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Positive Almost Periodic Solutions for a Class of Nonlinear Duffing Equations with a Deviating Argument*

Lequn Peng
Department of Mathematics, Hunan University of Arts and Science,
Changde, Hunan 415000, P. R. China
E-mail: penglq1956@yahoo.com.cn
Wentao Wang†
College of Mathematics and Information Engineering, Jiaxing University,
Jiaxing, Zhejiang 314001, P. R. China
E-mail: wentaowang2009@yahoo.com.cn

Abstract: In this paper, we study a class of nonlinear Duffing equations with a deviating argument and establish some sufficient conditions for the existence of positive almost periodic solutions of the equation. These conditions are new and complement to previously known results.

Keywords: Nonlinear Duffing equations; Almost periodic solution; Existence; Deviating argument.
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1. Introduction

Consider the following model for nonlinear Duffing equation with a deviating argument

$$x''(t) + cx'(t) - ax(t) + bx^m(t - \tau(t)) = p(t), \tag{1.1}$$

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† Corresponding author.

where $\tau(t)$ and $p(t)$ are almost periodic functions on R, $m > 1$, a, b and c are constants.

In recent years, the dynamic behaviors of nonlinear Duffing equations have been widely investigated in [1-4] due to the application in many fields such as physics, mechanics, engineering, other scientific fields. In such applications, it is important to know the existence of the almost periodic solutions for nonlinear Duffing equations. Some results on existence of the almost periodic solutions were obtained in the literature. We refer the reader to [5−7] and the references cited therein. Suppose that the following condition holds:

(H_0) $a = b = 1,\ c = 0,\ \tau : R \to R$ is a constant function, $m > 1$ is an integer, and

$$\sup_{t\in R} |p(t)| \leq (\frac{1}{m})^{\frac{1}{m-1}}(1 - \frac{1}{m}). \tag{1.2}$$

The authors of [6] and [7] obtained some sufficient conditions ensuring the existence of almost periodic solutions for Eq. (1.1). However, to the best of our knowledge, few authors have considered the problem of almost periodic solutions for Eq. (1.1) without the assumption (H_0). Thus, it is worthwhile to continue to investigate the existence of almost periodic solutions Eq. (1.1) in this case.

A primary purpose of this paper is to study the problem of positive almost periodic solutions of (1.1). Without assuming (H_0), we derive some sufficient conditions ensuring the existence of positive almost periodic solutions for Eq. (1.1), which are new and complement to previously known results. Moreover, an example is also provided to illustrate the effectiveness of our results.

Let $Q_1(t)$ be a continuous and differentiable function on R. Define

$$y = \frac{dx}{dt} + \xi x - Q_1(t),\ Q_2(t) = p(t) + (\xi - c)Q_1(t) - Q_1'(t) \tag{1.3}$$

where $\xi > 1$ is a constant, then we can transform (1.1) into the following system

$$\begin{cases} \dfrac{dx(t)}{dt} = -\xi x(t) + y(t) + Q_1(t), \\ \dfrac{dy(t)}{dt} = -(c - \xi)y(t) + (a - \xi(\xi - c))x(t) - bx^m(t - \tau(t)) + Q_2(t). \end{cases} \tag{1.4}$$

Definition 1 [see 8, 9]. Let $u(t) : R \longrightarrow R^n$ be continuous in t. $u(t)$ is said to be almost periodic on R if, for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\delta : \|u(t + \delta) - u(t)\| < \varepsilon \text{ for all } t \in R\}$ is relatively dense, i.e., for any $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$, for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $\|u(t + \delta) - u(t)\| < \varepsilon$ for all $t \in R$.

