

EXISTENCE RESULTS FOR A CLASS OF NONLINEAR PARABOLIC EQUATIONS IN ORLICZ SPACES

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ABSTRACT. An existence result of a renormalized solution for a class of non-linear parabolic equations in Orlicz spaces is proved. No growth assumption is made on the nonlinearities.

1. INTRODUCTION

In this paper we consider the following problem:

$$(1.1) \quad \frac{\partial b(x, u)}{\partial t} - \operatorname{div} \left(a(x, t, u, \nabla u) + \Phi(u) \right) = f \quad \text{in } \Omega \times (0, T),$$

$$(1.2) \quad b(x, u)(t = 0) = b(x, u_0) \quad \text{in } \Omega,$$

$$(1.3) \quad u = 0 \quad \text{on } \partial\Omega \times (0, T),$$

where Ω is a bounded open subset of \mathbb{R}^N and $T > 0$, $Q = \Omega \times (0, T)$. Let b be a Carathéodory function (see assumptions (3.1)-(3.2) of Section 3), the data f and $b(x, u_0)$ in $L^1(Q)$ and $L^1(\Omega)$ respectively, $Au = -\operatorname{div} \left(a(x, t, u, \nabla u) \right)$ is a Leray-Lions operator defined on $W_0^{1,x}L_M(\Omega)$, M is an appropriate N -function and which grows like $\bar{M}^{-1}M(\beta_K^4|\nabla u|)$ with respect to ∇u , but which is not restricted by any growth condition with respect to u (see assumptions (3.3)-(3.6)). The function Φ is just assumed to be continuous on \mathbb{R} .

Under these assumptions, the above problem does not admit, in general, a weak solution since the fields $a(x, t, u, \nabla u)$ and $\Phi(u)$ do not belong in $(L_{loc}^1(Q))^N$ in general. To overcome this difficulty we use in this paper the framework of renormalized solutions. This notion was introduced by Lions and DiPerna [31] for the study of Boltzmann equation (see also [27], [11], [29], [28], [2]).

A large number of papers was devoted to the study the existence of renormalized solution of parabolic problems under various assumptions and in different contexts: for a review on classical results see [7], [30], [9], [8], [4], [5], [34], [12], [13], [14].

The existence and uniqueness of renormalized solution of (1.1)-(1.3) has been proved in H. Redwane [34, 35] in the case where $Au = -\operatorname{div} \left(a(x, t, u, \nabla u) \right)$ is a Leray-Lions operator defined on $L^p(0, T; W_0^{1,p}(\Omega))$, the existence of renormalized solution in Orlicz spaces has been proved in E. Azroul, H. Redwane and M.

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Rhoudaf [32] in the case where $b(x, u) = b(u)$ and where the growth of $a(x, t, u, \nabla u)$ is controlled with respect to u . Note that here we extend the results in [34, 32] in three different directions: we assume $b(x, u)$ depend on x , and the growth of $a(x, t, u, \nabla u)$ is not controlled with respect to u and we prove the existence in Orlicz spaces.

The paper is organized as follows. In section 2 we give some preliminaries and gives the definition of N -function and the Orlicz-Sobolev space. Section 3 is devoted to specifying the assumptions on b , a , Φ , f and $b(x, u_0)$. In Section 4 we give the definition of a renormalized solution of (1.1)-(1.3). In Section 5 we establish (Theorem 5.1) the existence of such a solution.

2. PRELIMINARIES

Let $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be an N -function, i.e., M is continuous, convex, with $M(t) > 0$ for $t > 0$, $\frac{M(t)}{t} \rightarrow 0$ as $t \rightarrow 0$ and $\frac{M(t)}{t} \rightarrow \infty$ as $t \rightarrow \infty$. Equivalently, M admits the representation : $M(t) = \int_0^t a(s) ds$ where $a : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is non-decreasing, right continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \rightarrow \infty$ as $t \rightarrow \infty$. The N -function \bar{M} conjugate to M is defined by $\bar{M}(t) = \int_0^t \bar{a}(s) ds$, where $\bar{a} : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is given by $\bar{a}(t) = \sup\{s : a(s) \leq t\}$.

The N -function M is said to satisfy the Δ_2 condition if, for some $k > 0$,

$$(2.1) \quad M(2t) \leq k M(t) \quad \text{for all } t \geq 0.$$

When this inequality holds only for $t \geq t_0 > 0$, M is said to satisfy the Δ_2 -condition near infinity.

Let P and Q be two N -functions. $P \ll Q$ means that P grows essentially less rapidly than Q ; i.e., for each $\varepsilon > 0$,

$$(2.2) \quad \frac{P(t)}{Q(\varepsilon t)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

This is the case if and only if,

$$(2.3) \quad \frac{Q^{-1}(t)}{P^{-1}(t)} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

We will extend these N -functions into even functions on all \mathbb{R} . Let Ω be an open subset of \mathbb{R}^N . The Orlicz class $\mathcal{L}_M(\Omega)$ (resp. the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that :

$$(2.4) \quad \int_{\Omega} M(u(x)) dx < +\infty \quad (\text{resp. } \int_{\Omega} M(\frac{u(x)}{\lambda}) dx < +\infty \text{ for some } \lambda > 0).$$

Note that $L_M(\Omega)$ is a Banach space under the norm

$$(2.5) \quad \|u\|_{M,\Omega} = \inf \left\{ \lambda > 0 : \int_{\Omega} M(\frac{u(x)}{\lambda}) dx \leq 1 \right\}$$

and $\mathcal{L}_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_M(\Omega)$. The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if M satisfies the Δ_2 -condition, for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_M(\Omega)$ can be identified with $L_{\overline{M}}(\Omega)$ by means of the pairing $\int_{\Omega} u(x)v(x)dx$, and the dual norm on $L_{\overline{M}}(\Omega)$ is equivalent to $\|\cdot\|_{\overline{M},\Omega}$. The space $L_M(\Omega)$ is reflexive if and only if M and \overline{M} satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

We now turn to the Orlicz-Sobolev space. $W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order 1 lie in $L_M(\Omega)$ (resp. $E_M(\Omega)$). This is a Banach space under the norm

$$(2.6) \quad \|u\|_{1,M,\Omega} = \sum_{|\alpha| \leq 1} \|\nabla^\alpha u\|_{M,\Omega}.$$

Thus $W^1L_M(\Omega)$ and $W^1E_M(\Omega)$ can be identified with subspaces of the product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by ΠL_M , we will use the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. The space $W_0^1E_M(\Omega)$ is defined as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^1E_M(\Omega)$ and the space $W_0^1L_M(\Omega)$ as the $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ closure of $\mathcal{D}(\Omega)$ in $W^1L_M(\Omega)$. We say that u_n converges to u for the modular convergence in $W^1L_M(\Omega)$ if for some $\lambda > 0$, $\int_{\Omega} M\left(\frac{\nabla^\alpha u_n - \nabla^\alpha u}{\lambda}\right)dx \rightarrow 0$ for all $|\alpha| \leq 1$. This implies convergence for $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If M satisfies the Δ_2 condition on \mathbb{R}^+ (near infinity only when Ω has finite measure), then modular convergence coincides with norm convergence.

Let $W^{-1}L_{\overline{M}}(\Omega)$ (resp. $W^{-1}E_{\overline{M}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives of order ≤ 1 of functions in $L_{\overline{M}}(\Omega)$ (resp. $E_{\overline{M}}(\Omega)$). It is a Banach space under the usual quotient norm.

