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EXISTENCE OF SOLUTIONS OF ABSTRACT FRACTIONAL IMPULSIVE SEMILINEAR EVOLUTION EQUATIONS

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Abstract

In this paper we prove the existence of solutions of fractional impulsive semilinear evolution equations in Banach spaces. A nonlocal Cauchy problem is discussed for the evolution equations. The results are obtained using fractional calculus and fixed point theorems. An example is provided to illustrate the theory.

Keywords : Existence of solution, evolution equation, nonlocal condition, fractional calculus, fixed point theorems.

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1 Introduction

Fractional differential equations are increasingly used for many mathematical models in science and engineering. In fact fractional differential equations are considered as an alternative model to nonlinear differential equations [8]. The theory of fractional differential equations has been extensively studied by several authors [11, 16-19]. In [12, 14] the authors have proved the existence of solutions of abstract differential equations by using semigroup theory and fixed point theorem. Many partial fractional differential equations can be expressed as fractional differential equations in some Banach spaces [13].

The nonlocal Cauchy problem for abstract evolution differential equation was first studied by Byszewski [9]. Subsequently several authors have investigated the problem for different types of nonlinear differential equations and integrodifferential equations including functional differential equations in Banach spaces [2-4, 10, 20]. Mophou and N'Guérékata [22, 23, 24] and Balachandran and Park [5] discussed the existence of solutions of abstract fractional differential equations with nonlocal initial conditions.

Impulsive differential equations have become important in recent years as mathematical models of phenomena in both physical and social sciences. There has been a

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significant development in impulsive theory especially in the area of impulsive differential equations with fixed moments. Recently Benchohra and Slimani [6,7] discussed the existence and uniqueness of solutions of impulsive fractional differential equations and Ahmad and Sivasundaram [1] discussed the existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations. Motivated by this work we study in this paper the existence of solutions of fractional impulsive semilinear evolution equations in Banach spaces by using fractional calculus and fixed point theorems.

2 Preliminaries

We need some basic definitions and properties of fractional calculus which are used in this paper. By $C(J, \mathbb{X})$ we denote the Banach space of continuous functions $x(t)$ with $x(t) \in \mathbb{X}$ for $t \in J$, a compact interval in R and $\|x\|_{C(J,\mathbb{X})} = \max_{t \in J} \|x(t)\|$.

Definition2.1. A real function $f(t)$ is said to be in the space C_α, $\alpha \in R$ if there exists a real number $p > \alpha$, such that $f(t) = t^p g(t)$, where $g \in C[0, \infty)$ and it is said to be in the space C_α^m iff $f^{(m)} \in C_\alpha$, $m \in N$.

Definition2.2. The Riemann-Liouville fractional integral operator of order $\beta > 0$ of function $f \in C_\alpha$, $\alpha \geq -1$ is defined as

$$I^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds$$

where $\Gamma(.)$ is the Euler gamma function.

Definition2.3. If the function $f \in C_{-1}^m$ and m is a positive integer then we define the fractional derivative of $f(t)$ in the Caputo sense as

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^m(s) ds, \quad \mathrm{m} - 1 < \alpha < m.$$

If $0 < \alpha < 1$, then

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{f'(s))}{(t-s)^\alpha} ds,$$

where $f'(s) = \dfrac{df(s)}{ds}$ and f is an abstract function with values in $\mathbb{X}$. For basic facts about fractional derivatives and fractional calculus one can refer to the books [15,21,25,26].

Consider the Banach space

$$\begin{aligned} PC(J, \mathbb{X}) \ &= \{u : J \to \mathbb{X} : u \in C((t_k, t_{k+1}], \mathbb{X}),\ k = 0, \ldots, m \text{ and there exist} \\ &\qquad u(t_k^-) \text{ and } u(t_k^+),\ k = 1, \ldots, m \text{ with } u(t_k^-) = u(t_k)\}. \end{aligned}$$

with the norm $\|u\|_{PC} = \sup_{t\in J} \|u(t)\|$. Set $J' := [0,T]\backslash\{t_1,\dots,t_m\}$.