Throughout this paper, it will be assumed that $\tau, Q_1, Q_2 : R \to [0, +\infty)$ are almost periodic functions. From the theory of almost periodic functions in [8,9], it follows that for any $\epsilon > 0$, it is possible to find a real number $l = l(\epsilon) > 0$, for any interval with length $l(\epsilon)$, there exists a number $\delta = \delta(\epsilon)$ in this interval such that

$$|Q_1(t+\delta) - Q_1(t)| < \epsilon, \ |Q_2(t+\delta) - Q_2(t)| < \epsilon, \ |\tau(t+\delta) - \tau(t)| < \epsilon, \tag{1.5}$$

for all $t \in R$. We suppose that there exist constants $\underline{L}$, L^+ and $\bar{\tau}$ such that

$$\underline{L} = \min\{\inf_{t\in R} Q_1(t), \ \inf_{t\in R} Q_2(t)\} > 0, \ L^+ > \max\{\sup_{t\in R} Q_1(t), \sup_{t\in R} Q_2(t)\}, \bar{\tau} = \sup_{t\in R} \tau(t). \tag{1.6}$$

Let $C([-\bar{\tau}, 0], R)$ denote the space of continuous functions $\varphi : [-\bar{\tau}, 0] \to R$ with the supremum norm $\|\cdot\|$. It is known in [1−4] that for τ, Q_1 and Q_2 continuous, given a continuous initial function $\varphi \in C([-\bar{\tau}, 0], R)$ and a number y_0, then there exists a solution of (1.4) on an interval $[0, T)$ satisfying the initial condition and satisfying (1.4) on $[0, T)$. If the solution remains bounded, then $T = +\infty$. We denote such a solution by $(x(t), y(t)) = (x(t, \varphi, y_0), y(t, \varphi, y_0))$. Let $y(s) = y(0)$ for all $s \in (-\infty, 0]$ and $x(s) = x(-\bar{\tau})$ for all $s \in (-\infty, -\bar{\tau}]$. Then $(x(t), y(t))$ can be defined on R.

We also assume that the following conditions hold.

(C_1) $\eta = \min\{\xi - 1, \ (c - \xi) - |a - \xi(\xi - c)| - |b|\} \geq L^+ > 0$.

(C_2) $a - \xi(\xi - c) \geq 0, \quad b \leq 0$.

(C_3) $(c - \xi) > |a - \xi(\xi - c)| + m|b|(2\frac{L^+}{\eta})^{m-1}$.

2. Preliminary Results

The following lemmas will be useful to prove our main results in Section 3.

Lemma 2.1. Let (C_1) hold. Suppose that $(\widetilde{x}(t), \widetilde{y}(t))$ is a solution of system (1.4) with initial conditions

$$\widetilde{x}(s) = \widetilde{\varphi}(s), \ \widetilde{y}(0) = y_0, \max\{|\widetilde{\varphi}(s)|, |y_0|\} < \frac{L^+}{\eta}, \ s \in [-\bar{\tau}, \ 0]. \tag{2.1}$$

Then

$$\max\{|\widetilde{x}(t)|, |\widetilde{y}(t)|\} < \frac{L^+}{\eta} \text{ for all } t \geq 0. \tag{2.2}$$

Proof. Assume, by way of contradiction, that (2.2) does not hold. Then, one of the following cases must occur.

Case 1: There exists $t_1 > 0$ such that

$$\max\{|\widetilde{x}(t_1)|, |\widetilde{y}(t_1)|\} = |\widetilde{x}(t_1)| = \frac{L^+}{\eta} \quad \text{and} \quad \max\{|\widetilde{x}(t)|, |\widetilde{y}(t)|\} < \frac{L^+}{\eta} \quad \text{for all} \quad t \in [-\bar{\tau}, t_1). \tag{2.3}$$

Case 2: There exists $t_2 > 0$ such that

$$\max\{|\widetilde{x}(t_2)|, |\widetilde{y}(t_2)|\} = |\widetilde{y}(t_2)| = \frac{L^+}{\eta} \quad \text{and} \quad \max\{|\widetilde{x}(t)|, |\widetilde{y}(t)|\} < \frac{L^+}{\eta} \quad \text{for all} \quad t \in [-\bar{\tau}, t_2). \tag{2.4}$$

If **Case 1** holds, calculating the upper right derivative of $|\widetilde{x}(t)|$, together with (C_1), (1.4), (1.6) and (2.3) imply that

$$0 \leq D^+(|\widetilde{x}(t_1)|) \leq -\xi|\widetilde{x}(t_1)| + |\widetilde{y}(t_1)| + Q_1(t_1) \leq -(\xi - 1)\frac{L^+}{\eta} + Q_1(t_1) < 0,$$

which is a contradiction and implies that (2.2) holds.

