = u_0, \ddot{x}(0) = v_0, \text{ and } \ddot{x}(t) \in S, \forall t \in [0,T], \end{cases}$$

where $F : [0,T] \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is a continuous set-valued mapping, $G : \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is an upper semi-continuous set-valued mapping with $G(x,y,z) \subset \partial^C g(z)$ where $g : \mathbb{H} \to \mathbb{R}$ is a uniformly regular function over S and locally Lipschitz and S is a ball compact subset of a separable Hilbert space $\mathbb{H}$.

1. Introduction

The origins of boundary and initial value problems for differential inclusions are in the theory of differential equations and serve as models for a variety of applications including control theory. In [6] Hopkins studied an existence result for the third order differential inclusion

$$(\text{ThODI}) \begin{cases} x^{(3)}(t) \in G(x(t), \dot{x}(t), \ddot{x}(t)), \\ x(0) = x_0, \dot{x}(0) = u_0, \ddot{x}(0) = v_0, \end{cases}$$

where G is set-valued mapping with an upper semi-continuous compact valued included in the subdifferential of a convex lower semi-continuous function $g : \mathbb{R}^d \to \mathbb{R}$, that is, $G(x,y,z) \subset \partial^C g(z)$. In this paper we prove the existence of viable solutions for the general form of the third order differential inclusion

$$(\text{GThODI}) \begin{cases} x^{(3)}(t) \in F(t, x(t), \dot{x}(t), \ddot{x}(t)) + G(x(t), \dot{x}(t), \ddot{x}(t)), \text{ a.e. on } [0,T] \\ x(0) = x_0, \dot{x}(0) = u_0, \ddot{x}(0) = v_0, \text{ and } \ddot{x}(t) \in S, \forall t \in [0,T], \end{cases}$$

where $F : [0,T] \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is a continuous set-valued mapping, $G : \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is an upper semi-continuous set-valued mapping with $G(x,y,z) \subset \partial^C g(z)$ where $g : \mathbb{H} \to \mathbb{R}$ is a uniformly regular function over S and locally Lipschitz, and S is a ball compact subset of a separable Hilbert space $\mathbb{H}$. This general problem covers $(ThODI)$ in three different ways. First, it extends $(ThODI)$ from finite dimensional setting to separable Hilbert spaces. Secondly, it extends g to the case of uniformly regular function (not necessary convex) and also it covers (ThODI)

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by taking $F = 0$. Problem (GThODI) includes as a special case the following differential variational inequality: Given $T > 0$ and three points $x_0, u_0, v_0 \in \mathbb{H}$.

$$\text{(DVI)} \begin{cases} \text{Find } b \in (0,T),\ x : [0,b] \to \mathbb{H} \text{ such that } \ddot{x}(t) \in S, \text{ on } [0,b] \text{ and } \forall w \in S \\ \langle x^{(3)}(t), w - \ddot{x}(t)\rangle \leq a(x(t) + \dot{x}(t) - \ddot{x}(t), w - \ddot{x}(t)), \text{ a.e. on } [0,b] \\ x(0) = x_0, \dot{x}(0) = u_0, \ddot{x}(0) = v_0. \end{cases}$$

where $S = \{x \in \mathbb{H} : \Lambda(x) \leq 0\}$ ($\Lambda : \mathbb{H} \to \mathbb{R}$ is a C^1 convex function), $a(\cdot,\cdot)$ is a real bilinear, symmetric, bounded, and elliptic form on $\mathbb{H} \times \mathbb{H}$. We use our main theorem to prove that (DVI) has at least one solution.

This paper is organized as follows. In Section 2, we recall some definitions and results that will be needed in the paper. In Section 3, we prove our main existence theorem, by constructing a sequence of approximate solutions and showing its convergence to the solution of the given problem. Section 4 contains the application to differential variational inequalities.

2. Preliminaries

Throughout the paper $\mathbb{H}$ will denote a separable Hilbert space. We need to recall, from [1], some notation and definitions that will be used in all the paper.

Definition 2.1. ([1]) Let $f : \mathbb{H} \to R \cup \{+\infty\}$ be a l.s.c. function and $O \subset dom f$ be a nonempty open subset. We will say that f is *uniformly regular* over O with respect to $\beta \geq 0$ (we will also say β-uniformly regular) if for all $\bar{x} \in O$ and for all $\xi \in \partial^P f(\bar{x})$ one has

$$\langle \xi, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \beta \|x - \bar{x}\|^2 \quad \forall x \in O.$$

Here $\partial^P f(\bar{x})$ denotes the proximal subdifferential of f at x (for its definition the reader is refereed for instance to [3]). We say that f is uniformly regular over a closed set S if there exists an open set O containing S such that f is uniformly regular over O.

