Electronic Journal of Qualitative Theory of Differential Equations
2010, No. **5**, 1-6; **http://www.math.u-szeged.hu/ejqtde/**

EXISTENCE OF A POSITIVE SOLUTION TO A RIGHT FOCAL BOUNDARY VALUE PROBLEM

RICHARD I. AVERY, JOHNNY HENDERSON AND DOUGLAS R. ANDERSON

Abstract. In this paper we apply the recent extension of the Leggett-Williams Fixed Point Theorem which requires neither of the functional boundaries to be invariant to the second order right focal boundary value problem. We demonstrate a technique that can be used to deal with a singularity and provide a non-trivial example.

1. Introduction

The recent topological proof and extension of the Leggett-Williams fixed point theorem [3] does not require either of the functional boundaries to be invariant with respect to a functional wedge and its proof uses topological methods instead of axiomatic index theory. Functional fixed point theorems (including [2, 4, 5, 6, 8]) can be traced back to Leggett and Williams [7] when they presented criteria which guaranteed the existence of a fixed point for a completely continuous map that did not require the operator to be invariant with regard to the concave functional boundary of a functional wedge. Avery, Henderson, and O'Regan [1], in a dual of the Leggett-Williams fixed point theorem, gave conditions which guaranteed the existence of a fixed point for a completely continuous map that did not require the operator to be invariant relative to the concave functional boundary of a functional wedge. We will demonstrate a technique to take advantage of the added flexibility of the new fixed point theorem for a right focal boundary value problem.

2. Preliminaries

In this section we will state the definitions that are used in the remainder of the paper.

Definition 1. *Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a* cone *if it satisfies the following two conditions:*

(*i*) $x \in P, \lambda \geq 0$ *implies* $\lambda x \in P$;
(*ii*) $x \in P, -x \in P$ *implies* $x = 0$.

Every cone $P \subset E$ induces an ordering in E given by

$$x \leq y \text{ if and only if } y - x \in P.$$

Definition 2. *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

2000 *Mathematics Subject Classification.* 34B10.
Key words and phrases. Fixed-point theorems, positive solutions, singularities, cone.

Definition 3. *A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0, \infty)$ is continuous and*

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly we say the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Let α and ψ be non-negative continuous concave functionals on P and δ and β be non-negative continuous convex functionals on P; then, for non-negative real numbers a, b, c and d, we define the following sets:

$$A := A(\alpha, \beta, a, d) = \{x \in P : a \leq \alpha(x) \text{ and } \beta(x) \leq d\}, \tag{1}$$

$$B := B(\alpha, \delta, \beta, a, b, d) = \{x \in A : \delta(x) \leq b\}, \tag{2}$$

and

$$C := C(\alpha, \psi, \beta, a, c, d) = \{x \in A : c \leq \psi(x)\}. \tag{3}$$

We say that A is a *functional wedge with concave functional boundary* defined by the concave functional α and convex functional boundary defined by the convex functional β. We say that an operator $T : A \to P$ is *invariant with respect to the concave functional boundary*, if $a \leq \alpha(Tx)$ for all $x \in A$, and that T is *invariant with respect to the convex functional boundary*, if $\beta(Tx) \leq d$ for all $x \in A$. Note that A is a convex set. The following theorem is an extension of the original Leggett-Williams fixed point theorem [7].

Theorem 4. [Extension of Leggett-Williams] *Suppose P is a cone in a real Banach space E, α and ψ are non-negative continuous concave functionals on P, δ and β are non-negative continuous convex functionals on P, and for non-negative real numbers a, b, c and d the sets A, B and C are as defined in* (1), (2) *and* (3)*. Furthermore, suppose that A is a bounded subset of P, that $T : A \to P$ is completely continuous and that the following conditions hold:*

- *(A1) $\{x \in A : c < \psi(x)$ and $\delta(x) < b\} \neq \emptyset$ and $\{x \in P : \alpha(x) < a$ and $d < \beta(x)\} = \emptyset$;*
- *(A2) $\alpha(Tx) \geq a$ for all $x \in B$;*
- *(A3) $\alpha(Tx) \geq a$ for all $x \in A$ with $\delta(Tx) > b$;*
- *(A4) $\beta(Tx) \leq d$ for all $x \in C$; and,*
- *(A5) $\beta(Tx) \leq d$ for all $x \in A$ with $\psi(Tx) < c$.*

Then T has a fixed point $x^* \in A$.

3. Right Focal Boundary Value Problem

In this section we will illustrate the key techniques for verifying the existence of a positive solution for a boundary value problem using the newly developed extension of the Leggett-Williams fixed point theorem, applying the properties of a Green's function, bounding the nonlinearity by constants over some intervals, and using concavity to deal with a singularity. Consider the second order nonlinear focal boundary value problem

$$x''(t) + f(x(t)) = 0, \quad t \in (0,1), \tag{4}$$

$$x(0) = 0 = x'(1), \tag{5}$$

where $f : \mathbb{R} \to [0,\infty)$ is continuous. If x is a fixed point of the operator T defined by

$$Tx(t) := \int_0^1 G(t,s) f(x(s))ds,$$

where

$$G(t,s) = \begin{cases} t & : \ t \leq s, \\ s & : \ s \leq t, \end{cases}$$

is the Green's function for the operator L defined by

$$Lx(t) := -x'',$$

with right-focal boundary conditions

$$x(0) = 0 = x'(1),$$

then it is well known that x is a solution of the boundary value problem (4), (5). Throughout this section of the paper we will use the facts that $G(t,s)$ is nonnegative, and for each fixed $s \in [0,1]$, the Green's function is nondecreasing in t.