Consider the linear fractional impulsive evolution equation

$$\begin{aligned} &\frac{d^q u(t)}{dt^q} = A(t)u(t), \quad t\in J=[0,T],\ t\neq t_k, \\ &\Delta u\,|_{t=t_k} = I_k(u(t_k^-)), \\ &u(0)=u_0, \end{aligned} \tag{2.1}$$

where $0 < q \le 1$ and $A(t)$ is a bounded linear operator on a Banach space $\mathbb{X}$, $I_k : \mathbb{X} \to \mathbb{X},\ k = 1,2,\cdots,m$ and $u_0 \in \mathbb{X}$, $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = T$, $\Delta u|_{t=t_k} = u(t_k^+) - u(t_k^-)$, $u(t_k^+) = \lim\limits_{h\to 0^+} u(t_k + h)$ and $u(t_k^-) = \lim\limits_{h\to 0^-} u(t_k + h)$ represent the right and left limits of $u(t)$ at $t = t_k$.
Our Eq.(2.1) is equivalent to the integral equation

$$u(t) = \begin{cases} u_0 + \dfrac{1}{\Gamma(q)}\displaystyle\int_0^t (t-s)^{q-1}A(s)u(s)ds, \quad \text{if } t \in [0,t_1], \\ u_0 + \dfrac{1}{\Gamma(q)}\displaystyle\sum_{i=1}^{k}\int_{t_{i-1}}^{t_i} (t_i-s)^{q-1}A(s)u(s)ds \\ +\dfrac{1}{\Gamma(q)}\displaystyle\int_{t_k}^t (t-s)^{q-1}A(s)u(s)ds + \sum_{i=1}^{k} I_i(u(t_i^-)), \quad \text{if } t \in (t_k,t_{k+1}]. \end{cases} \tag{2.2}$$

Definition 2.4. By a solution of the abstract Cauchy problem (2.1), we mean an abstract function u such that the following conditions are satisfied:
(i) $u \in PC(J,\mathbb{X})$ and $u \in D(A(t))$ for all $t \in J'$;
(ii) $\dfrac{d^q u}{dt^q}$ exists on J' where $0 < q < 1$;
(iii) u satisfies Eq.(2.1) on J', and satisfy the conditions

$$\begin{aligned} &\Delta u|_{t=t_k} = I_k(u(t_k^-)),\ k = 1,\dots,m, \\ &u(0) = u_0. \end{aligned}$$

Now, we assume the following conditions to prove the existence of a solution of the evolution Eq.(2.1).
(HA) $A(t)$ is a bounded linear operator on $\mathbb{X}$ for each $t \in J$ and the function $t \to A(t)$ is continuous in the uniform operator topology.
(HI) The functions $I_k : \mathbb{X} \to \mathbb{X}$ are continuous and there exists a constant $L_1 > 0$ such that

$$\|I_k(u) - I_k(v)\| \le L_1\|u - v\|, \quad \text{for each } \ u,v \in \mathbb{X} \ \text{ and } \ k = 1,2\cdots,m.$$

For brevity let us take $\dfrac{T^q}{\Gamma(q+1)} = \gamma$.
Theorem 2.1 If the hypotheses (HA) and (HI) are satisfied, then Eq.(2.1) has a unique solution on J.

Proof: The proof is based on the application of Picard's iteration method. Let $M = \max_{0\le t\le T} \|A(t)\|$ and define a mapping $F : PC([0,T] : \mathbb{X}) \to PC([0,T] : \mathbb{X})$ by

$$\begin{aligned} Fu(t) &= u_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)u(s)ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s)u(s)ds + \sum_{0<t_k<t} I_k(u(t_k^-)). \end{aligned} \tag{2.3}$$

Let $u, v \in PC(J, \mathbb{X})$. Then from Eq.(2.3), we have for each $t \in J$

$$\|Fu(t) - Fv(t)\| \le \frac{T^q}{\Gamma(q+1)} M\big(m+1\big)\|u-v\| + mL_1\|u-v\|.$$

Then by induction we have

$$\|F^n u(t) - F^n v(t)\| \le \frac{\Big(M\gamma(m+1) + mL_1\Big)^n}{n!}\|u-v\|.$$

Since $\frac{1}{n!}\big(M\gamma(m+1) + mL_1\big)^n < 1$ for large n, then by the well-known generalization of the Banach contraction principle, F has a unique fixed point $u \in PC([0,T] : \mathbb{X})$. This fixed point is the solution of Eq.(2.1).