The class of functions that are uniformly regular over sets is so large, it contains convex sets, p-convex sets and epigraph of lower-C^2 functions. The following proposition gives some properties for uniformly regular locally Lipschitz functions over sets needed in the sequel. For the proof of these results we refer the reader to [1, 4].

Proposition 2.2. *Let $f : \mathbb{H} \to R$ be a locally Lipschitz function and $\emptyset \neq S \subset dom f$. If f is uniformly regular over S, then the following hold:*

(i) *The proximal subdifferential of f is closed over S as a set-valued mapping, that is, for every $x_n \to x$ with $x_n \in S$ and every $\zeta_n \to \zeta$ (weakly) with $\zeta_n \in \partial^P f(x_n)$ one has $\zeta \in \partial^P f(x)$;*

(ii) *The proximal subdifferential of f coincides with $\partial^C f(x)$ the Clark subdifferential of f (see for instance [3] for the definition of $\partial^C f(x)$), i.e., $\partial^C f(x) = \partial^P f(x)$ for all $x \in S$;*

(iii) *The proximal subdifferential of f is upper hemicontinuous over S, i.e, the support function $x \to \sigma(v, \partial^P f(x))$ is u.s.c. over S for every $v \in \mathbb{H}$ (where $\sigma(v,S) = \sup_{s \in S} \langle v, s \rangle$);*

(iv) *For any absolutely continuous map $x : [0,T] \to S$ one has*

$$\frac{d}{dt}(f \circ x)(t) = \langle \partial^C f(x(t)); \dot{x}(t) \rangle.$$

3. Existence results for third order differential inclusions

We start with the following technical lemmas. Their proofs follow the same lines as in the proof of Theorem 2.3 in [2].

Lemma 3.1. *Assume that*

(1) *S is nonempty subset in $\mathbb{H}$, $\upsilon_0 \in S$ and $K_0 = S \cap (\upsilon_0 + \rho B)$ is a compact set for some $\rho > 0$.*
(2) *$P : [0,T] \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is an u.s.c. set valued mapping with nonempty compact values.*
(3) *For any $(t,x,y,z) \in [0,T] \times S \times S \times S$ the following tangent condition holds*

$$\lim_{h \to 0^+} \inf \frac{1}{h} e(\upsilon + hP(t,x,y,z); S) = 0,$$

where $e(A,S) := sup_{a \in A} d_S(a)$.

Let $\alpha = \min\{T, \frac{\rho}{M+1}, 1\}$ where $M = \sup\{\|P(t,x,y,z)\| : (t,x,y,z) \in [0,T] \times K_0 \times K_0 \times K_0\}$. Then, there exists $a > 0$ such that we can construct sequences $\{w_i^m\}$, $\{t_i^m\}$, $\{\lambda_i^m\}$, $\{x_i^m\}$, $\{u_i^m\}$ and $\{\upsilon_i^m\}$ satisfying for some rank $\nu_m \geq 0$ the following assertions:

(1) $0 = t_0^m$, $t_{\nu_m}^m \leq a < T$ *with $a < \alpha$ and $t_i^m = \Sigma_{k=0}^{i-1} \lambda_k^m$ for all $i \in \{1,2,...,\nu_m\}$,*
(2) $\upsilon_i^m = \upsilon_0 + \Sigma_{k=0}^{i-1} \lambda_k^m w_k^m$ *and $(t_i^m, \upsilon_i^m) \in [0,T] \times K_0$ for all $i \in \{1,2,...,\nu_m\}$,*
(3) $w_i^m \in P(t_i^m, x_i^m, u_i^m, \upsilon_i^m) + \frac{1}{m}B$ *with $w_i^m = \frac{\psi_i^m - \upsilon_i^m}{\lambda_i^m}$ and $\psi_i^m \in S \cap B(\upsilon_i^m + \lambda_i^m b_i^m, M+1)$ for all $i \in \{1,2,...,\nu_m\}$, where*

$$\lambda_i^m = \max\{\zeta \in (0, \frac{\alpha}{2}] : \zeta \leq T - t_i^m \text{ and } d_S(\upsilon_i^m + \zeta b_i) < \frac{1}{m}\zeta\},$$

(4) $u_i^m = u_0 + \upsilon_i^m t_i^m - w_i^m \frac{t_i^{m\,2}}{2}$,
(5) $x_i^m = x_0 + u_0 t_i^m + \upsilon_i^m \frac{t_i^{m\,2}}{2} - w_i^m \frac{t_i^{m\,3}}{3}$.

