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On existence and uniqueness of positive solutions for integral boundary boundary value problems *

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Abstract: By applying the monotone iterative technique, we obtain the existence and uniqueness of $C^1[0,1]$ positive solutions in some set for singular boundary value problems of second order ordinary differential equations with integral boundary conditions.

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1 Introduction and the main result

In this paper, we consider the existence of positive solutions for the following nonlinear singular boundary value problem:

$$\begin{cases} -u'' + k^2 u = f(t,u), \ t \in (0,1), \\ u(0) = 0, \ u(1) = \displaystyle\int_0^1 u(t)dA(t), \end{cases} \tag{1.1}$$

where A is right continuous on $[0,1)$, left continuous at $t = 1$, and nondecreasing on $[0,1)$, with $A(0) = 0$. $\int_0^1 u(t)dA(t)$ denotes the Riemann-Stieltjes integral of u with respect to A. k is a constant. Problems involving Riemann-Stieltjes integral boundary condition have been studied in [3,7–9,13]. These boundary conditions includes multipoint and integral boundary conditions, and sums of these, in a single framework. By changing variables $t \mapsto 1 - t$, studying (1.1) also covers the case

$$u(0) = \int_0^1 u(t)dA(t), \ u(1) = 0.$$

For a comprehensive study of the case when there is a Riemann-Stieltjes integral boundary condition at both ends, see [7].

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In recent years, there are many papers investigating nonlocal boundary value problems of the second order ordinary differential equation $u''+f(t,u)=0$. For example, we refer the reader to [1,3–5,7–9,11,12] for some work on problems with integral type boundary conditions. However, there are fewer papers investigating boundary value problems of the equation $-u''+k^2u=f(t,u)$. In [6], Du and Zhao investigated the following multi-point boundary value problem

$$\begin{cases} -u''=f(t,u),\ t\in(0,1),\\ u(0)=\displaystyle\sum_{i=1}^{m-2}\alpha_i u(\eta_i),\ u(1)=0. \end{cases}$$

They assumed f is decreasing in u and get existence of $C[0,1]$ positive solutions ω with the property that $\omega(t)\geq m(1-t)$ for some $m>0$. In a recent paper [5], Webb and Zima studied the problem (1.1) (and others) when dA is allowed to be a signed measure, and obtained existence of multiple positive solutions under suitable conditions on $f(t,u)$. Here we only study the positive measure case. We impose stronger restrictions on f. We suppose f is increasing in u, satisfies a strong sublinear property and may be singular at $t=0,1$. By applying the monotone iterative technique, we obtain the existence and uniqueness of $C^1[0,1]$ positive solutions in some set D. Also, we use iterative methods, we establish uniqueness, obtain error estimates and the convergence rate of $C^1[0,1]$ positive solutions with the property that there exists $M>m>0$ such that $mt\leq u(t)\leq Mt$.

In this paper, we first introduce some preliminaries and lemmas in Section 2, and then we state our main results in Section 3.

2 Preliminaries and lemmas

We make the following assumptions:

(H_1) There exists $k>0$ such that $\sinh(k)>\displaystyle\int_0^1\sinh(k(1-t))dA(t)$;

(H_2) $f\in C((0,1)\times[0,+\infty),[0,+\infty))$, $f(t,u)$ is increasing in u and there exists a constant $b\in(0,1)$ such that

$$f(t,ru)\geq r^b f(t,u),\quad \text{for all } r\in(0,1) \text{ and } (t,u)\in(0,1)\times[0,+\infty). \tag{1.2}$$

Remark 2.1. If $M>1$, condition (1.2) is equivalent to

$$f(t,Mu)\leq M^b f(t,u),\quad \text{for all } (t,u)\in(0,1)\times[0,+\infty). \tag{1.3}$$

Our discussion is in the space $E = C[0,1]$ of continuous functions endowed with the usual supremum norm. Let $P = \{u \in C[0,1] : u \geq 0\}$ be the standard cone of nonnegative continuous functions.

Definition 2.1. A function $u \in C[0,1] \bigcap C^2(0,1)$ is called a $C[0,1]$ solution if it satisfies (1.1). A $C[0,1]$ solution u is called a $C^1[0,1]$ solution if both $u'(0+)$ and $u'(1-)$ exist. A solution u is called a positive solution if $u(t) > 0,\ t \in (0,1)$.

The Green's function for (1.1) is given in the following Lemma which was proved in [5] for the general case when dA is a signed measure.