3 Semilinear Evolution Equation

Now consider the semilinear fractional impulsive evolution equation

$$\begin{aligned} &\frac{d^q u(t)}{dt^q} = A(t)u(t) + f(t, u(t)), \quad t \in J = [0,T], \ t \ne t_k, \\ &\Delta u\,|_{t=t_k} = I_k(u(t_k^-)), \\ &u(0) = u_0, \end{aligned} \tag{3.1}$$

where $A(t)$ is a bounded linear operator and $f : J \times \mathbb{X} \to \mathbb{X}$ is continuous. This Eq.(3.1) is equivalent to the integral equation

$$\begin{aligned} u(t) &= u_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s)u(s)ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s)u(s)ds + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, u(s))ds \\ &\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, u(s))ds + \sum_{0<t_k<t} I_k(u(t_k^-)). \end{aligned} \tag{3.2}$$

We need the following additional assumptions to prove the existence of solution of the Eq.(3.1).
(Hf) $f : J \times \mathbb{X} \to \mathbb{X}$ is continuous and there exist constants $L_2 > 0, \ N > 0$ such that

$$\|f(t,u) - f(t,v)\| \le L_2\|u-v\| \text{ for all } \ u, v \in \mathbb{X}.$$

and $N = \max_{t \in J} \|f(t,0)\|$.

Theorem 3.1. If the hypotheses (HA),(HI),(Hf) are satisfied and if $\gamma(m+1)(M+L_2) + mL_1 < \frac{1}{2}$ then the fractional impulsive evolution Eq.(3.1) has a unique solution on J.

Proof. Let $Z = PC(J, \mathbb{X})$. Define the mapping $\Phi : Z \to Z$ by

$$\begin{aligned} \Phi u(t) &= u_0 + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s) u(s) ds \\ &+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s) u(s) ds + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, u(s)) ds \\ &+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, u(s)) ds + \sum_{0<t_k<t} I_k(u(t_k^-)). \end{aligned} \tag{3.3}$$

and we have to show that Φ has a fixed point. This fixed point is the solution of Eq.(3.1). Choose $r \geq 2\big(\|u_0\| + (m+1)N\gamma\big)$. Then we can show that $\Phi B_r \subset B_r$, where $B_r := \{u \in Z : \|u\| \leq r\}$. From the assumptions we have

$$\begin{aligned} \|\Phi u(t)\| &\leq \|u_0\| + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|A(s)\| \|u(s)\| ds \\ &+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|A(s)\| \|u(s)\| ds \\ &+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|f(s, u(s))\| ds \\ &+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|f(s, u(s))\| ds + \sum_{0<t_k<t} \|I_k(u(t_k^-))\| \\ &\leq \|u_0\| + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|A(s)\| \|u(s)\| ds \\ &+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|A(s)\| \|u(s)\| ds \\ &+ \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \big(\|f(s, u(s)) - f(s,0)\| + \|f(s,0)\|\big) ds \\ &+ \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \big(\|f(s, u(s)) - f(s,0)\| + \|f(s,0)\|\big) ds \\ &+ \sum_{0<t_k<t} \|I_k(u(t_k^-))\| \\ &\leq \|u_0\| + \frac{T^q}{\Gamma(q+1)} \big(mMr + Mr + m(L_2 r + N) + (L_2 r + N)\big) + mL_1 r \\ &= \|u_0\| + (m+1)\gamma N + r\big(\gamma(m+1)(M+L_2) + mL_1\big) \\ &\leq r. \end{aligned}$$