Lemma 3.2. *Let $P(t,x,y,z) = F(t,x,y,z) + G(x,y,z)$. Under the same assumptions in Lemma 3.1, we can construct sequence of the step functions υ_m, u_m, x_m, f_m, c_m and θ_m with the following properties:*

(1) $\upsilon_m(t) = \upsilon_i^m + (t - t_i^m) w_i^m$ *on $[t_i^m, t_{i+1}^m)$ for all $i \in \{1,2,...,\nu_m\}$,*
(2) $u_m(t) = u_0 + \int_0^t \upsilon_m(s)ds$, $x_m(t) = x_0 + \int_0^t u_m(s)ds$ *on $[0,a]$,*
(3) $f_m(t) = f_i^m \in F(\theta_m(t), x_m(\theta_m(t)), u_m(\theta_m(t)), \upsilon_m(\theta_m(t)))$ *on $[t_i^m, t_{i+1}^m)$, with $\theta_m(t) = t_i^m$ if $t \in [t_i^m, t_{i+1}^m)$ for all $i \in \{1,2,...,\nu_m\}$, $\theta_m(a) = a$,*
(4) $y_m(t) = y_m(\theta_m(t)) \in G(x_m(\theta_m(t)), u_m(\theta_m(t)), \upsilon_m(\theta_m(t)))$ *on $[t_i^m, t_{i+1}^m)$, where $y_i^m = w_i^m - f_i^m - c_i^m$,*
(5) $c_m(t) = c_i^m \in \frac{1}{m}B$ *if $t \in [t_i^m, t_{i+1}^m)$ for all $i \in \{1,2,...,\nu_m\}$, and*

$$\lim_{h \to \infty} \sup_{t \in [0,a]} \|c_m(t)\| = 0.$$

Also,

$$\|\upsilon_m(t_{i+1}^m) - \upsilon_m(t_i^m)\| \le (M+1)(t_{i+1}^m - t_i^m),$$
$$\|\dot{\upsilon}_m(t)\| = \|w_i^m\| \le M+1, \text{ a. e. on } [0,a],$$

and

$$\|\upsilon_m(t)\| < \rho + a(M+1) < 2\rho,$$

with M as in Lemma 3.1. Furthermore, $\psi_i^m \in K_1 = S \cap B(0,R)$ with $R = \|\upsilon_0\| + 2\rho + 2M + 1$.

Now we are in position to state and prove the main result in this section.

Theorem 3.3. *Let S be a nonempty subset of $\mathbb{H}$ and let $g : \mathbb{H} \to R$ be a locally Lipschitz function which is uniformly regular over S with constant $\beta \ge 0$. Assume that*

(1) *S is ball compact.*
(2) *$F : [0,T] \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is uniformly continuous set-valued mapping with compact values.*
(3) *$G : \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is an u.s.c. set valued mapping with compact values and $G(x,y,z) \subset \partial^P g(z)$, for all $x, y, z \in S$.*
(4) *For any $(t,x,y,z) \in [0,T] \times S \times S \times S$ the following tangent condition holds*

$$\liminf_{h\to 0} \frac{1}{h} e(x + h(F(t,x,y,z) + G(x,y,z)); S) = 0,$$

where $e(A,S) := \sup_{a\in A} d_S(a)$.

Then, for any $\upsilon_0 \in S$, $u_0, x_0 \in \mathbb{H}$, there exists $a \in (0,T)$ such that

$$\begin{cases} \dot{\upsilon}(t) \in F(t,x(t),u(t),\upsilon(t)) + G(x(t),u(t),\upsilon(t)) \text{ a.e. on } [0,a], \\ \upsilon(t) \in S \text{ on } [0,a], x(t) = x_0 + \int_0^t u(s)ds, \ u(t) = u_0 + \int_0^t \upsilon(s)ds, \\ x(0) = x_0, \ u(0) = u_0, \ \upsilon(0) = \upsilon_0, \end{cases}$$

has an absolutely continuous solution on $[0,a]$. In other words, there exists $a \in (0,T)$ such that (GThODI) has an absolutely continuous solution on $[0,a]$.