If the open set Ω has the segment property, then the space $\mathcal{D}(\Omega)$ is dense in $W_0^1L_M(\Omega)$ for the modular convergence and for the topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$ (cf. [21]). Consequently, the action of a distribution in $W^{-1}L_{\overline{M}}(\Omega)$ on an element of $W_0^1L_M(\Omega)$ is well defined. For more details see [1], [23].

For $K > 0$, we define the truncation at height K , $T_K : \mathbb{R} \rightarrow \mathbb{R}$ by

$$(2.7) \quad T_K(s) = \min(K, \max(s, -K)).$$

The following abstract lemmas will be applied to the truncation operators.

Lemma 2.1. [21] *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. Let M be an N -function and let $u \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$).*

Then $F(u) \in W^1L_M(\Omega)$ (resp. $W^1E_M(\Omega)$). Moreover, if the set of discontinuity points D of F' is finite, then

$$\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial u}{\partial x_i} & \text{a.e. in } \{x \in \Omega : u(x) \notin D\} \\ 0 & \text{a.e. in } \{x \in \Omega : u(x) \in D\} \end{cases}$$

Lemma 2.2. [21] *Let $F : \mathbb{R} \rightarrow \mathbb{R}$ be uniformly lipschitzian, with $F(0) = 0$. We suppose that the set of discontinuity points of F' is finite. Let M be an N -function, then the mapping $F : W^1L_M(\Omega) \rightarrow W^1L_M(\Omega)$ is sequentially continuous with respect to the weak* topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$.*

Let Ω be a bounded open subset of \mathbb{R}^N , $T > 0$ and set $Q = \Omega \times (0, T)$. M be an N -function. For each $\alpha \in \mathbb{N}^N$, denote by ∇_x^α the distributional derivative on Q of

order α with respect to the variable $x \in \mathbb{N}^N$. The inhomogeneous Orlicz-Sobolev spaces are defined as follows,

$$(2.8) \quad \begin{aligned} W^{1,x}L_M(Q) &= \{u \in L_M(Q) : \nabla_x^\alpha u \in L_M(Q) \forall |\alpha| \leq 1\} \\ \text{and } W^{1,x}E_M(Q) &= \{u \in E_M(Q) : \nabla_x^\alpha u \in E_M(Q) \forall |\alpha| \leq 1\} \end{aligned}$$

The last space is a subspace of the first one, and both are Banach spaces under the norm,

$$(2.9) \quad \|u\| = \sum_{|\alpha| \leq 1} \|\nabla_x^\alpha u\|_{M,Q}.$$

We can easily show that they form a complementary system when Ω satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_M(Q)$ which have as many copies as there is α -order derivatives, $|\alpha| \leq 1$. We shall also consider the weak topologies $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ and $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. If $u \in W^{1,x}L_M(Q)$ then the function $: t \mapsto u(t) = u(t, \cdot)$ is defined on $(0, T)$ with values in $W^1L_M(\Omega)$. If, further, $u \in W^{1,x}E_M(Q)$ then the concerned function is a $W^1E_M(\Omega)$ -valued and is strongly measurable. Furthermore the following imbedding holds: $W^{1,x}E_M(Q) \subset L^1(0, T; W^1E_M(\Omega))$. The space $W^{1,x}L_M(Q)$ is not in general separable, if $u \in W^{1,x}L_M(Q)$, we can not conclude that the function $u(t)$ is measurable on $(0, T)$. However, the scalar function $t \mapsto \|u(t)\|_{M,\Omega}$ is in $L^1(0, T)$. The space $W_0^{1,x}E_M(Q)$ is defined as the (norm) closure in $W^{1,x}E_M(Q)$ of $\mathcal{D}(Q)$. We can easily show as in [22] that when Ω has the segment property, then each element u of the closure of $\mathcal{D}(Q)$ with respect of the weak * topology $\sigma(\Pi L_M, \Pi E_{\overline{M}})$ is a limit, in $W^{1,x}L_M(Q)$, of some subsequence $(u_i) \subset \mathcal{D}(Q)$ for the modular convergence; i.e., there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$,

$$(2.10) \quad \int_Q M\left(\frac{\nabla_x^\alpha u_i - \nabla_x^\alpha u}{\lambda}\right) dx dt \rightarrow 0 \text{ as } i \rightarrow \infty.$$

This implies that (u_i) converges to u in $W^{1,x}L_M(Q)$ for the weak topology $\sigma(\Pi L_M, \Pi L_{\overline{M}})$. Consequently,

$$(2.11) \quad \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M, \Pi E_{\overline{M}})} = \overline{\mathcal{D}(Q)}^{\sigma(\Pi L_M, \Pi L_{\overline{M}})}.$$

This space will be denoted by $W_0^{1,x}L_M(Q)$. Furthermore, $W_0^{1,x}E_M(Q) = W_0^{1,x}L_M(Q) \cap \Pi E_M$. Poincaré's inequality also holds in $W_0^{1,x}L_M(Q)$, i.e., there is a constant $C > 0$ such that for all $u \in W_0^{1,x}L_M(Q)$ one has,

$$(2.12) \quad \sum_{|\alpha| \leq 1} \|\nabla_x^\alpha u\|_{M,Q} \leq C \sum_{|\alpha|=1} \|\nabla_x^\alpha u\|_{M,Q}.$$

Thus both sides of the last inequality are equivalent norms on $W_0^{1,x}L_M(Q)$. We have then the following complementary system

$$(2.13) \quad \begin{pmatrix} W_0^{1,x}L_M(Q) & F \\ W_0^{1,x}E_M(Q) & F_0 \end{pmatrix}$$

F being the dual space of $W_0^{1,x}E_M(Q)$. It is also, except for an isomorphism, the quotient of $\Pi L_{\overline{M}}$ by the polar set $W_0^{1,x}E_M(Q)^\perp$, and will be denoted by $F =$

$W^{-1,x}L_{\bar{M}}(Q)$ and it is shown that,

$$(2.14) \quad W^{-1,x}L_{\bar{M}}(Q) = \left\{ f = \sum_{|\alpha| \leq 1} \nabla_x^\alpha f_\alpha : f_\alpha \in L_{\bar{M}}(Q) \right\}.$$

This space will be equipped with the usual quotient norm

$$(2.15) \quad \|f\| = \inf \sum_{|\alpha| \leq 1} \|f_\alpha\|_{\bar{M},Q}$$

where the infimum is taken on all possible decompositions

$$(2.16) \quad f = \sum_{|\alpha| \leq 1} \nabla_x^\alpha f_\alpha, \quad f_\alpha \in L_{\bar{M}}(Q).$$

The space F_0 is then given by,

$$(2.17) \quad F_0 = \left\{ f = \sum_{|\alpha| \leq 1} \nabla_x^\alpha f_\alpha : f_\alpha \in E_{\bar{M}}(Q) \right\}$$

and is denoted by $F_0 = W^{-1,x}E_{\bar{M}}(Q)$.

Remark 2.3. We can easily check, using lemma 2.1, that each uniformly lipschitzian mapping F , with $F(0) = 0$, acts in inhomogeneous Orlicz-Sobolev spaces of order 1 : $W^{1,x}L_M(Q)$ and $W_0^{1,x}L_M(Q)$.

3. ASSUMPTIONS AND STATEMENT OF MAIN RESULTS

Throughout this paper, we assume that the following assumptions hold true: Ω is a bounded open set on \mathbb{R}^N ($N \geq 2$), $T > 0$ is given and we set $Q = \Omega \times (0, T)$. Let M and P be two N -function such that $P \ll M$.