Thus, Φ maps B_r into itself. Now, for $u_1, u_2 \in Z$, we have

$$\begin{aligned}
\|\Phi u_1(t) - \Phi u_2(t)\| &\leq \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|A(s)\big(u_1(s) - u_2(s)\big)\| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|A(s)\big(u_1(s) - u_2(s)\big)\| ds \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} \|f(s, u_1(s)) - f(s, u_2(s))\| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|f(s, u_1(s) - f(s, u_2(s))\| ds \\
&\quad + \sum_{0<t_k<t} \|I_k(u_1(t_k^-)) - I_k(u_2(t_k^-))\| \\
&\leq \frac{T^q}{\Gamma(q+1)} \big(m(M+L_2) + (M+L_2)\big) \|u_1 - u_2\| + mL_1 \|u_1 - u_2\| \\
&\leq \big(\gamma(m+1)(M+L_2) + mL_1\big) \|u_1 - u_2\| \\
&\leq \frac{1}{2} \|u_1 - u_2\|.
\end{aligned}$$

Hence Φ is a contraction mapping and therefore there exists a unique fixed point $u \in B_r$ such that $\Phi u(t) = u(t)$. Any fixed point of Φ is the solution of Eq.(3.1).

4 Nonlocal Cauchy Problem

In this section we discuss the existence of solution of the impulsive evolution equation (3.1) with nonlocal condition of the form

$$u(0) + g(u) = u_0 \tag{4.1}$$

where $g : PC(J, \mathbb{X}) \to \mathbb{X}$ is a given function which satisfies the following condition.
(Hg) $g : PC(J, \mathbb{X}) \to \mathbb{X}$ is continuous and there exists a constant $G > 0$ such that

$$\|g(u) - g(v)\| \leq G\|u - v\|_{PC} \quad \text{for} \quad u, v \in PC(J, \mathbb{X}).$$

Theorem 4.1. If the hypotheses (HA),(HI),(Hf) and (Hg) are satisfied and if $\gamma(m+1)(M+L_2)+mL_1+G < \frac{1}{2}$ then the fractional impulsive evolution Eq.(3.1) with nonlocal condition (4.1) has a unique solution on J.

Proof: We want to prove that the operator $\Psi : Z \to Z$ defined by

$$\begin{aligned}
\Psi u(t) &= u_0 - g(u) + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} A(s) u(s) ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} A(s) u(s) ds + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k - s)^{q-1} f(s, u(s)) ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} f(s, u(s)) ds + \sum_{0<t_k<t} I_k(u(t_k^-))
\end{aligned} \tag{4.2}$$

has a fixed point. This fixed point is then a solution of Eq.(3.1) and Eq.(4.1). Choose $r \geq 2\big(\|u_0\| + \|g(0)\| + (m+1)N\gamma\big)$. Then it is easy to see that $\Psi B_r \subset B_r$. Further, for $u_1, u_2 \in Z$, we have

$$\begin{aligned}\|\Psi u_1(t) - \Psi u_2(t)\| &\leq \big(G+\gamma(m+1)(M+L_2)+mL_1\big)\|u_1(t)-u_2(t)\| \\ &\leq \frac{1}{2}\|u_1(t)-u_2(t)\|.\end{aligned}$$

The result follows by the application of contraction mapping principle.

Our next result is based on the following well-known fixed point theorem.

Krasnoselskii Theorem. Let S be a closed convex nonempty subset of a Banach space X. Let P, Q be two operators such that (i) $Px + Qy \in S$ whenever $x, y \in S$;
(ii) P is a contraction mapping;
(iii) Q is compact and continuous.
Then there exists $z \in S$ such that $z = Pz + Qz$.

Now, we assume the following condition instead of (Hf) and apply the above fixed point theorem.

$(\mathrm{Hf})'$ $f : J \times \mathbb{X} \to \mathbb{X}$ is continuous and there exists a continuous function $\mu \in L^1(J)$ such that $\|f(t,u)\| \leq \mu(t)$, for all $(t,u) \in J \times \mathbb{X}$.

Theorem 4.2. Assume that (HA),(HI),$(\mathrm{Hf})'$,(Hg) hold. If $G+\gamma(m+1)M+mL_1 < 1$, then the fractional evolution Eq.(3.1) with nonlocal condition (4.1) has a solution on J.