If **Case 2** holds, calculating the upper right derivative of $|\widetilde{y}(t)|$, together with (C_1), (1.4), (1.6) and (2.4) imply that

$$\begin{aligned} 0 &\leq D^+(|\widetilde{y}(t_2)|) \\ &\leq -(c-\xi)|\widetilde{y}(t_2)| + |a - \xi(\xi - c)||\widetilde{x}(t_2)| + |b||\widetilde{x}^m(t_2 - \tau(t_2))| + Q_2(t_2) \\ &\leq -[(c-\xi) - |a - \xi(\xi - c)| - |b|(\frac{L^+}{\eta})^{m-1}]\frac{L^+}{\eta} + Q_2(t_2) \\ &\leq -[(c-\xi) - |a - \xi(\xi - c)| - |b|]\frac{L^+}{\eta} + Q_2(t_2) \\ &< 0, \end{aligned}$$

which is a contradiction and implies that (2.2) holds. The proof of Lemma 2.1 is now completed.

Lemma 2.2. Suppose that (C_1) and (C_2) hold. Moreover, we choose a sufficiently large constant $\theta > 0$ such that for all $t > 0, \zeta = \frac{L^+}{\eta\theta} < \frac{L^+}{\eta}$, and

$$-Q_1(t) < -\xi\zeta + \zeta, \quad \text{and} \quad -Q_2(t) < -(c - \xi)\zeta + (a - \xi(\xi - c))\zeta - b\zeta^m. \tag{2.5}$$

If $(\overline{x}(t), \overline{y}(t))$ is a solution of system (1.4) with initial conditions

$$\overline{x}(s) = \overline{\varphi}(s), \ \overline{y}(0) = y_0, \ \min\{\overline{\varphi}(s), y_0\} > \zeta, \ s \in [-\bar{\tau}, \ 0]. \tag{2.6}$$

Then

$$\min\{\overline{x}(t), \overline{y}(t)\} > \zeta, \ \text{for all} \ t \geq 0. \tag{2.7}$$

Proof. Contrarily, one of the following cases must occur.

Case I: There exists $t_3 > 0$ such that

$$\min\{\overline{x}(t_3), \overline{y}(t_3)\} = \overline{x}(t_3) = \zeta, \quad \text{and} \quad \min\{\overline{x}(t), \overline{y}(t)\} > \zeta \ \text{ for all } \ t \in [-\bar{\tau}, t_3). \tag{2.8}$$

Case II: There exists $t_4 > 0$ such that

$$\min\{\overline{x}(t_4), \overline{y}(t_4)\} = \overline{y}(t_4) = \zeta, \quad \text{and} \quad \min\{\overline{x}(t), \overline{y}(t)\} > \zeta \ \text{ for all } \ t \in [-\bar{\tau}, t_4). \tag{2.9}$$

If **Case I** holds, together with (C_1), (1.4), (2.5) and (2.8) imply that

$$0 \geq \overline{x}'(t_3) = -\xi\overline{x}(t_3) + \overline{y}(t_3) + Q_1(t_3) \geq -\xi\zeta + \zeta + Q_1(t_3) > 0,$$

which is a contradiction and implies that (2.7) holds.

If **Case II** holds, together with (C_2) , (1.4), (2.5) and (2.9) imply that

$$\begin{aligned} 0 &\geq \overline{y}'(t_4) \\ &= -(c-\xi)\overline{y}(t_4) + (a - \xi(\xi - c))\overline{x}(t_4) - b\overline{x}^m(t_4 - \tau(t_4)) + Q_2(t_4) \\ &\geq -(c-\xi)\zeta + (a - \xi(\xi - c))\zeta - b\zeta^m + Q_2(t_4) \\ &> 0, \end{aligned}$$

which is a contradiction and implies that (2.7) holds. The proof of Lemma 2.2 is now completed.