(3.1) $b : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function such that,

for every $x \in \Omega$: $b(x, s)$ is a strictly increasing C^1 -function, with $b(x, 0) = 0$.

For any $K > 0$, there exists $\lambda_K > 0$, a function A_K in $L^\infty(\Omega)$ and a function B_K in $L_M(\Omega)$ such that

$$(3.2) \quad \lambda_K \leq \frac{\partial b(x, s)}{\partial s} \leq A_K(x) \quad \text{and} \quad \left| \nabla_x \left(\frac{\partial b(x, s)}{\partial s} \right) \right| \leq B_K(x),$$

for almost every $x \in \Omega$, for every s such that $|s| \leq K$.

Consider a second order partial differential operator $A : D(A) \subset W^{1,x}L_M(Q) \rightarrow W^{-1,x}L_{\bar{M}}(Q)$ in divergence form,

$$A(u) = -\text{div} \left(a(x, t, u, \nabla u) \right)$$

where

(3.3) $a : \Omega \times (0, T) \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function satisfying

for any $K > 0$, there exist $\beta_K^i > 0$ (for $i = 1, 2, 3, 4$) and a function $C_K \in E_{\bar{M}}(Q)$ such that:

$$(3.4) \quad |a(x, t, s, \xi)| \leq C_K(x, t) + \beta_K^1 \bar{M}^{-1} P(\beta_K^2 |s|) + \beta_K^3 \bar{M}^{-1} M(\beta_K^4 |\xi|)$$

for almost every $(x, t) \in Q$ and for every $|s| \leq K$ and for every $\xi \in \mathbb{R}^N$.

$$(3.5) \quad \left[a(x, t, s, \xi) - a(x, t, s, \xi^*) \right] \left[\xi - \xi^* \right] > 0$$

$$(3.6) \quad a(x, t, s, \xi) \xi \geq \alpha M(|\xi|)$$

for almost every $(x, t) \in Q$, for every $s \in \mathbb{R}$ and for every $\xi \neq \xi^* \in \mathbb{R}^N$, where $\alpha > 0$ is a given real number.

$$(3.7) \quad \Phi : \mathbb{R} \rightarrow \mathbb{R}^N \text{ is a continuous function}$$

$$(3.8) \quad f \text{ is an element of } L^1(Q).$$

$$(3.9) \quad u_0 \text{ is an element of } L^1(\Omega) \text{ such that } b(x, u_0) \in L^1(\Omega).$$

Remark 3.1. As already mentioned in the introduction, problem (1.1)-(1.3) does not admit a weak solution under assumptions (3.1)-(3.9) (even when $b(x, u) = u$) since the growths of $a(x, t, u, Du)$ and $\Phi(u)$ are not controlled with respect to u (so that these fields are not in general defined as distributions, even when u belongs to $W_0^{1,x}L_M(Q)$).

4. DEFINITION OF A RENORMALIZED SOLUTION

The definition of a renormalized solution for problem (1.1)-(1.3) can be stated as follows.

Definition 4.1. A measurable function u defined on Q is a renormalized solution of Problem (1.1)-(1.3) if

$$(4.1) \quad T_K(u) \in W_0^{1,x}L_M(Q) \quad \forall K \geq 0 \text{ and } b(x, u) \in L^\infty(0, T; L^1(\Omega)),$$

$$(4.2) \quad \int_{\{(t,x) \in Q ; m \leq |u(x,t)| \leq m+1\}} a(x, t, u, \nabla u) \nabla u \, dx \, dt \longrightarrow 0 \quad \text{as } m \rightarrow +\infty ;$$

and if, for every function S in $W^{2,\infty}(\mathbb{R})$, which is piecewise C^1 and such that S' has a compact support, we have

$$(4.3) \quad \frac{\partial B_S(x, u)}{\partial t} - \operatorname{div} \left(S'(u) a(x, t, u, \nabla u) \right) + S''(u) a(x, t, u, \nabla u) \nabla u \\ - \operatorname{div} \left(S'(u) \Phi(u) \right) + S''(u) \Phi(u) \nabla u = f S'(u) \quad \text{in } D'(Q),$$

and

$$(4.4) \quad B_S(x, u)(t = 0) = B_S(x, u_0) \text{ in } \Omega,$$

where $B_S(x, z) = \int_0^z \frac{\partial b(x, r)}{\partial r} S'(r) \, dr$.

The following remarks are concerned with a few comments on definition 4.1.

Remark 4.2. Equation (4.3) is formally obtained through pointwise multiplication of equation (1.1) by $S'(u)$. Note that due to (4.1) each term in (4.3) has a meaning in $L^1(Q) + W^{-1,x}L_{\overline{M}}(Q)$.

Indeed, if K is such that $\text{supp}S' \subset [-K, K]$, the following identifications are made in (4.3).

★ $B_S(x, u) \in L^\infty(Q)$, because $|B_S(x, u)| \leq K\|A_K\|_{L^\infty(\Omega)}\|S'\|_{L^\infty(\mathbb{R})}$.

★ $S'(u)a(x, t, u, \nabla u)$ identifies with $S'(u)a(x, t, T_K(u), \nabla T_K(u))$ a.e. in Q . Since indeed $|T_K(u)| \leq K$ a.e. in Q . Since $S'(u) \in L^\infty(Q)$ and with (3.4), (4.1) we obtain that

$$S(u)a(x, t, T_K(u), \nabla T_K(u)) \in (L_{\overline{M}}(Q))^N.$$

★ $S'(u)a(x, t, u, \nabla u)\nabla u$ identifies with $S'(u)a(x, t, T_K(u), \nabla T_K(u))\nabla T_K(u)$ and in view of (3.2) and (4.1) one has

$$S'(u)a(x, t, T_K(u), \nabla T_K(u))\nabla T_K(u) \in L^1(Q).$$

★ $S'(u)\Phi(u)$ and $S''(u)\Phi(u)\nabla u$ respectively identify with $S'(u)\Phi(T_K(u))$ and $S''(u)\Phi(T_K(u))\nabla T_K(u)$. Due to the properties of S and (3.7), the functions S' , S'' and $\Phi \circ T_K$ are bounded on \mathbb{R} so that (4.1) implies that $S'(u)\Phi(T_K(u)) \in (L^\infty(Q))^N$, and $S''(u)\Phi(T_K(u))\nabla T_K(u) \in (L_M(Q))^N$.

The above considerations show that equation (4.3) takes place in $D'(Q)$ and that

$$(4.5) \quad \frac{\partial B_S(x, u)}{\partial t} \text{ belongs to } W^{-1,x}L_{\overline{M}}(Q) + L^1(Q).$$

Due to the properties of S and (3.2), we have

$$(4.6) \quad \left| \nabla B_S(x, u) \right| \leq \|A_K\|_{L^\infty(\Omega)} |\nabla T_K(u)| \|S'\|_{L^\infty(\Omega)} + K \|S'\|_{L^\infty(\Omega)} B_K(x)$$

and

$$(4.7) \quad B_S(x, u) \text{ belongs to } W_0^{1,x}L_M(Q).$$

Moreover (4.5) and (4.7) implies that $B_S(x, u)$ belongs to $C^0([0, T]; L^1(\Omega))$ (for a proof of this trace result see [30]), so that the initial condition (4.4) makes sense.