Proof. Let $L > 0$ and $\rho > 0$ be two positive scalars such that g is Lipschitz over $\upsilon_0 + \rho B$ with ratio L. Since S is ball compact, $K_0 = S \cap (\upsilon_0 + \rho B)$ is compact in $\mathbb{H}$. Let M and a be two positive scalars such that

$$\|F(t,x,y,z) + G(x,y,z) + H(t,x,y,z)\| \le M,$$

for all $(t,x,y,z) \in [0,T] \times K_0 \times K_0 \times K_0$ and $a = \min\{T, \frac{\rho}{M+1}, 1\}$. By applying Lemma 3.2, there exist sequences of step functions υ_m, u_m, x_m, f_m, y_m, c_m and θ_m with the following properties:

(1) $\upsilon_m(t) = \upsilon_i^m + (t - t_i^m)w_i^m$ on $[t_i^m, t_{i+1}^m)$ for all $i \in \{1,2,...,\nu_m\}$,
(2) $u_m(t) = u_0 + \int_0^t \upsilon_m(s)ds$, $x_m(t) = x_0 + \int_0^t u_m(s)ds$ on $[0,a]$,
(3) $f_m(t) = f_m(\theta_m(t)) \in F(\theta_m(t), x_m(\theta_m(t)), u_m(\theta_m(t)), \upsilon_m(\theta_m(t)))$ on $[t_i^m, t_{i+1}^m)$, with $\theta_m(t) = t_i^m$ if $t \in [t_i^m, t_{i+1}^m)$ for all $i \in \{1,2,...,\nu_m\}$, $\theta_m(a) = a$,
(4) $y_m(t) = y_m(\theta_m(t)) \in G(x_m(\theta_m(t)), u_m(\theta_m(t)), \upsilon_m(\theta_m(t)))$ on $[t_i^m, t_{i+1}^m)$, where $y_i^m = w_i^m - f_i^m - h_i^m - c_i^m$

(5) $c_m(t) = c_i^m \in \frac{1}{m}B$ if $t \in [t_i^m, t_{i+1}^m)$ for all $i \in \{1, 2, ..., \nu_m\}$, and

$$\lim_{m\to\infty} \sup_{t\in[0,a]} \|c_m(t)\| = 0.$$

Also,

$$\|\upsilon_m(t_{i+1}^m) - \upsilon_m(t_i^m)\| \le (M+1)(t_{i+1}^m - t_i^m),$$
$$\|\dot{\upsilon}_m(t)\| = \|w_i^m\| \le M+1,$$

and

$$\|\upsilon_m(t)\| < \rho + a(M+1) < 2\rho.$$

We want to prove that υ_m converges to a solution of the given differential inclusion. First, we mention that the sequence f_m can be constructed with the relative compactness property in the space of bounded functions (see [7]). Therefore, without loss of generality we can suppose that there is a bounded function f such that

$$\lim_{m\to\infty} \sup_{t\in[0,a]} \|f_m(t) - f(t)\| = 0.$$

We note that

$$\upsilon_m(t) = \upsilon_i^m + (t - t_i^m)w_i^m = \upsilon_i^m + \frac{t - t_i^m}{\lambda_i^m}(\psi_i^m - \upsilon_i^m), \text{ for all } t \in [t_i^m, t_{i+1}^m).$$

Clearly υ_m is continuous on all the interval $[0, a]$. Indeed, it is continuous on $[t_i^m, t_{i+1}^m)$ and $\upsilon_m(t_i^m) = \lim_{t \overset{\geq}{\to} t_i^m} \upsilon_m(t) = \upsilon_i^m$ and

$$\lim_{t \overset{\leq}{\to} t_i^m} \upsilon_m(t) = \lim_{t \overset{\leq}{\to} t_i^m} \left[\upsilon_{i-1}^m + (t - t_{i-1}^m)w_{i-1}^m\right] = \upsilon_{i-1}^m + \lambda_{i-1}^m w_{i-1}^m = \upsilon_i^m,$$

and hence υ_m is continuous on the nodes t_i^m. Therefore, the sequence of mappings υ_m is equi-Lipschitz with ratio $M+1$ on all $[0, a]$. On the other hand, we have $0 \le t - t_i^m \le t_{i+1}^m - t_i^m = \lambda_i^m$ and so $0 \le \frac{t-t_i^m}{\lambda_i^m} \le 1$, and hence we get

$$\frac{t - t_i^m}{\lambda_i^m}(\psi_i^m - \upsilon_i^m) \in \overline{co}\big[\{0\} \cup \{K_1 - K_0\}\big].$$

Thus

$$\upsilon_m(t) \in K := K_0 + \overline{co}\big[\{0\} \cup \{K_1 - K_0\}\big].$$