Lemma 2.3. Suppose that (C_1), (C_2) and (C_3) hold. Moreover, assume that $(x(t), y(t))$ is a solution of system (1.4) with initial conditions

$$x(s) = \varphi(s), \ y(0) = y_0, \ \zeta < \min\{\varphi(s), y_0\} \leq \max\{\varphi(s), y_0\} < \frac{L^+}{\eta}, \ s \in [-\bar{\tau}, \ 0]. \tag{2.10}$$

Then for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$, such that every interval $[\alpha, \alpha + l]$ contains at least one number δ for which there exists $N > 0$ satisfies

$$\max\{|x(t+\delta) - x(t)|, |y(t+\delta) - y(t)|\} \leq \epsilon \ \text{ for all } \ t > N. \tag{2.11}$$

Proof. Since

$$\min\{\xi - 1, \ (c-\xi) - |a - \xi(\xi - c)| - m|b|(2\frac{L^+}{\eta})^{m-1}\} > 0,$$

it follows that there exist constants $\lambda > 0$ and γ such that

$$\gamma = \min\{(\xi - 1) - \lambda, \ ((c-\xi) - |a - \xi(\xi - c)| - m|b|(2\frac{L^+}{\eta})^{m-1}e^{\lambda\bar{\tau}}) - \lambda\} > 0. \tag{2.12}$$

Let

$$\begin{cases} \epsilon_1(\delta,t) = Q_1(t+\delta) - Q_1(t), \\ \epsilon_2(\delta,t) = -b[x^m(t-\tau(t+\delta)+\delta) - x^m(t-\tau(t)+\delta)] + Q_2(t+\delta) - Q_2(t). \end{cases} \tag{2.13}$$

By Lemmas 2.1 and 2.2, the solution $(x(t), y(t))$ is bounded and

$$\zeta < \min\{x(t), y(t)\} \leq \max\{x(t), y(t)\} < \frac{L^+}{\eta}, \quad \text{for all} \quad t \in [0, +\infty). \tag{2.14}$$

Thus, the right side of (1.4) is also bounded, which implies that $x(t)$ and $y(t)$ are uniformly continuous on $[-\bar{\tau}, +\infty)$. From (1.5), for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$, such that every interval $[\alpha, \alpha + l], \alpha \in R$, contains a δ for which

$$|\epsilon_i(\delta,t)| \leq \frac{1}{2}\gamma\epsilon, \ i = 1, 2, \ t \geq K_0, \quad \text{where} \quad K_0 \geq 0 \quad \text{is a sufficently large constant.} \tag{2.15}$$

Denote $u(t) = x(t+\delta) - x(t)$ and $v(t) = y(t+\delta) - y(t)$. Let $K_1 > \max\{K_0, -\delta\}$. Then, for $t \geq K_1$, we obtain

$$\frac{du(t)}{dt} = -\xi[x(t+\delta) - x(t)] + y(t+\delta) - y(t) + \epsilon_1(\delta, t), \tag{2.16}$$

and

$$\begin{aligned} \frac{dv(t)}{dt} &= -(c-\xi)[y(t+\delta) - y(t)] + [a - \xi(\xi - c)][x(t+\delta) - x(t)] \\ &\quad -b[x^m(t-\tau(t)+\delta) - x^m(t-\tau(t))] + \epsilon_2(\delta,t). \end{aligned} \tag{2.17}$$

Calculating the upper right derivative of $e^{\lambda s}|u(s)|$ and $e^{\lambda s}|v(s)|$, in view of (2.16), (2.17), (C_1), (C_2) and (C_3), for $t \geq K_1$, we have

$$\begin{aligned} & D^+(e^{\lambda s}|u(s)|)|_{s=t} \\ = \ & \lambda e^{\lambda t}|u(t)| + e^{\lambda t}\ \mathrm{sgn}(u(t))\{-\xi[x(t+\delta) - x(t)] + y(t+\delta) - y(t) + \epsilon_1(\delta,t)\} \\ \leq \ & e^{\lambda t}\{(\lambda - \xi)|u(t)| + |v(t)|\} + \frac{1}{2}\gamma\epsilon e^{\lambda t}, \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} & D^+(e^{\lambda s}|v(s)|)|_{s=t} \\ = \ & \lambda e^{\lambda t}|v(t)| + e^{\lambda t}\ \mathrm{sgn}(v(t))\{-(c-\xi)[y(t+\delta) - y(t)] + (a - \xi(\xi - c))[x(t+\delta) - x(t)] \\ & -b[x^m(t-\tau(t)+\delta) - x^m(t-\tau(t))] + \epsilon_2(\delta,t)\} \\ \leq \ & e^{\lambda t}\{(\lambda - (c-\xi))|v(t)| + |a - \xi(\xi - c)||u(t)| \\ & +|b||x^m(t-\tau(t)+\delta) - x^m(t-\tau(t))|\} + \frac{1}{2}\gamma\epsilon e^{\lambda t}. \end{aligned} \tag{2.19}$$