Remark 4.3. For every $S \in W^{2,\infty}(\mathbb{R})$, nondecreasing function such that $\text{supp}S' \subset [-K, K]$ and (3.2), we have

$$(4.8) \quad \lambda_K |S(r) - S(r')| \leq \left| B_S(x, r) - B_S(x, r') \right| \leq \|A_K\|_{L^\infty(\Omega)} |S(r) - S(r')|$$

for almost every $x \in \Omega$ and for every $r, r' \in \mathbb{R}$.

5. EXISTENCE RESULT

This section is devoted to establish the following existence theorem.

Theorem 5.1. *Under assumption (3.1)-(3.9) there exists at least a renormalized solution of Problem (1.1)-(1.3).*

Proof. The proof is divided into 5 steps. □

★ **Step 1.** For $n \in \mathbb{N}^*$, let us define the following approximations of the data:

$$(5.1) \quad b_n(x, r) = b(x, T_n(r)) + \frac{1}{n} r \quad \text{a.e. in } \Omega, \forall s \in \mathbb{R},$$

$$(5.2) \quad a_n(x, t, r, \xi) = a(x, t, T_n(r), \xi) \quad \text{a.e. in } Q, \forall s \in \mathbb{R}, \forall \xi \in \mathbb{R}^N,$$

$$(5.3) \quad \Phi_n \text{ is a Lipschitz continuous bounded function from } \mathbb{R} \text{ into } \mathbb{R}^N,$$

such that Φ_n uniformly converges to Φ on any compact subset of \mathbb{R} as n tends to $+\infty$.

$$(5.4) \quad f_n \in C_0^\infty(Q) : \|f_n\|_{L^1} \leq \|f\|_{L^1} \text{ and } f_n \longrightarrow f \text{ in } L^1(Q) \text{ as } n \text{ tends to } +\infty,$$

$$(5.5) \quad u_{0n} \in C_0^\infty(\Omega) : \|b_n(x, u_{0n})\|_{L^1} \leq \|b(x, u_0)\|_{L^1} \text{ and } b_n(x, u_{0n}) \longrightarrow b(x, u_0) \text{ in } L^1(\Omega)$$

as n tends to $+\infty$.

Let us now consider the following regularized problem:

$$(5.6) \quad \frac{\partial b_n(x, u_n)}{\partial t} - \operatorname{div} \left(a_n(x, t, u_n, \nabla u_n) + \Phi_n(u_n) \right) = f_n \quad \text{in } Q,$$

$$(5.7) \quad u_n = 0 \quad \text{on } (0, T) \times \partial\Omega,$$

$$(5.8) \quad b_n(x, u_n)(t = 0) = b_n(x, u_{0n}) \quad \text{in } \Omega.$$

As a consequence, proving existence of a weak solution $u_n \in W_0^{1,x} L_M(Q)$ of (5.6)-(5.8) is an easy task (see e.g. [25], [33]).

★ **Step 2.** The estimates derived in this step rely on usual techniques for problems of the type (5.6)-(5.8).

Proposition 5.2. *Assume that (3.1)-(3.9) hold true and let u_n be a solution of the approximate problem (5.6) – (5.8). Then for all K , $n > 0$, we have*

$$(5.9) \quad \|T_K(u_n)\|_{W_0^{1,x} L_M(Q)} \leq K \left(\|f\|_{L^1(Q)} + \|b(x, u_0)\|_{L^1(\Omega)} \right) \equiv CK,$$

where C is a constant independent of n .

$$(5.10) \quad \int_{\Omega} B_K^n(x, u_n)(\tau) dx \leq K (\|f\|_{L^1(Q)} + \|b(x, u_0)\|_{L^1(\Omega)}) \equiv CK,$$

for almost any τ in $(0, T)$, and where $B_K^n(x, r) = \int_0^r T_K(s) \frac{\partial b_n(x, s)}{\partial s} ds$.

$$(5.11) \quad \lim_{K \rightarrow \infty} \operatorname{meas} \left\{ (x, t) \in Q : |u_n| > K \right\} = 0 \quad \text{uniformly with respect to } n.$$

Proof. We take $T_K(u_n) \chi_{(0, \tau)}$ as test function in (5.6), we get for every $\tau \in (0, T)$

$$(5.12) \quad \left\langle \frac{\partial b_n(x, u_n)}{\partial t}, T_K(u_n) \chi_{(0, \tau)} \right\rangle + \int_{Q_\tau} a_n(x, t, T_K(u_n), \nabla T_K(u_n)) \nabla T_K(u_n) dx dt \\ + \int_{Q_\tau} \Phi_n(u_n) \nabla T_K(u_n) dx dt = \int_{Q_\tau} f_n T_K(u_n) dx dt,$$

which implies that,

$$(5.13) \quad \int_{\Omega} B_K^n(x, u_n)(\tau) dx + \int_{Q_\tau} a_n(x, t, T_K(u_n), \nabla T_K(u_n)) \nabla T_K(u_n) dx dt + \int_{Q_\tau} \Phi_n(u_n) \nabla T_K(u_n) dx dt = \int_{Q_\tau} f_n T_K(u_n) dx dt + \int_{\Omega} B_K^n(x, u_{0n}) dx$$

where, $B_K^n(x, r) = \int_0^r T_K(s) \frac{\partial b_n(x, s)}{\partial s} ds$.

The Lipschitz character of Φ_n , Stokes formula together with the boundary condition (5.7), make it possible to obtain

$$(5.14) \quad \int_{Q_\tau} \Phi_n(u_n) \nabla T_K(u_n) dx dt = 0.$$

Due to the definition of B_K^n we have,

$$(5.15) \quad 0 \leq \int_{\Omega} B_K^n(x, u_{0n}) dx \leq K \int_{\Omega} |b_n(x, u_{0n})| dx \leq K \|b(x, u_0)\|_{L^1(\Omega)}.$$

By using (5.14), (5.15) and the fact that $B_K^n(x, u_n) \geq 0$, permit to deduce from (5.13) that

$$(5.16) \quad \int_Q a_n(x, t, T_K(u_n), \nabla T_K(u_n)) \nabla T_K(u_n) dx dt \leq K(\|f_n\|_{L^1(Q)} + \|b_n(x, u_{0n})\|_{L^1(\Omega)}) \leq CK,$$

which implies by virtue of (3.6), (5.4) and (5.5) that,

$$(5.17) \quad \int_Q M(\nabla T_K(u_n)) dx dt \leq K(\|f\|_{L^1(Q)} + \|b(x, u_0)\|_{L^1(\Omega)}) \equiv CK.$$

We deduce from that above inequality (5.13) and (5.15) that

$$(5.18) \quad \int_{\Omega} B_K^n(x, u_n)(\tau) dx \leq (\|f\|_{L^1(Q)} + \|b(x, u_0)\|_{L^1(\Omega)}) \equiv CK.$$

for almost any τ in $(0, T)$.