Therefore, since the set K is compact (because K_0 and K_1 are compact) and $\|\dot{\upsilon}_m(t)\| \le M+1$, for all $t \in [0, a]$, then, the assumptions of Arzela-Ascoli theorem are satisfied. Hence a subsequence of υ_m may be extracted (still denoted υ_m) that converges to a Lipschitz function $\upsilon : [0, a] \to \mathbb{H}$ such that

$$\lim_{m\to\infty} \max_{t\in[0,a]} \|\upsilon_m(t) - \upsilon(t)\| = 0,$$

and

$$\dot{\upsilon}_m \to \dot{\upsilon} \text{ in the weak topology of } L^2([0, a]; \mathbb{H}).$$

Since $\|\upsilon_m(t)\| < 2\rho$ we have

$$\|\dot{u}_m(t)\| = \|\upsilon_m(t)\| < 2\rho,$$

and

$$\|\dot{x}_m(t)\| = \|u_m(t)\| \quad = \quad \|u_0 + \int_0^t \upsilon_m(s)ds\| \le \|u_0\| + 2\rho T,$$

On the other hand

$$u_m(t) = u_0 + \int_0^t \upsilon_m(s)ds \in K_0 + aK := K_2,$$

and

$$x_m(t) = x_0 + \int_0^t u_m(s)ds \in K_0 + aK_2 := K_3.$$

Clearly, K_2 and K_3 are compact sets in $\mathbb{H}$, then by Arzela-Ascoli theorem, there are subsequences of u_m and x_m (still denoted u_m and x_m respectively) that converge to absolutely continuous mappings $u : [0, a] \to \mathbb{H}$ and $x : [0, a] \to \mathbb{H}$ respectively, such that

$$\lim_{m\to\infty} \sup_{t\in[0,a]} \|u_m(t) - u(t)\| = 0,$$
$$\lim_{m\to\infty} \sup_{t\in[0,a]} \|x_m(t) - x(t)\| = 0,$$

and

$$\dot{u}_m \to \dot{u},\ \dot{x}_m \to \dot{x} \text{ in the weak topology of } L^2([0, a]; \mathbb{H}).$$

Also, we have that $(\upsilon_m \circ \theta_m)$, $(u_m \circ \theta_m)$ and $(x_m \circ \theta_m)$ converge uniformly on $[0, a]$ to υ, u, and x respectively. Since $\upsilon_m(\theta_m(t)) = \upsilon_i^m \in K_0$, by closedness of K_0 we get, $\upsilon(t) \in K_0 \subset S$. Indeed,

$$d(\upsilon(t); K_0) \le d(\upsilon_m(\theta_m(t)); K_0) + \|\upsilon(t) - \upsilon_m(\theta_m(t))\| \to 0, \text{ as } m \to \infty.$$

By construction, we have

$$f_m(t) = f_m(\theta_m(t)) \in F(\theta_m(t), x_m(\theta_m(t)), u_m(\theta_m(t)), \upsilon_m(\theta_m(t))),$$

and so by the continuity of F and the closedness of its values we obtain

$$f(t) \in F(t, x(t), u(t), \upsilon(t)).$$

Now, put $y(t) = \dot{\upsilon}(t) - f(t)$, we must prove that

$$y(t) \in G(x(t), u(t), \upsilon(t)) \text{ a.e. on } [0, a].$$

By construction, we have for a.e. $t \in [0, a]$

$$y_m(t) \in G(x_m(\theta_m(t)), u_m(\theta_m(t)), \upsilon_m(\theta_m(t))) \subset \partial^C g(\upsilon_m(\theta_m(t)) = \partial^P g(\upsilon_m(\theta_m(t)),$$

where the above equality follows from the uniform regularity of g over S and the part (ii) in Proposition 2.2. The weak convergence of $\dot{\upsilon}_m$ and Mazur's lemma entail for almost all $t \in [0, a]$

$$\dot{\upsilon}(t) \in \bigcap_m \overline{co}\{\dot{\upsilon}_k(t) : k \ge m\}.$$