Let

$$M(t) = \max_{-\bar{\tau} \le s \le t} \{e^{\lambda s} \max\{|u(s)|, |v(s)|\}\}. \tag{2.20}$$

It is obvious that $e^{\lambda t} \max\{|u(t)|, |v(t)|\} \le M(t)$, and $M(t)$ is non-decreasing.

Now, we consider two cases.

Case (i):

$$M(t) > e^{\lambda t} \max\{|u(t)|, |v(t)|\} \quad \text{for all} \quad t \ge K_1. \tag{2.21}$$

We claim that

$$M(t) \equiv M(K_1) \quad \text{is a constant for all} \quad t \ge K_1. \tag{2.22}$$

Assume, by way of contradiction, that (2.22) does not hold. Then, there exists $t_5 > 0$ such that $M(t_5) > M(K_1)$. Since

$$e^{\lambda t} \max\{|u(t)|, |v(t)|\} \le M(K_1) \quad \text{for all} \quad -\bar{\tau} \le t \le K_1.$$

There must exist $\beta \in (K_1,\ t_5)$ such that

$$e^{\lambda \beta} \max\{|u(\beta)|, |v(\beta)|\} = M(t_5) \ge M(\beta),$$

which contradicts (2.21). This contradiction implies that (2.22) holds. It follows that there exists $t_6 > K_1$ such that

$$\max\{|u(t)|, |v(t)|\} \le e^{-\lambda t} M(t) = e^{-\lambda t} M(K_1) < \epsilon \quad \text{for all} \quad t \ge t_6. \tag{2.23}$$

Case (ii): There is a point $t_0 \ge K_1$ such that $M(t_0) = e^{\lambda t_0} \max\{|u(t_0)|, |v(t_0)|\}$. Then, if $M(t_0) = e^{\lambda t_0} \max\{|u(t_0)|, |v(t_0)|\} = e^{\lambda t_0} |u(t_0)|$, in view of (2.18) and (2.19), we get

$$\begin{aligned} D^+(e^{\lambda s}|u(s)|)|_{s=t_0} &\le [\lambda - \xi]|u(t_0)|e^{\lambda t_0} + |v(t_0)|e^{\lambda t_0} + \frac{1}{2}\gamma\epsilon e^{\lambda t_0} \\ &\le [\lambda - (\xi - 1)]M(t_0) + \frac{1}{2}\gamma\epsilon e^{\lambda t_0} \\ &< -\gamma M(t_0) + \gamma\epsilon e^{\lambda t_0}, \end{aligned} \tag{2.24}$$

and

$$\begin{aligned} & D^+(e^{\lambda s}|v(s)|)|_{s=t_0} \\ \le\ & [\lambda - (c - \xi)]|v(t_0)|e^{\lambda t_0} + |a - \xi(\xi - c)||u(t_0)|e^{\lambda t_0} \\ & + |b||x^m(t_0 - \tau(t_0) + \delta) - x^m(t_0 - \tau(t_0))|e^{\lambda((t_0 - \tau(t_0))} e^{\lambda\tau(t_0)} + \frac{1}{2}\gamma\epsilon e^{\lambda t_0} \\ \le\ & [\lambda - (c - \xi)]|v(t_0)|e^{\lambda t_0} + |a - \xi(\xi - c)||u(t_0)|e^{\lambda t_0} + |b|m|(x(t_0 - \tau(t_0)) \\ & + h(t_0)(x(t_0 - \tau(t_0) + \delta) - x(t_0 - \tau(t_0))))^{m-1}(x(t_0 - \tau(t_0) + \delta) \\ & - x(t_0 - \tau(t_0)))|e^{\lambda((t_0 - \tau(t_0))} e^{\lambda\tau(t_0)} + \frac{1}{2}\gamma\epsilon e^{\lambda t_0}, \end{aligned}$$