We prove (5.11). Indeed, thanks to lemma 5.7 of [21], there exist two positive constants δ, λ such that,

$$(5.19) \quad \int_Q M(v) dx dt \leq \delta \int_Q M(\lambda |\nabla v|) dx dt \text{ for all } v \in W_0^{1,x} L_M(Q).$$

Taking $v = \frac{T_K(u_n)}{\lambda}$ in (5.19) and using (5.17), one has

$$(5.20) \quad \int_Q M\left(\frac{T_K(u_n)}{\lambda}\right) dx dt \leq CK,$$

where C is a constant independent of K and n . Which implies that,

$$(5.21) \quad \text{meas}\left\{(x, t) \in Q : |u_n| > K\right\} \leq \frac{C'K}{M\left(\frac{K}{\lambda}\right)}.$$

where C' is a constant independent of K and n . Finally,

$$\lim_{K \rightarrow \infty} \text{meas} \left\{ (x, t) \in Q : |u_n| > K \right\} = 0 \quad \text{uniformly with respect to } n.$$

□

We prove the following proposition:

Proposition 5.3. *Let u_n be a solution of the approximate problem (5.6)-(5.8), then*

$$(5.22) \quad u_n \rightarrow u \quad \text{a.e. in } Q,$$

$$(5.23) \quad b_n(x, u_n) \rightarrow b(x, u) \quad \text{a.e. in } Q,$$

$$(5.24) \quad b(x, u) \in L^\infty(0, T; L^1(\Omega)),$$

$$(5.25) \quad a_n(x, t, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \varphi_k \quad \text{in } (L_{\overline{M}}(Q))^N \quad \text{for } \sigma(\Pi L_{\overline{M}}, \Pi E_M)$$

for some $\varphi_k \in (L_{\overline{M}}(Q))^N$.

$$(5.26) \quad \lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} \int_{\{m \leq |u_n| \leq m+1\}} a_n(x, t, u_n, \nabla u_n) \nabla u_n \, dx \, dt = 0.$$

Proof. Proceeding as in [5, 9, 7], we have for any $S \in W^{2,\infty}(\mathbb{R})$ such that S' has a compact support ($\text{supp } S' \subset [-K, K]$)

$$(5.27) \quad B_S^n(x, u_n) \text{ is bounded in } W_0^{1,x} L_M(Q),$$

and

$$(5.28) \quad \frac{\partial B_S^n(x, u_n)}{\partial t} \text{ is bounded in } L^1(Q) + W^{-1,x} L_{\overline{M}}(Q),$$

independently of n .

As a consequence of (4.6) and (5.17) we then obtain (5.27). To show that (5.28) holds true, we multiply the equation for u_n in (5.6) by $S'(u_n)$ to obtain

$$(5.29) \quad \frac{\partial B_S^n(x, u_n)}{\partial t} = \text{div} \left(S'(u_n) a_n(t, x, u_n, \nabla u_n) \right) - S''(u_n) a_n(x, t, u_n, \nabla u_n) \nabla u_n + \text{div} \left(S'(u_n) \Phi_n(u_n) \right) + f_n S'(u_n) \quad \text{in } D'(Q).$$

Where $B_S^n(x, r) = \int_0^r S'(s) \frac{\partial b_n(x, s)}{\partial s} \, ds$. Since $\text{supp } S'$ and $\text{supp } S''$ are both included in $[-K, K]$, u^ε may be replaced by $T_K(u_n)$ in each of these terms. As a consequence, each term in the right hand side of (5.29) is bounded either in $W^{-1,x} L_{\overline{M}}(Q)$ or in $L^1(Q)$. As a consequence of (3.2), (5.29) we then obtain (5.28). Arguing again as in [5, 7, 6, 9] estimates (5.27), (5.28) and (4.8), we can show (5.22) and (5.23).

We now establish that $b(x, u)$ belongs to $L^\infty(0, T; L^1(\Omega))$. To this end, recalling (5.23) makes it possible to pass to the limit-inf in (5.18) as n tends to $+\infty$ and to obtain

$$\frac{1}{K} \int_\Omega B_K(x, u)(\tau) \, dx \leq (\|f\|_{L^1(Q)} + \|b(x, u_0)\|_{L^1(\Omega)}) \equiv C,$$

for almost any τ in $(0, T)$. Due to the definition of $B_K(x, s)$, and because of the pointwise convergence of $\frac{1}{K}B_K(x, u)$ to $b(x, u)$ as K tends to $+\infty$, which shows that $b(x, u)$ belongs to $L^\infty(0, T; L^1(\Omega))$.

We prove (5.25). Let $\varphi \in (E_M(Q))^N$ with $\|\varphi\|_{M, Q} = 1$. In view of the monotonicity of a one easily has,

$$(5.30) \quad \int_Q a_n(x, t, T_k(u_n), \nabla T_k(u_n)) \varphi \, dx \, dt \leq \int_Q a_n(x, t, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \, dt \\ + \int_Q a_n(x, t, T_k(u_n), \varphi) [\nabla T_k(u_n) - \varphi] \, dx \, dt.$$

and

$$(5.31) \quad - \int_Q a_n(x, t, T_k(u_n), \nabla T_k(u_n)) \varphi \, dx \, dt \leq \int_Q a_n(x, t, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \, dt \\ - \int_Q a_n(x, t, T_k(u_n), -\varphi) [\nabla T_k(u_n) + \varphi] \, dx \, dt,$$

since $T_k(u_n)$ is bounded in $W_0^{1,x}L_M(Q)$, one easily deduce that $a_n(x, t, T_k(u_n), \nabla T_k(u_n))$ is a bounded sequence in $(L_{\overline{M}}(Q))^N$, and we obtain (5.25).

Now we prove (5.26). We take of $T_1(u_n - T_m(u_n))$ as test function in (5.6), we obtain

$$(5.32) \quad \left\langle \frac{\partial b_n(x, u_n)}{\partial t}, T_1(u_n - T_m(u_n)) \right\rangle + \int_{\{m \leq |u_n| \leq m+1\}} a_n(x, t, u_n, \nabla u_n) \nabla u_n \, dx \, dt \\ + \int_Q \operatorname{div} \left[\int_0^{u_n} \Phi(r) T_1'(r - T_m(r)) \right] \, dx \, dt = \int_Q f_n T_1(u_n - T_m(u_n)) \, dx \, dt.$$

Using the fact that $\int_0^{u_n} \Phi(r) T_1'(r - T_m(r)) \, dx \, dt \in W_0^{1,x}L_M(Q)$ and Stokes formula, we get

$$(5.33) \quad \int_\Omega B_n^m(x, u_n(T)) \, dx + \int_{\{m \leq |u_n| \leq m+1\}} a(x, t, u_n, \nabla u_n) \nabla u_n \, dx \, dt \\ \leq \int_Q |f_n T_1(u_n - T_m(u_n))| \, dx \, dt + \int_\Omega B_n^m(x, u_{0n}) \, dx,$$

where $B_n^m(x, r) = \int_0^r \frac{\partial b_n(x, s)}{\partial s} T_1(s - T_m(s)) \, ds$.

In order to pass to the limit as n tends to $+\infty$ in (5.33), we use $B_n^m(x, u_n(T)) \geq 0$ and (5.4)-(5.5), we obtain that

$$(5.34) \quad \lim_{n \rightarrow +\infty} \int_{\{m \leq |u_n| \leq m+1\}} a_n(x, t, u_n, \nabla u_n) \nabla u_n \, dx \, dt \\ \leq \int_{\{|u| > m\}} |f| \, dx \, dt + \int_{\{|u_0| > m\}} |b(x, u_0)| \, dx.$$

Finally by (3.8), (3.9) and (5.34) we obtain (5.26). □

★ **Step 3.** This step is devoted to introduce for $K \geq 0$ fixed, a time regularization $w_{\mu,j}^i$ of the function $T_K(u)$ and to establish the following proposition:

Proposition 5.4. *Let u_n be a solution of the approximate problem (5.6)-(5.8). Then, for any $k \geq 0$:*

$$(5.35) \quad \nabla T_k(u_n) \rightarrow \nabla T_k(u) \quad \text{a.e. in } Q,$$

$$(5.36) \quad a_n(x, t, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup a(x, t, T_k(u), \nabla T_k(u)) \quad \text{weakly in } (L_{\overline{M}}(Q))^N,$$

$$(5.37) \quad M(|\nabla T_k(u_n)|) \rightarrow M(|\nabla T_k(u)|) \quad \text{strongly in } L^1(Q),$$

as n tends to $+\infty$.