Define the cone $P \subset E = C[0,1]$ by

$$P := \{x \in E \ : \ x \text{ is nonnegative, nondecreasing, and concave}\}.$$

For fixed $\nu, \tau, \mu \in [0,1]$ and $x \in P$, define the concave functionals α and ψ on P by

$$\alpha(x) := \min_{t\in[\tau,1]} x(t) = x(\tau), \quad \psi(x) := \min_{t\in[\mu,1]} x(t) = x(\mu),$$

and the convex functionals δ and β on P by

$$\delta(x) := \max_{t\in[0,\nu]} x(t) = x(\nu), \quad \beta(x) := \max_{t\in[0,1]} x(t) = x(1).$$

In the following theorem, we demonstrate how to apply the Extension of the Leggett-Williams Fixed Point Theorem (Theorem 4), to prove the existence of at least one positive solution to (4), (5).

Theorem 5. *If $\tau, \nu, \mu \in (0,1]$ are fixed with $\tau \leq \mu < \nu \leq 1$, d and m are positive real numbers with $0 < m \leq d\mu$ and $f : [0,\infty) \to [0,\infty)$ is a continuous function such that*

(a) $f(w) \geq \frac{d}{\nu-\tau}$ for $w \in [\tau d, \nu d]$,
(b) $f(w)$ is decreasing for $w \in [0,m]$ with $f(m) \geq f(w)$ for $w \in [m,d]$, and
(c) $\int_0^{\mu} s\, f\left(\frac{ms}{\mu}\right)\, ds \leq \frac{2d - f(m)(1-\mu^2)}{2}$,

then the operator T has at least one positive solution $x^* \in A(\alpha, \beta, \tau d, d)$.

Proof. Let $a = \tau d$, $b = \nu d = \frac{a\nu}{\tau}$, and $c = d\mu$. Let $x \in A(\alpha, \beta, a, d)$ then if $t \in (0,1)$, by the properties of the Green's function $(Tx)''(t) = -f(x(t))$ and $Tx(0) = 0 = (Tx)'(1)$, thus

$$T : A(\alpha, \beta, a, d) \to P.$$

We will also take advantage of the following property of the Green's function. For any $y, w \in [0,1]$ with $y \leq w$ we have

$$\min_{s\in[0,1]} \frac{G(y,s)}{G(w,s)} \geq \frac{y}{w}. \tag{6}$$

By the Arzela-Ascoli Theorem it is a standard exercise to show that T is a completely continuous operator using the properties of G and f, and by the definition of β, we have that A is a bounded subset of the cone P. Also, if $x \in P$ and $\beta(x) > d$, then by the properties of the cone P,

$$\alpha(x) = x(\tau) \geq \left(\frac{\tau}{1}\right) x(1) = \tau\beta(x) > \tau d = a.$$

Therefore,

$$\{x \in P \ : \ \alpha(x) < a \text{ and } d < \beta(x)\} = \emptyset.$$

For any $K \in \left(\frac{2d}{2-\mu}, \frac{2d}{2-\nu}\right)$ the function x_K defined by

$$x_K(t) \equiv \int_0^1 KG(t,s)ds = \frac{Kt(2-t)}{2} \in A,$$

since

$$\alpha(x_K) = x_K(\tau) = \frac{K\tau(2-\tau)}{2} > \frac{d\tau(2-\tau)}{2-\mu} \geq d\tau = a,$$

$$\beta(x_K) = x_K(1) = \frac{K}{2} < \frac{d}{2-\nu} \leq d,$$

and x_K has the properties that

$$\psi(x_K) = x_K(\mu) = \frac{K\mu(2-\mu)}{2} > \left(\frac{2d}{2-\mu}\right)\left(\frac{\mu(2-\mu)}{2}\right) = d\mu = c$$

and

$$\delta(x_K) = x_K(\nu) = \frac{K\nu(2-\nu)}{2} < \left(\frac{2d}{2-\nu}\right)\left(\frac{\nu(2-\nu)}{2}\right) = d\nu = b.$$

Hence

$$\{x \in A \ : \ c < \psi(x) \text{ and } \delta(x) < b\} \neq \emptyset.$$

Claim 1: $\alpha(Tx) \geq a$ for all $x \in B$.

Let $x \in B$. Thus by condition (a),

$$\begin{aligned} \alpha(Tx) &= \int_0^1 G(\tau,s)\ f(x(s))\ ds \geq \left(\frac{a}{\tau(\nu-\tau)}\right) \int_\tau^\nu G(\tau,s)\ ds \\ &= \left(\frac{a}{\tau(\nu-\tau)}\right)(\tau(\nu-\tau)) = a. \end{aligned}$$

Claim 2: $\alpha(Tx) \geq a$, for all $x \in A$ with $\delta(Tx) > b$.