where $0 < h(t_0) < 1$, it follows that

$$
\begin{aligned}
& D^+(e^{\lambda s}|v(s)|)|_{s=t_0} \\
\leq\ & [\lambda - (c-\xi)]|v(t_0)|e^{\lambda t_0} + |a - \xi(\xi - c)||u(t_0)|e^{\lambda t_0} + |b|m|(1-h(t_0))x(t_0-\tau(t_0)) \\
& + h(t_0)x(t_0 - \tau(t_0)+\delta))^{m-1}||u(t_0-\tau(t_0))|e^{\lambda((t_0-\tau_j(t_0))}e^{\lambda\tau_j(t_0)} + \frac{1}{2}\gamma\epsilon e^{\lambda t_0} \\
\leq\ & [\lambda - (c-\xi)]|v(t_0)|e^{\lambda t_0} + |a - \xi(\xi - c)||u(t_0)|e^{\lambda t_0} + |b|m(2\frac{L^+}{\eta})^{m-1}|u(t_0-\tau(t_0))| \\
& \cdot e^{\lambda((t_0-\tau_j(t_0))}e^{\lambda\tau_j(t_0)} + \frac{1}{2}\gamma\epsilon e^{\lambda t_0} \\
\leq\ & [\lambda - ((c-\xi) - |a-\xi(\xi-c)| - |b|m(2\frac{L^+}{\eta})^{m-1}e^{\lambda\bar{\tau}})]M(t_0) + \frac{1}{2}\gamma\epsilon e^{\lambda t_0} \\
<\ & -\gamma M(t_0) + \gamma\epsilon e^{\lambda t_0}. \qquad (2.25)
\end{aligned}
$$

In addition, if $M(t_0) \geq \epsilon e^{\lambda t_0}$, (2.24) and (2.25) imply that $M(t)$ is strictly decreasing in a small neighborhood $(t_0,\ t_0+\delta_0)$. This contradicts that $M(t)$ is non-decreasing. Hence,

$$
e^{\lambda t_0}\max\{|u(t_0)|, |v(t_0)|\} = M(t_0) < \epsilon e^{\lambda t_0}, \quad \text{and} \quad \max\{|u(t_0)|, |v(t_0)|\} < \epsilon. \qquad (2.26)
$$

For any $t > t_0$, by the same approach used in the proof of (2.26), we have

$$
e^{\lambda t}\max\{|u(t)|, |v(t)|\} < \epsilon e^{\lambda t}, \quad \text{and} \quad \max\{|u(t)|, |v(t)|\} < \epsilon \quad \text{if} \quad M(t) = e^{\lambda t}\max\{|u(t)|, |v(t)|\}. \qquad (2.27)
$$

On the other hand, if $M(t) > e^{\lambda t}\max\{|u(t)|, |v(t)|\}$ for all $t > t_0$, we can choose $t_0 \leq t_7 < t$ such that

$$
M(t_7) = e^{\lambda t_7}\max\{|u(t_7)|, |v(t_7)|\} < e^{\lambda t_7}\epsilon \ \text{ and } \ M(s) > e^{\lambda s}\max\{|u(s)|, |v(s)|\} \quad \text{for all} \quad s \in (t_7,\ t].
$$

Using a similar argument as in the proof of **Case (i)** , we can show that

$$
M(s) \equiv M(t_7) \quad \text{is a constant for all} \quad s \in (t_7,\ t], \qquad (2.28)
$$

which implies that

$$
\max\{|u(t)|, |v(t)|\} < e^{-\lambda t}M(t) = e^{-\lambda t}M(t_7) < \epsilon.
$$

In summary, there must exist $N > 0$ such that $\max\{|u(t)|, |v(t)|\} \leq \epsilon$ holds for all $t > N$. The proof of Lemma 2.3 is now completed.