Lemma 2.1 [5] Suppose that $g \in C(0,1)$ and (H_1) holds. Then the following linear boundary value problem

$$\begin{cases} -u'' + k^2 u = g(t),\ t \in (0,1), \\ u(0) = 0,\ u(1) = \displaystyle\int_0^1 u(t)dA(t) \end{cases} \tag{2.1}$$

has a unique positive solution u and u can be expressed in the form

$$u(t) = \int_0^1 F(t,s)g(s)ds,$$

where

$$F(t,s) = G(t,s) + \frac{\sinh(kt)}{\sinh(k) - \int_0^1 \sinh(k\tau)dA(\tau)} \int_0^1 G(\tau,s)dA(\tau),\ s,\ t \in [0,1], \tag{2.2}$$

$$G(t,s) = \begin{cases} \dfrac{\sinh(ks)\sinh(k(1-t))}{\sinh(k)\ \ k}, & 0 \leq s \leq t, \\ \dfrac{\sinh(kt)\sinh(k(1-s))}{\sinh(k)\ \ k}, & t \leq s \leq 1. \end{cases} \tag{2.3}$$

Remark 2.2. We call $F(t,s)$ the Green's function of problem (1.1). Suppose that (H_1), (H_2) hold. Then solutions of (1.1) are equivalent to continuous solutions of the integral equation

$$u(t) = \int_0^1 F(t,s)f(s,u(s))ds,$$

where $F(t,s)$ is mentioned in (2.2).

Lemma 2.2 For any $t,\ s \in [0,1]$, there exist constants $c_1,\ c_2 > 0$ such that

$$c_2 e(t)e(s) \leq F(t,s) \leq c_1 e(s),\ s,\ t \in [0,1], \tag{2.4}$$

where $e(s) = s(1-s)$.

Proof. Suppose that

$$I(t) = \sinh(k)t - \sinh(kt),\ t \in [0,1].$$

Then $I(0) = I(1) = 0$ and $I''(t) = -k^2 \sinh(kt) \le 0,\ t \in [0,1]$. So $I(t) \ge 0$, i.e.

$$\sinh(kt) \le \sinh(k)t, \quad t \in [0,1]. \tag{2.5}$$

Similarly we have

$$kt \le \sinh(kt), \quad t \in [0,1]. \tag{2.6}$$

From (2.3) we know

$$\frac{k}{\sinh(k)} G(t,t)G(s,s) \le G(t,s) \le G(t,t). \tag{2.7}$$

By using (2.3), (2.5) and (2.6) we obtain

$$G(t,t) \ge \frac{(kt)(k(1-t))}{\sinh(k)k} = \frac{ke(t)}{\sinh(k)}, \tag{2.8}$$

and

$$G(t,t) \le \frac{(\sinh(k)t)(\sinh(k)(1-t))}{\sinh(k)k} = \frac{\sinh(k)e(t)}{k}. \tag{2.9}$$

From (2.2), (2.7), (2.8) and (2.9) we have

$$F(t,s) \ge G(t,s) \ge \frac{k}{\sinh(k)} G(t,t)G(s,s) \ge (\frac{k}{\sinh(k)})^3 e(t)e(s) \tag{2.10}$$

and

$$\begin{aligned} F(t,s) &\le G(s,s) + G(s,s)\frac{\sinh(k)}{\sinh(k) - \int_0^1 \sinh(k\tau)dA(\tau)} \int_0^1 dA(\tau) \\ &\le \frac{\sinh(k)}{k} e(s)[1 + \frac{\sinh(k)}{\sinh(k) - \int_0^1 \sinh(k\tau)dA(\tau)} \int_0^1 dA(\tau)]. \end{aligned} \tag{2.11}$$

Letting $c_1 = \frac{\sinh(k)}{k}[1 + \frac{\sinh(k)}{\sinh(k) - \int_0^1 \sinh(k\tau)dA(\tau)} \int_0^1 dA(\tau)]$ and $c_2 = (\frac{k}{\sinh(k)})^3$, we have $c_2 e(t)e(s) \le F(t,s) \le c_1 e(s)$.

Thus, (2.4) holds.

3 Main results

Now we state the main results as follows.