Let us give the following lemma which will be needed later:

Lemma 5.5. *Under assumptions (3.1) – (3.9), and let (z_n) be a sequence in $W_0^{1,x}L_M(Q)$ such that,*

$$(5.38) \quad z_n \rightharpoonup z \quad \text{in } W_0^{1,x}L_M(Q) \quad \text{for } \sigma(\Pi L_M(Q), \Pi E_{\overline{M}}(Q)),$$

$$(5.39) \quad (a_n(x, t, z_n, \nabla z_n))_n \text{ is bounded in } (L_{\overline{M}}(Q))^N,$$

$$(5.40) \quad \int_Q [a_n(x, t, z_n, \nabla z_n) - a_n(x, t, z_n, \nabla z \chi_s)] [\nabla z_n - \nabla z \chi_s] dx dt \rightarrow 0,$$

as n and s tend to $+\infty$, and where χ_s is the characteristic function of

$$Q_s = \{(x, t) \in Q ; |\nabla z| \leq s\}.$$

Then,

$$(5.41) \quad \nabla z_n \rightarrow \nabla z \quad \text{a.e. in } Q,$$

$$(5.42) \quad \lim_{n \rightarrow \infty} \int_Q a_n(x, t, z_n, \nabla z_n) \nabla z_n dx dt = \int_Q a(x, t, z, \nabla z) \nabla z dx dt,$$

$$(5.43) \quad M(|\nabla z_n|) \rightarrow M(|\nabla z|) \quad \text{in } L^1(Q).$$

Proof. See [32]. □

Proof. (Proposition 5.4). The proof is almost identical of the one given in, e.g. [32], where the result is established for $b(x, u) = u$ and where the growth of $a(x, t, u, Du)$ is controlled with respect to u . This proof is devoted to introduce for $k \geq 0$ fixed, a time regularization of the function $T_k(u)$, this notion, introduced by R. Landes (see Lemma 6 and Proposition 3, p. 230 and Proposition 4, p. 231 in [24]). More recently, it has been exploited in [10] and [15] to solve a few nonlinear evolution problems with L^1 or measure data.

Let $v_j \in D(Q)$ be a sequence such that $v_j \rightarrow u$ in $W_0^{1,x}L_M(Q)$ for the modular convergence and let $\psi_i \in D(\Omega)$ be a sequence which converges strongly to u_0 in $L^1(\Omega)$.

Let $w_{i,j}^\mu = T_k(v_j)_\mu + e^{-\mu t} T_k(\psi_i)$ where $T_k(v_j)_\mu$ is the mollification with respect to time of $T_k(v_j)$, note that $w_{i,j}^\mu$ is a smooth function having the following properties:

$$(5.44) \quad \frac{\partial w_{i,j}^\mu}{\partial t} = \mu(T_k(v_j) - w_{i,j}^\mu), \quad w_{i,j}^\mu(0) = T_k(\psi_i), \quad |w_{i,j}^\mu| \leq k,$$

$$(5.45) \quad w_{i,j}^\mu \rightarrow T_k(u)_\mu + e^{-\mu t} T_k(\psi_i) \quad \text{in } W_0^{1,x} L_M(Q),$$

for the modular convergence as $j \rightarrow \infty$.

$$(5.46) \quad T_k(u)_\mu + e^{-\mu t} T_k(\psi_i) \rightarrow T_k(u) \quad \text{in } W_0^{1,x} L_M(Q),$$

for the modular convergence as $\mu \rightarrow \infty$.

Let now the function h_m defined on \mathbb{R} with $m \geq k$ by: $h_m(r) = 1$ if $|r| \leq m$, $h(r) = -|r| + m + 1$ if $m \leq |r| \leq m + 1$ and $h(r) = 0$ if $|r| \geq m + 1$.

Using the admissible test function $\varphi_{n,j,m}^{\mu,i} = (T_k(u_n) - w_{i,j}^\mu) h_m(u_n)$ as test function in (5.6) leads to

$$(5.47) \quad \begin{aligned} & \left\langle \frac{\partial b_n(x, u_n)}{\partial t}, \varphi_{n,j,m}^{\mu,i} \right\rangle + \int_Q a_n(x, t, u_n, \nabla u_n) (\nabla T_k(u_n) - \nabla w_{i,j}^\mu) h_m(u_n) \, dx \, dt \\ & \quad + \int_Q a_n(x, t, u_n, \nabla u_n) (T_k(u_n) - w_{i,j}^\mu) \nabla u_n h'_m(u_n) \, dx \, dt \\ & \quad + \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n h'_m(u_n) (T_k(u_n) - w_{i,j}^\mu) \, dx \, dt \\ & \quad + \int_Q \Phi_n(u_n) h_m(u_n) (\nabla T_k(u_n) - \nabla w_{i,j}^\mu) \, dx \, dt = \int_Q f_n \varphi_{n,j,m}^{\mu,i} \, dx \, dt. \end{aligned}$$

Denoting by $\epsilon(n, j, \mu, i)$ any quantity such that,

$$\lim_{i \rightarrow \infty} \lim_{\mu \rightarrow \infty} \lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} \epsilon(n, j, \mu, i) = 0.$$

The very definition of the sequence $w_{i,j}^\mu$ makes it possible to establish the following lemma.

Lemma 5.6. *Let $\varphi_{n,j,m}^{\mu,i} = (T_k(u_n) - w_{i,j}^\mu) h_m(u_n)$, we have for any $k \geq 0$:*

$$(5.48) \quad \left\langle \frac{\partial b_n(x, u_n)}{\partial t}, \varphi_{n,j,m}^{\mu,i} \right\rangle \geq \epsilon(n, j, \mu, i),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $L^1(Q) + W^{-1,x} L_M(Q)$ and $L^\infty(Q) \cap W_0^{1,x} L_M(Q)$.

Proof. See [34, 32]. □

Now, we turn to complete the proof of proposition 5.4. First, it is easy to see that (see also [32]):

$$(5.49) \quad \int_Q f_n \varphi_{n,j,m}^{\mu,i} \, dx \, dt = \epsilon(n, j, \mu),$$

$$(5.50) \quad \int_Q \Phi_n(u_n) h_m(u_n) (\nabla T_k(u_n) - \nabla w_{i,j}^\mu) \, dx \, dt = \epsilon(n, j, \mu),$$

and

$$(5.51) \quad \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(u_n) \nabla u_n (T_k(u_n) - w_{i,j}^\mu) \, dx \, dt = \epsilon(n, j, \mu).$$

Concerning the third term of the right hand side of (5.47) we obtain that

$$(5.52) \quad \int_{\{m \leq |u_n| \leq m+1\}} a_n(x, t, u_n, \nabla u_n) \nabla u_n h'_m(u_n) (T_k(u_n) - w_{i,j}^\mu) \, dx \, dt \\ \leq 2k \int_{\{m \leq |u_n| \leq m+1\}} a_n(x, t, u_n, \nabla u_n) \nabla u_n \, dx \, dt.$$

Then by (5.26). we deduce that,

$$(5.53) \quad \int_{\{m \leq |u_n| \leq m+1\}} a_n(x, t, u_n, \nabla u_n) \nabla u_n h'_m(u_n) (T_k(u_n) - w_{i,j}^\mu) \, dx \, dt \leq \epsilon(n, \mu, m).$$