3. Main Results

In this section, we establish some results for the existence of the positive almost periodic solution of system (1.4).

Theorem 3.1. Suppose that (C_1), (C_2) and (C_3) are satisfied. Then system (1.4) has at least one positive almost periodic solution $Z^*(t) = (x^*(t),\ y^*(t))$.

Proof. Let $(x(t), y(t))$ be a solution of system (1.4) with initial conditions (2.10). Set

$$\begin{cases} \epsilon_{1,k}(t) = Q_1(t+t_k) - Q_1(t), \\ \epsilon_{2,k}(t) = -b[x^m(t-\tau(t+t_k)+t_k) - x^m(t-\tau(t)+t_k)] \\ \qquad\qquad +Q_2(t+t_k) - Q_2(t), \end{cases} \tag{3.1}$$

where t_k is any sequence of real numbers. By Lemmas 2.1 and 2.2, the solution $(x(t), y(t))$ is bounded and (2.14) holds. Again from (1.5), we can select a sequence $\{t_k\} \to +\infty$ such that

$$|\epsilon_{1,k}(t)| \le \frac{1}{k}, |\epsilon_{2,k}(t)| \le \frac{1}{k} \quad \text{for all} \quad t \ge 0. \tag{3.2}$$

Since $\{(x(t+t_k), y(t+t_k))\}_{k=1}^{+\infty}$ is uniformly bounded and equiuniformly continuous, by Arzela-Ascoli Lemma and diagonal selection principle, we can choose a subsequence $\{t_{k_j}\}$ of $\{t_k\}$, such that $(x(t+t_{k_j}), y(t+t_{k_j}))$(for convenience, we still denote by $(x(t+t_k), y(t+t_k))$) uniformly converges to a continuous function $Z^*(t) = (x^*(t), y^*(t))$ on any compact set of R, and

$$\zeta \le \min\{x^*(t), y^*(t)\} \le \max\{x^*(t), y^*(t)\} \le \frac{L^+}{\eta}, \quad \text{for all} \quad t \in R. \tag{3.3}$$

Now, we prove that $Z^*(t)$ is a positive solution of (1.4). In fact, for any $t > 0$ and $\Delta t \in R$, we have

$$\begin{aligned} & x^*(t+\Delta t) - x^*(t) \\ = & \lim_{k\to+\infty} [x(t+\Delta t+t_k) - x(t+t_k)] \\ = & \lim_{k\to+\infty} \int_t^{t+\Delta t} \{-\xi x(\mu+t_k) + y(\mu+t_k) + Q_1(\mu+t_k)\} d\mu \\ = & \int_t^{t+\Delta t} \{-\xi x^*(\mu) + y^*(\mu) + Q_1(\mu)\} d\mu + \lim_{k\to+\infty} \int_t^{t+\Delta t} \epsilon_{1,k}(\mu) d\mu \\ = & \int_t^{t+\Delta t} \{-\xi x^*(\mu) + y^*(\mu) + Q_1(\mu)\} d\mu, \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} & y^*(t+\Delta t) - y^*(t) \\ = & \lim_{k\to+\infty} [y(t+\Delta t+t_k) - y(t+t_k)] \end{aligned}$$

$$
\begin{aligned}
&= \lim_{k\to+\infty}\int_t^{t+\Delta t}\{-(c-\xi)y(\mu+t_k)+(a-\xi(\xi-c))x(\mu+t_k) \\
&\quad -bx^m(\mu-\tau(\mu+t_k)+t_k)+Q_2(\mu+t_k)\}d\mu \\
&= \int_t^{t+\Delta t}\{-(c-\xi)y^*(\mu)+(a-\xi(\xi-c))x^*(\mu)-b(x^*(\mu-\tau(\mu)))^m+Q_2(\mu)\}d\mu \\
&\quad +\lim_{k\to+\infty}\int_t^{t+\Delta t}\epsilon_{2,k}(\mu)d\mu \\
&= \int_t^{t+\Delta t}\{-(c-\xi)y^*(\mu)+(a-\xi(\xi-c))x^*(\mu)-b(x^*(\mu-\tau(\mu)))^m+Q_2(\mu)\}d\mu, \quad (3.5)
\end{aligned}
$$

which imply that

$$
\begin{cases}
\dfrac{dx^*(t)}{dt} = -\xi x^*(t)+y^*(t)+Q_1(t), \\
\dfrac{dy^*(t)}{dt} = -(c-\xi)y^*(t)+(a-\xi(\xi-c))x^*(t)-b(x^*(t-\tau(t)))^m+Q_2(t).
\end{cases} \quad (3.6)
$$

Therefore, $Z^*(t)$ is a positive solution of (1.4).