Theorem 3.1 Suppose that (H_1), (H_2) hold. Let $D = \{u(t) \in C[0,1] \mid \exists L_u \ge l_u > 0,\ l_u t \le u(t) \le L_u t,\ t \in [0,1]\}$. If

$$0 < \int_0^1 f(t,t)dt < +\infty \tag{3.1}$$

holds. Then problem (1.1) has a unique $C^1[0,1]$ positive solution u^* in D. Moreover, for any initial $x_0 \in D$, the sequence of functions defined by

$$x_n = \int_0^1 F(t,s)f(s,x_{n-1}(s))ds,\ n = 1,2,\ldots$$

converges uniformly to the unique solution $u^*(t)$ on $[0,1]$ as $n \to \infty$. Furthermore, we have the error estimation

$$\|x_n(t) - u^*(t)\| \le 2(1 - (t_0^2)^{b^n})\|v_0\|, \tag{3.2}$$

where $t_0,\ v_0$ are defined below, and $F(t,s)$ is mentioned in (2.2).

Proof. From $u(t) \in D$ we know there exists $L_u > 1 > l_u > 0$ such that

$$l_u s \le u(s) \le L_u s,\ \ s \in [0,1].$$

This, together with (H_2), (1.2) and (1.3), implies that

$$(l_u)^b f(s,s) \le f(s,u(s)) \le f(s, L_u s) \le (L_u)^b f(s,s), \quad s \in (0,1). \tag{3.3}$$

Let us define an operator T by

$$Tu = \int_0^1 F(t,s) f(s,u(s)) ds,\ \ u \in D. \tag{3.4}$$

From (3.1) and (3.3) and Lemma 2.2 we can have

$$\int_0^1 F(t,s) f(s,u(s)) ds \le c_1 (L_u)^b \int_0^1 s(1-s) f(s,s) ds < +\infty.$$

So the integral operator T makes sense. By (2.2), (2.3), (2.5), (2.6) and (2.7), we have that

$$\begin{aligned} F(t,s) &\ge \sinh(kt) \frac{\displaystyle\int_0^1 G(\tau,s) dA(\tau)}{\sinh(k) - \displaystyle\int_0^1 \sinh(k\tau) dA(\tau)} \\ &\ge kt \frac{\displaystyle\int_0^1 G(\tau,s) dA(\tau)}{\sinh(k) - \displaystyle\int_0^1 \sinh(k\tau) dA(\tau)}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} F(t,s) &\le G(t,t) + \frac{\sinh(kt)}{\sinh(k) - \displaystyle\int_0^1 \sinh(k\tau) dA(\tau)} \int_0^1 G(\tau,s) dA(\tau) \\ &= \sinh(kt) \left(\frac{\sinh(k(1-t))}{\sinh(k) k} + \frac{\displaystyle\int_0^1 G(\tau,s) dA(\tau)}{\sinh(k) - \displaystyle\int_0^1 \sinh(k\tau) dA(\tau)} \right) \\ &\le t \sinh(k) \left(\frac{1}{k} + \frac{\displaystyle\int_0^1 G(\tau,s) dA(\tau)}{\sinh(k) - \displaystyle\int_0^1 \sinh(k\tau) dA(\tau)} \right). \end{aligned} \tag{3.6}$$

Thus

$$Tu(t) \geq t\frac{k\,(l_u)^b \int_0^1 \left(\int_0^1 G(\tau,s)f(s,s)ds\right)dA(\tau)}{\sinh(k) - \int_0^1 \sinh(k\tau)dA(\tau)}, \quad t \in [0,1], \tag{3.7}$$

$$\begin{aligned} Tu(t) \leq \quad & t\,(L_u)^b \sinh(k)\times \\ & \int_0^1 \left(\frac{1}{k} + \frac{\int_0^1 G(\tau,s)dA(\tau)}{\sinh(k) - \int_0^1 \sinh(k\tau)dA(\tau)}\right) f(s,s)ds,\ t \in [0,1]. \end{aligned} \tag{3.8}$$

Thus, from (3.1), (3.7) and (3.8), we obtain

$$T : D \to D.$$

It is known from Remark 2.2 that a fixed point of the operator T is a solution of BVP (1.1).