Let $x \in A$ with $\delta(Tx) > b$. Thus by the properties of G (6),

$$\begin{aligned}\alpha(Tx) &= \int_0^1 G(\tau,s)\, f(x(s))\, ds \geq \left(\frac{\tau}{\nu}\right)\int_0^1 G(\nu,s)\, f(x(s))\, ds \\ &= \left(\frac{\tau}{\nu}\right)\delta(Tx) > \left(\frac{\tau}{\nu}\right)(d\nu) = a.\end{aligned}$$

Claim 3: $\beta(Tx) \leq d$, for all $x \in C$.

Let $x \in C$, thus by the concavity of x, for $s \in [0,\mu]$ we have

$$x(s) \geq \frac{cs}{\mu} \geq \frac{ms}{\mu}.$$

Hence by properties (b) and (c),

$$\begin{aligned}\beta(Tx) &= \int_0^1 G(1,s)\, f(x(s))\, ds = \int_0^1 s\, f(x(s))\, ds \\ &= \int_0^\mu s\, f(x(s))\, ds + \int_\mu^1 s\, f(x(s))\, ds \\ &\leq \int_0^\mu s\, f\left(\frac{ms}{\mu}\right)\, ds + f(m)\int_\mu^1 s\, ds \\ &\leq \frac{2d - f(m)(1-\mu^2)}{2} + \frac{f(m)(1-\mu^2)}{2} = d.\end{aligned}$$

Claim 4: $\beta(Tx) \leq d$, for all $x \in A$ with $\psi(Tx) < c$.

Let $x \in A$ with $\psi(Tx) < c$. Thus by the properties of G (6),

$$\begin{aligned}\beta(Tx) &= \int_0^1 G(1,s)\, f(x(s))\, ds \leq \left(\frac{1}{\mu}\right)\int_0^1 G(\mu,s)\, f(x(s))\, ds \\ &= \left(\frac{1}{\mu}\right)Tx(\mu) = \left(\frac{1}{\mu}\right)\psi(Tx) \leq \left(\frac{1}{\mu}\right)c = d.\end{aligned}$$

Therefore, the hypotheses of Theorem 4 have been satisfied; thus the operator T has at least one positive solution $x^* \in A(\alpha, \beta, a, d)$. □

We note that because of the concavity of solutions, the proof of Theorem 5 remains valid for certain singular nonlinearities as presented in this example.

Example: Let

$$d = \frac{5}{4},\ \tau = \frac{1}{16},\ \mu = \frac{3}{4},\ \text{ and }\ \nu = \frac{15}{16}.$$

Then the boundary value problem

$$x'' + \frac{1}{\sqrt{x}} + \sqrt{x} = 0,$$

with right-focal boundary conditions

$$x(0) = 0 = x'(1),$$

has at least one positive solution x^* which can be verified by the above theorem, with

$$5/64 \leq x^*(1/16) \quad \text{and} \quad x^*(1) \leq 5/4.$$

References

[1] R. I. Avery, J. Henderson and D. O'Regan, Dual of the compression-expansion fixed point theorems, *Fixed Point Theory and Applications*, **2007** (2007), Article ID 90715, 11 pages.

[2] D. R. Anderson and R. I. Avery, Fixed point theorem of cone expansion and compression of functional type, *J. Difference Equations Appl.*, **8**(2002), 1073–1083.

[3] D. R. Anderson, R. I. Avery and J. Henderson, A topological proof and extension of the Leggett-Williams fixed point theorem, *Communications on Applied Nonlinear Analysis*, **16**(2009), 39–44.

[4] D. Guo, A new fixed point theorem, *Acta Math. Sinica*, **24**(1981), 444–450.

[5] D. Guo, Some fixed point theorems on cone maps, *Kexeu Tongbao*, **29**(1984), 575–578.

[6] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations*, P. Noordhoff, Groningen, The Netherlands, 1964.

[7] R. W. Leggett and L. R. Williams, Multiple positive fixed points of nonlinear operators on ordered Banach spaces, *Indiana Univ. Math. J.*, **28**(1979), 673–688.

[8] J. Sun and G. Zhang, A generalization of the cone expansion and compression fixed point theorem and applications, *Nonlin. Anal.*, **67**(2007), 579-586.

[9] E. Zeidler, *Nonlinear Functional Analysis and its Applications I: Fixed-Point Theorems*, Springer-Verlag, New York, 1986.

(Received November 23, 2009)

College of Arts and Sciences, Dakota State University, Madison, South Dakota 57042 USA

E-mail address: rich.avery@dsu.edu

Department of Mathematics, Baylor University, Waco, Texas 76798 USA

E-mail address: Johnny_Henderson@baylor.edu

Department of Mathematics and Computer Science, Concordia College, Moorhead, MN 56562 USA

E-mail address: andersod@cord.edu