Proof. Choose

$$r \geq \frac{\|u_0\| + \|g(0)\| + \gamma(m+1)\mu_0}{1-(G+\gamma(m+1)M+mL_1)}$$

where $\mu_0 = \sup_{t\in J}\mu(t)$ and define the operators P and Q on B_r as

$$\begin{aligned}Pu(t) &= u_0 - g(u) + \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}A(s)u(s)ds \\ &\quad + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}A(s)u(s)ds + \sum_{0<t_k<t} I_k(u(t_k^-)) \quad \text{and} \\ Qu(t) &= \frac{1}{\Gamma(q)}\sum_{0<t_k<t}\int_{t_{k-1}}^{t_k}(t_k-s)^{q-1}f(s,u(s))ds + \frac{1}{\Gamma(q)}\int_{t_k}^{t}(t-s)^{q-1}f(s,u(s))ds.\end{aligned}$$

For any $u, v \in B_r$, we have

$$\begin{aligned}
&\|Pu(t)+Qv(t)\| \\
&\quad \leq \|u_0\| + \|g(u)-g(0)\| + \|g(0)\| + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k-s)^{q-1} \|A(s)\| \|u(s)\| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|A(s)\| \|u(s)\| ds + \sum_{0<t_k<t} \|I_k(u(t_k^-)\| \\
&\quad + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k-s)^{q-1} \|f(s,v(s))\| ds + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|f(s,v(s))\| ds \\
&\quad \leq \|u_0\| + \|g(0)\| + \gamma(m+1)\mu_0 + r(G + (m+1)\gamma M + mL_1) \\
&\quad \leq r.
\end{aligned}$$

Hence, we deduce that $\|Pu + Qv\| \leq r$.
Next, for any $t \in I$, $u, v \in \mathbb{X}$ we have

$$\begin{aligned}
&\|Pu(t) - Pv(t)\| \\
&\quad \leq \|g(u)-g(v)\| + \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k-s)^{q-1} \|A(s)\| \|u(s)-v(s)\| ds \\
&\quad + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|A(s)\| u(s)-v(s)\| ds + \sum_{0<t_k<t} \|I_k(u(t_k^-)) - I_k(v(t_k^-))\| \\
&\quad \leq G\|u-v\| + \gamma(m+1)M\|u-v\| + mL_1\|u-v\| \\
&\quad \leq (G + \gamma(m+1)M + mL_1)\|u-v\|.
\end{aligned}$$

And since $G + \gamma(m+1)M + mL_1 < 1$, then P is a contraction mapping.
Now, let us prove that Q is continuous and compact.

Let $\{u_n\}$ be a sequence in B_r, such that $u_n \to u$ in B_r. Then

$$f(s, u_n(s)) \to f(s, u(s)), \quad n \to \infty$$

because the function f is continuous on $I \times \mathbb{X}$. Now, for each $t \in I$, we have

$$\begin{aligned}
\|Qu_n(t) - Qu(t)\| \leq\ & \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k-s)^{q-1} \|f(s,u_n(s)) - f(s,u(s))\| ds \\
& + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|f(s,u_n(s) - f(s,u(s))\| ds \\
& \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Consequently, $\lim_{n\to\infty} \|Qu_n(t) - Qu(t)\| = 0$. In other words, Q is continuous.
Let's now note that Q is uniformly bounded on B_r. This follows from the inequality

$$\begin{aligned}
\|Qu(t)\| \leq\ & \frac{1}{\Gamma(q)} \sum_{0<t_k<t} \int_{t_{k-1}}^{t_k} (t_k-s)^{q-1} \|f(s,u(s))\| ds \\
& + \frac{1}{\Gamma(q)} \int_{t_k}^{t} (t-s)^{q-1} \|f(s,u(s))\| ds. \\
\leq\ & \gamma(m+1)\mu_0.
\end{aligned}$$