So, for any $\xi \in \mathbb{H}$ we have

$$\begin{aligned} \langle \xi, \dot{\upsilon}(t) \rangle &\le \inf_m \sup_{k\ge m} \langle \xi, \dot{\upsilon}_k(t) \rangle \\ &\le \limsup_m \left[\sigma(\xi, \partial^P g(\upsilon_m(\theta_m(t)))) + \langle \xi, f_m(t) + c_m(t) \rangle\right] \\ &\le \sigma(\xi, \partial^P g(\upsilon(t))) + \langle \xi, f(t) \rangle = \sigma\big(\xi, \partial^P g(\upsilon(t)) + f(t)\big), \end{aligned}$$

where the last inequality follows from upper hemicontinuity of the proximal subdifferential of uniformly regular functions (part (iii) in Proposition 2.2) and the uniform convergence on $[0,a]$ of f_m and c_m to f and 0 respectively, and the fact that $\upsilon_m(\theta_m(t)) \to \upsilon(t)$ on K_0. Thus, by the convexity and the closedness of proximal subdifferential of uniformly regular functions, we have

$$y(t) = \dot{\upsilon}(t) - f(t) \in \partial^P g(\upsilon(t)).$$

As υ is absolutely continuous and g is uniformly regular locally Lipschitz function over S we get by part (iv) in Proposition 2.2

$$\begin{aligned} \frac{d}{dt}(g \circ \upsilon)(t) &= \big\langle \partial^P g(\upsilon(t)), \dot{\upsilon}(t) \big\rangle \\ &= \big\langle \dot{\upsilon}(t) - f(t), \dot{\upsilon}(t) \big\rangle \\ &= \|\dot{\upsilon}(t)\|^2 - \big\langle f(t), \dot{\upsilon}(t) \big\rangle. \end{aligned}$$

Consequently

$$g(\upsilon(a)) - g(\upsilon_0) = \int_0^a \|\dot{\upsilon}(t)\|^2 dt - \int_0^a \big\langle f(t), \dot{\upsilon}(t) \big\rangle dt. \tag{3.1}$$

On the other hand, since $y_i^m \in G(x_i^m, u_i^m, \upsilon_i^m) \subset \partial^P g(\upsilon_i^m)$ then

$$\begin{aligned} g(\upsilon_{i+1}^m) - g(\upsilon_i^m) &\geq \big\langle y_i^m, \upsilon_{i+1}^m - \upsilon_i^m \big\rangle - \beta \|\upsilon_{i+1}^m - \upsilon_i^m\|^2 \\ &= \Big\langle \dot{\upsilon}_m(t) - f_m(t) - c_m(t), \int_{t_i^m}^{t_{i+1}^m} \dot{\upsilon}_m(s) ds \Big\rangle \\ &- \beta \|\upsilon_{i+1}^m - \upsilon_i^m\|^2 \\ &\geq \int_{t_i^m}^{t_{i+1}^m} \|\dot{\upsilon}_m(s)\|^2 ds - \int_{t_i^m}^{t_{i+1}^m} \big\langle f_m(s), \dot{\upsilon}_m(s) \big\rangle ds \\ &- \int_{t_i^m}^{t_{i+1}^m} \big\langle c_m(s), \dot{\upsilon}_m(s) \big\rangle ds - \beta (M+1)^2 (t_{i+1}^m - t_i^m)^2 \\ &\geq \int_{t_i^m}^{t_{i+1}^m} \|\dot{\upsilon}_m(s)\|^2 ds - \int_{t_i^m}^{t_{i+1}^m} \big\langle f_m(s), \dot{\upsilon}_m(s) \big\rangle ds \\ &- \int_{t_i^m}^{t_{i+1}^m} \big\langle c_m(s), \dot{\upsilon}_m(s) \big\rangle ds - \frac{\beta (M+1)^2}{m} (t_{i+1}^m - t_i^m). \end{aligned}$$

By adding, we obtain

$$\begin{aligned} g(\upsilon_m(a)) - g(\upsilon_0) &\geq \int_0^a \|\dot{\upsilon}_m(s)\|^2 ds - \int_0^a \big\langle f_m(s), \dot{\upsilon}_m(s) \big\rangle ds \\ &- \int_0^a \big\langle c_m(s), \dot{\upsilon}_m(s) \big\rangle ds - \frac{\beta a (M+1)^2}{m}. \end{aligned}$$

Passing to the limit superior as $m \to \infty$

$$\begin{aligned} g(\upsilon(a)) - g(\upsilon_0) &\geq \limsup_m \int_0^a \|\dot{\upsilon}_m(s)\|^2 ds - \lim_m \int_0^a \big\langle f_m(s), \dot{\upsilon}_m(s) \big\rangle ds \\ &\geq \limsup_m \int_0^a \|\dot{\upsilon}_m(s)\|^2 ds - \int_0^a \big\langle f(s), \dot{\upsilon}(s) \big\rangle ds. \end{aligned}$$