Finally, by means of (5.47)-(5.53), we obtain,

$$(5.54) \quad \int_Q a_n(x, t, u_n, \nabla u_n) (\nabla T_k(u_n) - \nabla w_{i,j}^\mu) h_m(u_n) \, dx \, dt \leq \epsilon(n, j, \mu, m).$$

Splitting the first integral on the left hand side of (5.54) where $|u_n| \leq k$ and $|u_n| > k$, we can write,

$$\int_Q a_n(x, t, u_n, \nabla u_n) (\nabla T_k(u_n) - \nabla w_{i,j}^\mu) h_m(u_n) \, dx \, dt \\ = \int_Q a_n(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla w_{i,j}^\mu) h_m(u_n) \, dx \, dt \\ - \int_{\{|u_n| > k\}} a_n(x, t, u_n, \nabla u_n) \nabla w_{i,j}^\mu h_m(u_n) \, dx \, dt.$$

Since $h_m(u_n) = 0$ if $|u_n| \geq m + 1$, one has

$$(5.55) \quad \int_Q a_n(x, t, u_n, \nabla u_n) (\nabla T_k(u_n) - \nabla w_{i,j}^\mu) h_m(u_n) \, dx \, dt \\ = \int_Q a_n(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla w_{i,j}^\mu) h_m(u_n) \, dx \, dt \\ - \int_{\{|u_n| > k\}} a_n(x, t, T_{m+1}(u_n), \nabla T_{m+1}(u_n)) \nabla w_{i,j}^\mu h_m(u_n) \, dx \, dt = I_1 + I_2$$

In the following we pass to the limit in (5.55) as n tends to $+\infty$, then j then μ and then m tends to $+\infty$. We prove that

$$I_2 = \int_Q \varphi_m \nabla T_k(u)_\mu h_m(u) \chi_{\{|u| > k\}} \, dx \, dt + \epsilon(n, j, \mu).$$

Using now the term I_1 of (5.55), we conclude that, it is easy to show that,

$$(5.56) \quad \int_Q a_n(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla w_{i,j}^\mu) h_m(u_n) \, dx \, dt \\ = \int_Q \left[a_n(x, t, T_k(u_n), \nabla T_k(u_n)) - a_n(x, t, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \right]$$

$$\begin{aligned}
& \times \left[\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s \right] h_m(u_n) \, dx \, dt \\
& + \int_Q a_n \left(x, t, T_k(u_n), \nabla T_k(v_j) \chi_j^s \right) \left[\nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s \right] h_m(u_n) \, dx \, dt \\
& \quad + \int_Q a_n \left(x, t, T_k(u_n), \nabla T_k(u_n) \right) \nabla T_k(v_j) \chi_j^s h_m(u_n) \, dx \, dt \\
& - \int_Q a_n \left(x, t, T_k(u_n), \nabla T_k(u_n) \right) \nabla w_{i,j}^\mu h_m(u_n) \, dx \, dt = J_1 + J_2 + J_3 + J_4,
\end{aligned}$$

where χ_j^s denotes the characteristic function of the subset

$$\Omega_s^j = \left\{ (x, t) \in Q : |\nabla T_k(v_j)| \leq s \right\}$$

In the following we pass to the limit in (5.56) as n tends to $+\infty$, then j then μ then m tends and then s tends to $+\infty$ in the last three integrals of the last side. We prove that

$$(5.57) \quad J_2 = \epsilon(n, j),$$

$$(5.58) \quad J_3 = \int_Q \varphi_k \nabla T_k(u) \chi_s \, dx \, dt + \epsilon(n, j),$$

and

$$(5.59) \quad J_4 = - \int_Q \varphi_k \nabla T_k(u) \, dx \, dt + \epsilon(n, j, \mu, s).$$

We conclude then that,

$$\begin{aligned}
(5.60) \quad & \int_Q \left[a_n \left(x, t, T_k(u_n), \nabla T_k(u_n) \right) - a_n \left(x, t, T_k(u_n), \nabla T_k(u) \chi_s \right) \right] \left[\nabla T_k(u_n) - \nabla T_k(u) \chi_s \right] \, dx \, dt \\
& = \int_Q \left[a_n \left(x, t, T_k(u_n), \nabla T_k(u_n) \right) - a_n \left(x, t, T_k(u_n), \nabla T_k(u) \chi_s \right) \right] \\
& \quad \times \left[\nabla T_k(u_n) - \nabla T_k(u) \chi_s \right] h_m(u_n) \, dx \, dt \\
& + \int_Q a_n \left(x, t, T_k(u_n), \nabla T_k(u_n) \right) \left[\nabla T_k(u_n) - \nabla T_k(u) \chi_s \right] (1 - h_m(u_n)) \, dx \, dt \\
& - \int_Q a_n \left(x, t, T_k(u_n), \nabla T_k(u) \chi_s \right) \left[\nabla T_k(u_n) - \nabla T_k(u) \chi_s \right] (1 - h_m(u_n)) \, dx \, dt.
\end{aligned}$$

Combining (5.48), (5.56), (5.57), (5.58), (5.59) and (5.60) we deduce,

$$\begin{aligned}
(5.61) \quad & \int_Q \left[a_n \left(x, t, T_k(u_n), \nabla T_k(u_n) \right) - a_n \left(x, t, T_k(u_n), \nabla T_k(u) \chi_s \right) \right] \left[\nabla T_k(u_n) - \nabla T_k(u) \chi_s \right] \, dx \, dt \\
& \leq \epsilon(n, j, \mu, m, s).
\end{aligned}$$

To pass to the limit in (5.61) as n, j, m, s tends to infinity, we obtain

$$\begin{aligned}
(5.62) \quad & \lim_{s \rightarrow \infty} \lim_{n \rightarrow \infty} \int_Q \left[a_n \left(x, t, T_k(u_n), \nabla T_k(u_n) \right) - a_n \left(x, t, T_k(u_n), \nabla T_k(u) \chi_s \right) \right] \\
& \quad \times \left[\nabla T_k(u_n) - \nabla T_k(u) \chi_s \right] \, dx \, dt = 0.
\end{aligned}$$

This implies by the lemma 5.5, the desired statement and hence the proof of Proposition 5.4 is achieved. \square

★ **Step 4.** In this step we prove that u satisfies (4.2).

Lemma 5.7. *The limit u of the approximate solution u_n of (5.6)-(5.8) satisfies*

$$(5.63) \quad \lim_{m \rightarrow +\infty} \int_{\{m \leq |u| \leq m+1\}} a(x, t, u, \nabla u) \nabla u \, dx \, dt = 0.$$

Proof. Remark that for any fixed $m \geq 0$ one has

$$\begin{aligned} & \int_{\{m \leq |u_n| \leq m+1\}} a_n(x, t, u_n, \nabla u_n) \nabla u_n \, dx \, dt \\ &= \int_Q a_n(x, t, u_n, \nabla u_n) \left[\nabla T_{m+1}(u_n) - \nabla T_m(u_n) \right] \, dx \, dt \\ &= \int_Q a_n \left(x, t, T_{m+1}(u_n), \nabla T_{m+1}(u_n) \right) \nabla T_{m+1}(u_n) \, dx \, dt \\ & \quad - \int_Q a_n \left(x, t, T_m(u_n), \nabla T_m(u_n) \right) \nabla T_m(u_n) \, dx \, dt \end{aligned}$$