Secondly, we prove that $Z^*(t)$ is a positive almost periodic solution of (1.4). From Lemma 2.3, for any $\epsilon > 0$, there exists $l = l(\epsilon) > 0$, such that every interval $[\alpha, \alpha + l]$ contains at least one number δ for which there exists $N > 0$ satisfies

$$\max\{|x(t+\delta)-x(t)|, |y(t+\delta)-y(t)|\} \le \epsilon \quad \text{for all} \quad t > N. \quad (3.7)$$

Then, for any fixed $s \in R$, we can find a sufficient large positive integer $N_0 > N$ such that for any $k > N_0$

$$s+t_k > N, \quad \max\{|x(s+t_k+\delta)-x(s+t_k)|, |y(s+t_k+\delta)-y(s+t_k)|\} \le \epsilon. \quad (3.8)$$

Let $k \to +\infty$, we obtain

$$|x^*(s+\delta)-x^*(s)| \le \epsilon n \quad \text{and} \quad |y^*(s+\delta)-y^*(s)| \le \epsilon,$$

which imply that $Z^*(t)$ is a positive almost periodic solution of (1.4). This completes the proof.

4. An Example

Example 4.1. Nonlinear Duffing equation with a deviating argument

$$x''(t)+28x'(t)-192x(t)+2x^3(t-\sin^2(t)) = 12(1+0.9\sin t)+1.8\cos t+1+0.01\sin\sqrt{2}t, \quad (4.1)$$

has at least one positive almost periodic solution $x^*(t)$.

Proof. Set

$$y = \frac{dx}{dt} + 16x - 1 - 0.9 \sin t, \tag{4.2}$$

we can transform (4.1) into the following system

$$\begin{cases} \dfrac{dx(t)}{dt} = -16x(t) + y(t) + 1 + 0.9 \sin t, \\ \dfrac{dy(t)}{dt} = -12y(t) + 2x^3(t - \sin^2(t)) + 1 + 0.9 \cos t + 0.01 \sin \sqrt{2}t. \end{cases} \tag{4.3}$$

Since

$$a = 192, \quad b = -2, \quad c = 28, \quad m = 3, \quad \xi = 16,$$

$$Q_1(t) = 1 + 0.9 \sin t, \quad Q_2(t) = 1 + 0.9 \cos t + 0.01 \sin \sqrt{2}t.$$

Then

$$\eta = \min\{\xi - 1, \quad (c - \xi) - |a - \xi(\xi - c)| - |b|\} = 10 > 0,$$

$$L^+ = 2, \quad a - \xi(\xi - c) = 0, \quad b = -2 \leq 0,$$

$$(c - \xi) > |a - \xi(\xi - c)| + m|b|(2\frac{L^+}{\eta})^{m-1} = 11.04 > 0.$$

It is straightforward to check that all assumptions needed in Theorem 3.1 are satisfied. Hence, system (4.3) has at least one positive almost periodic solution. It follows that nonlinear Duffing equation (4.1) has at least one positive almost periodic solution.

Remark 4.1. Since

$$\tau(t) = \sin^2 t, p(t) = 12(1 + 0.9 \sin t) + 1.8 \cos t + 1 + 0.01 \sin \sqrt{2}t,$$

it is clear that the condition (H_0) is not satisfied. Therefore, all the results in [1-7] and the references therein can not be applicable to prove that the existence of positive almost periodic solutions for nonlinear Duffing equation (4.1). Moreover, we propose a totally new approach to proving the existence of positive almost periodic solutions of nonlinear Duffing equation, which is different from [1-9] and the references therein. This implies that the results of this paper are essentially new.

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