This inequality compared with (3.1) yields

$$\int_0^a \|\dot{\upsilon}(t)\|^2 dt \geq \limsup_m \int_0^a \|\dot{\upsilon}_m(s)\|^2 ds,$$

and so

$$\|\dot{\upsilon}\|_{L^2}^2 \geq \limsup_m \|\dot{\upsilon}_m\|_{L^2}^2.$$

On the other hand, the weak lower semi-continuity of the norm ensures

$$\|\dot{\upsilon}\|_{L^2} \leq \liminf_m \|\dot{\upsilon}_m\|_{L^2}.$$

Therefore, we get

$$\|\dot{\upsilon}\|_{L^2} = \lim_m \|\dot{\upsilon}_m\|_{L^2},$$

which ensures that $\dot{\upsilon}_m$ converges uniformly to $\dot{\upsilon}$ in $L^2([0,a];\mathbb{H})$. By construction we have for almost all t in $[0,a]$

$$\big((x_m(\theta_m(t)), u_m(\theta_m(t)), \upsilon_m(\theta_m(t))), \dot{\upsilon}_m(t) - f_m(t) - c_m(t)\big) \in \text{graph } G,$$

and since G has closed graph, we conclude that

$$\big((x(t), u(t), \upsilon(t)), \dot{\upsilon}(t) - f(t)\big) \in \text{graph } G,$$

that is,

$$\dot{\upsilon}(t) - f(t) \in G(x(t), u(t), \upsilon(t)) \text{ a.e. on } [0,a].$$

Thus, for almost all t in $[0,a]$

$$\dot{\upsilon}(t) \in G(x(t), u(t), \upsilon(t)) + f(t) \subset G(x(t), u(t), \upsilon(t)) + F(t, x(t), u(t), \upsilon(t)).$$

The proof then is complete. □

We end this section with some important corollaries. The first one is an extension of the main result in [6] from finite dimensional spaces to separable Hilbert spaces and from the case of convex functions to uniform regular functions. Our proof is completely different to the one given in [6].

Corollary 3.4. *Let S be a nonempty closed subset of $\mathbb{H}$ and let $g : \mathbb{H} \to R$ be a locally Lipschitz function which is uniformly regular over S with constant $\beta \geq 0$. Assume that*

1. *S is ball compact.*
2. *$G : \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is a u.s.c. set valued mapping with compact values and $G(x,y,z) \subset g(z)$, for all $x, y, z \in S$.*
3. *For any $(t,x,y,z) \in [0,T] \times S \times S \times S$ the following tangent condition holds*

$$\lim_{h \to 0^+} \inf \frac{1}{h} e(x + h(G(x,y,z)); S) = 0.$$

Then, for every $x_0, u_0 \in \mathbb{H}$, and every $v_0 \in S$, there exists $a \in (0,T)$ and an absolutely continuous solution of the following third order differential inclusion

$$\begin{cases} x^{(3)}(t) \in G(x(t), \dot{x}(t), \ddot{x}(t)), \ \text{a.e. on } [0,a], \\ x(0) = x_0, \dot{x}(0) = u_0, \ddot{x}(0) = v_0, \ \text{and } \ddot{x}(t) \in S \end{cases}$$

Proof. It is a direct application of Theorem 3.3 with $F \equiv \{0\}$. □

Now, we are going to prove the existence of solution for third order nonconvex sweeping processes with a perturbation in separable Hilbert spaces, that is,

$$\text{(ThOSPP)}\begin{cases} x^{(3)}(t) \in N_S^C(\ddot{x}(t)) + F(t, x(t), \dot{x}(t), \ddot{x}(t)), \\ x(0) = x_0, \dot{x}(0) = u_0, \ddot{x}(0) = v_0. \end{cases}$$

Corollary 3.5. *Let $\mathbb{H}$ be a separable Hilbert. Assume that*

(1) *S is a nonempty uniformly prox-regular closed subset in $\mathbb{H}$;*
(2) *$F : [0, T] \times \mathbb{H} \times \mathbb{H} \times \mathbb{H} \to \mathbb{H}$ is an uniformly continuous set-valued mapping with compact values;*
(3) *For any $(t, x, y, z) \in [0, T] \times S \times S \times S$ the following tangent condition holds*

$$\liminf_{h\to 0^+} \frac{1}{h} e(x + h(\partial^C d_S(z) + F(t, x, y, z); S) = 0.$$

Then, for any $x_0, u_0 \in \mathbb{H}$ and every $v_0 \in S$, there exists $a \in (0, T)$ such that (ThOSPP) has at least one absolutely continuous solution on $[0, a]$.