According to (5.42) (with $z_n = T_m(u_n)$ or $z_n = T_{m+1}(u_n)$), one is at liberty to pass to the limit as n tends to $+\infty$ for fixed $m \geq 0$ and to obtain

$$(5.64) \quad \begin{aligned} & \lim_{n \rightarrow +\infty} \int_{\{m \leq |u_n| \leq m+1\}} a_n(x, t, u_n, \nabla u_n) \nabla u_n \, dx \, dt \\ &= \int_Q a \left(x, t, T_{m+1}(u), \nabla T_{m+1}(u) \right) \nabla T_{m+1}(u) \, dx \, dt \\ & \quad - \int_Q a \left(x, t, T_m(u), \nabla T_m(u) \right) \nabla T_m(u) \, dx \, dt \\ &= \int_{\{m \leq |u| \leq m+1\}} a(x, t, u, \nabla u) \nabla u \, dx \, dt \end{aligned}$$

Taking the limit as m tends to $+\infty$ in (5.64) and using the estimate (5.26) it possible to conclude that (5.63) holds true and the proof of Lemma 5.7 is complete. \square

★ **Step 5.** In this step, u is shown to satisfies (4.3) and (4.4). Let S be a function in $W^{2,\infty}(\mathbb{R})$ such that S' has a compact support. Let K be a positive real number such that $\text{supp}(S') \subset [-K, K]$. Pointwise multiplication of the approximate equation (5.6) by $S'(u_n)$ leads to

$$(5.65) \quad \begin{aligned} & \frac{\partial B_S^n(x, u_n)}{\partial t} - \text{div} \left(S'(u_n) a_n(x, t, u_n, \nabla u_n) \right) + S''(u_n) a_n(x, t, u_n, \nabla u_n) \nabla u_n \\ & \quad - \text{div} \left(S'(u_n) \Phi(u_n) \right) + S''(u_n) \Phi(u_n) \nabla u_n = f S'(u_n) \quad \text{in } D'(Q), \end{aligned}$$

where $B_S^n(x, z) = \int_0^z S'(r) \frac{\partial b_n(x, r)}{\partial r} \, dr$.

It what follows we pass to the limit as n tends to $+\infty$ in each term of (5.65).

★ Since S' is bounded, and $B_S^n(x, u_n)$ converges to $B_S(x, u)$ a.e. in Q and in $L^\infty(Q)$ weak ★. Then $\frac{\partial B_S^n(x, u_n)}{\partial t}$ converges to $\frac{\partial B_S(x, u)}{\partial t}$ in $D'(Q)$ as n tends to $+\infty$.

★ Since $\text{supp}S \subset [-K, K]$, we have

$$S'(u_n)a_n(x, t, u_n, \nabla u_n) = S'(u_n)a_n(x, t, T_K(u_n), \nabla T_K(u_n)) \quad \text{a.e. in } Q.$$

The pointwise convergence of u_n to u as n tends to $+\infty$, the bounded character of S' , (5.22) and (5.36) of Lemma 5.4 imply that

$$S'(u_n)a_n(x, t, T_K(u_n), \nabla T_K(u_n)) \rightharpoonup S'(u)a(x, t, T_K(u), \nabla T_K(u)) \quad \text{weakly in } (L_{\overline{M}}(Q))^N,$$

for $(\Pi L_{\overline{M}}, \Pi E_M)$ as n tends to $+\infty$, because $S(u) = 0$ for $|u| \geq K$ a.e. in Q . And the term $S'(u)a(x, t, T_K(u), \nabla T_K(u)) = S'(u)a(x, t, u, \nabla u)$ a.e. in Q .

★ Since $\text{supp}S' \subset [-K, K]$, we have

$$S''(u_n)a_n(x, t, u_n, \nabla u_n)\nabla u_n = S''(u_n)a_n(x, t, T_K(u_n), \nabla T_K(u_n))\nabla T_K(u_n) \quad \text{a.e. in } Q.$$

The pointwise convergence of $S''(u_n)$ to $S''(u)$ as n tends to $+\infty$, the bounded character of S'' and (5.22)-(5.36) of Lemma 5.4 allow to conclude that

$$S''(u_n)a_n(x, t, u_n, \nabla u_n)\nabla u_n \rightharpoonup S''(u)a(x, t, T_K(u), \nabla T_K(u))\nabla T_K(u) \quad \text{weakly in } L^1(Q),$$

as n tends to $+\infty$. And

$$S''(u)a(x, t, T_K(u), \nabla T_K(u))\nabla T_K(u) = S''(u)a(x, t, u, \nabla u)\nabla u \quad \text{a.e. in } Q.$$

★ Since $\text{supp}S' \subset [-K, K]$, we have $S'(u_n)\Phi_n(u_n) = S'(u_n)\Phi_n(T_K(u_n))$ a.e. in Q . As a consequence of (3.7), (5.3) and (5.22), it follows that:

$$S'(u_n)\Phi_n(u_n) \rightarrow S'(u)\Phi(T_K(u)) \quad \text{strongly in } (E_M(Q))^N,$$

as n tends to $+\infty$. The term $S'(u)\Phi(T_K(u))$ is denoted by $S'(u)\Phi(u)$.

★ Since $S \in W^{1,\infty}(\mathbb{R})$ with $\text{supp}S' \subset [-K, K]$, we have $S''(u_n)\Phi_n(u_n)\nabla u_n = \Phi_n(T_K(u_n))\nabla S''(u_n)$ a.e. in Q , we have, $\nabla S''(u_n)$ converges to $\nabla S''(u)$ weakly in $L_M(Q)^N$ as n tends to $+\infty$, while $\Phi_n(T_K(u_n))$ is uniformly bounded with respect to n and converges a.e. in Q to $\Phi(T_K(u))$ as n tends to $+\infty$. Therefore

$$S''(u_n)\Phi_n(u_n)\nabla u_n \rightharpoonup \Phi(T_K(u))\nabla S''(u) \quad \text{weakly in } L_M(Q).$$

★ Due to (5.4) and (5.22), we have $f_n S(u_n)$ converges to $fS(u)$ strongly in $L^1(Q)$, as n tends to $+\infty$.

As a consequence of the above convergence result, we are in a position to pass to the limit as n tends to $+\infty$ in equation (5.65) and to conclude that u satisfies (4.3).

It remains to show that $B_S(x, u)$ satisfies the initial condition (4.4). To this end, firstly remark that, S' has a compact support, we have $B_S^n(x, u_n)$ is bounded in $L^\infty(Q)$. Secondly, (5.65) and the above considerations on the behavior of the terms

of this equation show that $\frac{\partial B_S^n(x, u_n)}{\partial t}$ is bounded in $L^1(Q) + W^{-1,x}L^1_{\overline{M}}(Q)$. As a consequence, an Aubin's type Lemma (see e.g., [36], Corollary 4) (see also [16]) implies that $B_S^n(x, u^n)$ lies in a compact set of $C^0([0, T]; L^1(\Omega))$. It follows that, $B_S^n(x, u_n)(t=0)$ converges to $B_S(x, u)(t=0)$ strongly in $L^1(\Omega)$. Due to (4.8) and (5.5), we conclude that $B_S^n(x, u_n)(t=0) = B_S^n(x, u_{0n})$ converges to $B_S(x, u)(t=0)$ strongly in $L^1(\Omega)$. Then we conclude that

$$B_S(x, u)(t=0) = B_S(x, u_0) \text{ in } \Omega.$$

As a conclusion of step 1 to step 5, the proof of theorem 5.1 is complete.

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