From condition (1.2) we obtain

$$T(ru) = \int_0^1 F(t,s)f(s,ru(s))ds \geq r^b \int_0^1 F(t,s)f(s,u(s))ds = r^b Tu, \tag{3.9}$$

Obviously T is an increasing operator and from (1.3) we have

$$T(Mu) \leq M^b Tu. \tag{3.10}$$

Let $x_0 \in D$ be given. Choose $t_0 \in (0,1)$ such that

$$t_0^{1-b}x_0 \leq Tx_0 \leq (\frac{1}{t_0})^{1-b}x_0.$$

Let us define $u_0 = t_0x_0,\ v_0 = \frac{1}{t_0}x_0,\ t_0 \in (0,1)$. Then $u_0 \leq v_0$ and from (3.9) and (3.10) we have

$$Tu_0 \geq t_0^bTx_0 \geq t_0x_0 = u_0,\ Tv_0 \leq (\frac{1}{t_0})^bTx_0 \leq \frac{1}{t_0}x_0 = v_0. \tag{3.11}$$

Now we define

$$u_n = Tu_{n-1},\ v_n = Tv_{n-1},\ (n = 1,2,3,\ldots).$$

It is easy to verify from (3.11) that

$$u_0 \leq u_1 \leq \ldots \leq u_n \leq \ldots \leq v_n \leq \ldots \leq v_1 \leq v_0. \tag{3.12}$$

Clearly, $u_0 = t_0^2v_0$. By induction, we see that

$$u_n \geq (t_0^2)^{b^n}v_n,\ (n = 0,1,2,\ldots). \tag{3.13}$$

Since P is a normal cone with normality constant 1, it follows that

$$\|v_n - u_n\| \le \|u_{n+p} - u_n\| \le (1 - (t_0^2)^{b^n})\|v_0\|. \tag{3.14}$$

So $\{u_n\}$ is a cauchy sequence, therefore u_n converges to some $u^* \in D$. From this inequality it also follows that $v_n \to u^*$.

We see that u^* is a fixed point of T. Thus, $u^* \in D$ from $u_0,\ v_0 \in D$ and $u^* \in [u_0, v_0]$. It follows from $u_0 \le x_0 \le v_0$ that $u_n \le x_n \le v_n,\ (n = 1, 2, 3, \ldots)$. So

$$\begin{aligned} \|x_n - u^*\| &\le \|x_n - u_n\| + \|u_n - u^*\| \le 2\|v_n - u_n\| \\ &\le 2(1 - (t_0^2)^{b^n})\|v_0\|. \end{aligned} \tag{3.15}$$

Next we prove the uniqueness of fixed points of T. Let $\overline{x} \in D$ be any fixed points of T. From $u^*,\ \overline{x} \in D$ and the definition of D, we can put $t_1 = \sup\{t > 0 \mid \overline{x} \ge tu^*\}$. Evidently $0 < t_1 < \infty$. We now prove $t_1 \ge 1$. In fact, if $0 < t_1 < 1$, then

$$\overline{x} = T\overline{x} \ge T(t_1 u^*) \ge (t_1)^b T u^* = (t_1)^b u^*,$$

which contradicts the definition of t_1 since $(t_1)^b > t_1$. Thus $t_1 \ge 1$ and $\overline{x} \ge u^*$. In the same way, we can prove $\overline{x} \le u^*$ and hence $\overline{x} = u^*$. The uniqueness of fixed points of A in D is proved. For any initial $z_0 \in D$, $z_n = T^n z_0 \to u^*$ with rate of convergence

$$\|z_n - u^*\| = o(1 - (t_0^2)^{b^n}) \tag{3.16}$$

from the results above. Choosing $z_0 = x_0$, we obtain

$$\|x_n - u^*\| = o(1 - (t_0^2)^{b^n}). \tag{3.17}$$

This completes the proof of Theorem 3.1.

Remark Suppose that $\beta_i(t)(i = 0, 1, 2, \ldots m)$ are nonnegative continuous functions on $(0, 1)$, which may be unbounded at the end points of $(0, 1)$. Ω is the set of functions $f(t, u)$ which satisfy the condition (H_2). Then we have the following conclusions:

(1) $\beta_i(t) \in \Omega,\ u^b \in \Omega$, where $0 < b < 1$;

(2) If $0 < b_i < +\infty(i = 1, 2, \ldots m)$ and $b > \max\limits_{1 \le i \le m}\{b_i\}$, then $[\beta_0(t) + \sum\limits_{i=1}^{m} \beta_i(t) u^{b_i}]^{\frac{1}{b}} \in \Omega$;

(3) If $f(t, u) \in \Omega$, then $\beta_i(t) f(t, u) \in \Omega$;

(4) If $f_i(t, u) \in \Omega(i = 1, 2, \ldots m)$, then $\max\limits_{1 \le i \le m}\{f_i(t, u)\} \in \Omega,\ \min\limits_{1 \le i \le m}\{f_i(t, u)\} \in \Omega$.

The above four facts can be verified directly. This indicates that there are many kinds of functions which satisfy the condition (H_2).

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