Now, let's prove that $Qu,\ u \in B_r$ is equicontinuous. Let $t_1, t_2 \in J,\ t_1 < t_2$, and let B_r be a bounded set in $\mathbb{X}$. Let $u \in B_r$, we have

$$\begin{aligned}
&\|Qu(t_2) - Qu(t_1)\| \\
&\quad \leq \frac{1}{\Gamma(q)} \int_0^{t_2} (t_2 - s)^{q-1} \|f(s, u(s))\| ds + \frac{1}{\Gamma(q)} \int_0^{t_1} (t_2 - s)^{q-1} \|f(s, u(s))\| ds \\
&\quad \leq \frac{1}{\Gamma(q)} \int_0^{t_1} \big((t_2 - s)^{q-1} - (t_1 - s)^{q-1}\big) \|f(s, u(s))\| ds \\
&\qquad + \frac{1}{\Gamma(q)} \int_{t_1}^{t_2} (t_2 - s)^{q-1} \|f(s, u(s))\| ds \\
&\quad \leq \frac{\mu_0}{\Gamma(q+1)} \big(2(t_2 - t_1)^q - t_1^q + t_2^q\big).
\end{aligned}$$

As $t_1 \to t_2$, the right hand side of the above inequality tends to zero. In short we have proven that $Q(B_r)$ is relatively compact for $t \in I$. By Arzela Ascoli's theorem, Q is compact. Hence by the Krasnoselskii theorem there exists a solution of the problem (3.1) with nonlocal condition (4.1).

5 Example

Consider the following impulsive fractional differential equation

$$\frac{d^q u(t)}{dt^q} = \frac{1}{20} e^{-t} |u(t)| + \frac{e^{-t} |u(t)|}{(9 + e^t)(1 + |u(t)|)}, \quad t \in J,\ t \neq \frac{1}{2},\ 0 < q \leq 1, \tag{5.1}$$

$$\Delta u|_{t=\frac{1}{2}} = \frac{|u(\frac{1}{2}^-)|}{8 + |u(\frac{1}{2}^-)|}, \tag{5.2}$$

$$u(0) = u_0. \tag{5.3}$$

Take $J := [0, 1] \subset R^+$,

$$\begin{aligned}
A(t) &= \frac{1}{20} e^{-t} I, \\
f(t, u) &= \frac{e^{-t} u}{(9 + e^t)(1 + u)}, \quad (t, u) \in J \times \mathbb{X}, \quad \text{and} \\
I_k(u) &= \frac{u}{8 + u}, \quad u \in \mathbb{X}.
\end{aligned}$$

Let $u, v \in \mathbb{X}$ and $t \in J$. Then we have

$$\begin{aligned}
\|f(t, u) - f(t, v)\| &= \frac{e^{-t}}{(9 + e^t)} \left| \frac{u}{(1 + u)} - \frac{v}{(1 + v)} \right| \\
&= \frac{e^{-t} |u - v|}{(9 + e^t)(1 + u)(1 + v)} \\
&\leq \frac{e^{-t}}{(9 + e^t)} |u - v| \\
&\leq \frac{1}{10} |u - v|.
\end{aligned}$$

Hence the condition (Hf) holds with $L_2 = \frac{1}{10}$. Let $u, v \in \mathbb{X}$. Then we have

$$\|I_k(u) - I_k(v)\| = |\frac{u}{8+u} - \frac{v}{8+v}| = \frac{8|u-v|}{(8+u)(8+v)} \leq \frac{1}{8}|u-v|.$$

Hence the condition (HI) holds with $L_1 = \frac{1}{8}$. Here $M = \frac{1}{20}$. We shall check that condition $\gamma(m+1)(M+L_2) + mL_1 < \frac{1}{2}$ is satisfied with $m = 1$. Indeed

$$\gamma(m+1)(M+L_2) + mL_1 < 1/2 \quad \Leftrightarrow \Gamma(q+1) > \frac{4}{5}, \tag{5.4}$$

which is satisfied for some $q \in (0,1]$. Then by Theorem 3.1 the problem (5.1)-(5.3) has a unique solution on [0,1] for the values of q satisfying (5.4).

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