Proof. In [1], the author proved in Theorem 4.1 that the function d_S is uniformly regular over S. Thus, we can apply Theorem 3.3 with $g := d_S$ and the set-valued mapping $G := \partial^C d_S$ which satisfies the hypothesis of Theorem 3.3 then we get a solution x of the third order differential inclusion

$$\begin{cases} x^{(3)}(t) \in \partial^C d_S(\ddot{x}(t)) + F(t, x(t), \dot{x}(t), \ddot{x}(t)), \\ x(0) = x_0, \dot{x}(0) = u_0, \ddot{x}(0) = v_0, \text{ and } \ddot{x}(t) \in S, \forall t \in [0, a]. \end{cases}$$

Now, since $\ddot{x}(t) \in S$ we have $\partial^C d_S(\ddot{x}(t)) \subset N_S^C(\ddot{x}(t))$ and so x is a solution of $(ThOSPP)$. □

4. Application to differential variational inequalities

In this section we are interested with the following differential variational inequality: Given $T > 0$ and three points $x_0, u_0, v_0 \in \mathbb{H}$.

$$\text{(DVI)}\begin{cases} \text{Find } b \in (0, T),\ x : [0, b] \to \mathbb{H} \text{ such that } \ddot{x}(t) \in S, \text{ on } [0, b] \text{ and } \forall w \in S \\ \langle x^{(3)}(t), w - \ddot{x}(t)\rangle \le a(x(t) + \dot{x}(t) - \ddot{x}(t), w - \ddot{x}(t)), \text{ a.e. on } [0, b], \\ x(0) = x_0, \dot{x}(0) = u_0, \ddot{x}(0) = v_0. \end{cases}$$

where $S = \{x \in \mathbb{H} : \Lambda(x) \le 0\}$ ($\Lambda : \mathbb{H} \to \mathbb{R}$ is a C^1 convex function), $a(\cdot, \cdot)$ is a real bilinear, symmetric, bounded, and elliptic form on $\mathbb{H} \times \mathbb{H}$. Let A be a linear and bounded operator on $\mathbb{H}$ associated with $a(\cdot, \cdot)$, that is, $a(u, v) = \langle Au, v\rangle$, $\forall u, v \in \mathbb{H}$. We prove the existence of solutions of (DVI). To do that, we recall first (see for example [5]) that

$$T(C; x) = \{v \in \mathbb{H} : \lim_{h\to 0^+} h^{-1} d_C(x + hv) = 0\}, \forall x \in C, \tag{4.1}$$

for any closed convex set C. In our case we can check (see for instance [5]) that

$$T(S; x) = \{v \in \mathbb{H} : \langle \bigtriangledown\Lambda(x), v\rangle = 0\}, \forall x \in S. \tag{4.2}$$

Proposition 4.1. *Assume that* $\mathbb{H}$ *is a separable Hilbert space and that* Λ, *and* A *satisfy*

$$\langle \nabla\Lambda(x), \partial d_S(z) + A(y - z)\rangle = 0, \quad \forall x, y, z \in \mathbb{H}. \tag{4.3}$$

Then, for any $x_0, u_0, v_0 \in \mathbb{H}$ *with* $\Lambda(v_0) = 0$, *there exists* $b \in (0, T)$ *such that (DVI) has at least one absolutely continuous solution on* $[0, b]$.

Proof. Since S is a closed convex set, the variational inequality of type (DVI) can be rewritten in the form of (ThOSPP) as follows

$$\begin{cases} x^{(3)}(t) - A(x(t) + \dot{x}(t) - \ddot{x}(t)) \in N_S^C(\ddot{x}(t)), \\ \\ x(0) = x_0, \dot{x}(0) = u_0, \ddot{x}(0) = v_0, \text{ and } \ddot{x}(t) \in S \text{ on } [0, b]. \end{cases}$$

Take $F(t, x, y, z) = \{A(x + y - z)\}$. Clearly F is uniformly continuous (since A is bounded linear operator) with compact values. By (4.3), we have

$$\langle \nabla\Lambda(x), \partial d_S(z) + A(x + y - z)\rangle = 0, \quad \forall x, y, z \in \mathbb{H}.$$

and so (4.2) yields

$$\partial d_S(z) + A(x + y - z) \subset T(x; S)$$

and hence we obtain by (4.1)

$$\liminf_{h\to 0^+} \frac{1}{h} e(x + h(\partial d_S(z) + A(x + y - z)); S) = 0.$$

Consequently, all the assumptions of Corollary 3.5 are satisfied and so the proof is complete. ☐

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