Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **72**, 1-14; **http://www.math.u-szeged.hu/ejqtde/**

ON A NONSTANDARD VOLTERRA TYPE DYNAMIC INTEGRAL EQUATION ON TIME SCALES

DEEPAK B. PACHPATTE

Abstract. The main objective of the present paper is to study some basic qualitative properties of solutions of a nonstandard Volterra type dynamic integral equation on time scales. The tools employed in the analysis are based on the applications of the Banach fixed point theorem and a certain integral inequality with explicit estimate on time scales.

1. Introduction

Many physical systems can be modeled via dynamical systems on time scales. As a response to the needs of the diverse applications, recently many authors have studied the qualitative properties of solutions of Volterra type integral equations on time scales, see [3, 6, 7, 8]. In [1] the authors have studied the Fredholm integral equation in which the functions involved under the integral sign contains the derivative of a unknown functions, using Pervo's fixed point theorem, the method of successive approximation and trapezoidal quadrature rule. In view of the importance of the equation studied in [1], Pachpatte [10, 11, 12] has studied the existence uniqueness and other properties of solutions of more general integral equations using Banach fixed point theorem and suitable integral inequalities with explicit estimates. Motivated by the results in [1, 10, 11, 12], in this paper we consider the nonstandard Volterra type dynamic equation on time scales of the form

$$x(t) = g(t) + \int_{t_0}^{t} f\Big(t, \tau, x(\tau), x^{\Delta}(\tau)\Big)\Delta\tau, \tag{1.1}$$

where g,f are given functions and x is the unknown function to be found, $g : I_{\mathbb{T}} \to \mathbb{R}^n$, $f : I_{\mathbb{T}}^2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, t is from a time scale $\mathbb{T}$, which is nonempty closed subset of $\mathbb{R}$, the set of real numbers, $\tau \leq t$ and $I_{\mathbb{T}} = I \cap \mathbb{T}$, $I = [t_0, \infty)$ the given subset of $\mathbb{R}$, $\mathbb{R}^n$ the real n dimensional Euclidean space with appropriate norm defined by $|.|$. The integral sign represents the general type of operation known as delta integral (for details, see [2]).

In recent papers ([6, 7, 8]) the authors have studied existence and other qualitative properties of solutions of equation (1.1) when x^{Δ} in (1.1) is absent. In fact, the study of qualitative properties of solutions of (1.1) is challenging task because of the occurrence of the extra factor x^{Δ} in the integrand on the right hand side in (1.1). In this paper we offer sufficient conditions for the existence, uniqueness and other properties of solutions of (1.1). The main tools employed here are based

2000 *Mathematics Subject Classification.* 39A10, 39A12.
Key words and phrases. Integral equations, time scales, qualitative properties,
Banach fixed point theorem, explicit estimates, existence and uniqueness.

on the application of Banach fixed point theorem and a suitable integral inequality with explicit estimates on time scales. We hope that the results given here will encourage the further investigation and widen the scope of applications.

2. Preliminaries

In this section we introduce some basic definitions and results on time scales $\mathbb{T}$ needed in our subsequent discussion. The forward (backward) jump operator $\sigma(t)$ of t for $t <$ sup $\mathbb{T}$ (respectively $\rho(t)$ at t for $t >$inf $\mathbb{T}$) is given by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad (\rho(t) = \sup\{s \in \mathbb{T} : s < t\}),$$

for all $t \in \mathbb{T}$. The graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by $\mu\,(t) = \sigma\,(t) - t$. Throughout we assume that $\mathbb{T}$ has a topology that it inherits from the standard topology on the real number $\mathbb{R}$. The jump operators σ and ρ allow the classification of points in a time scale in the way: If $\sigma(t) > t$, then the point t is called right scattered ; while if $\rho(t) < t$, then t is termed left scattered. If $t <$ sup $\mathbb{T}$ and $\sigma(t) = t$, then the point t is called right dense: while if $t >$ inf $\mathbb{T}$ and $\rho(t) = t$, then we say t is left-dense. If $\mathbb{T}$ has a left-scattered maximum value m, then we define $\mathbb{T}^k := \mathbb{T} - m$, otherwise $\mathbb{T}^k := \mathbb{T}$. We say that $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous provided f is continuous at each right-dense point of $\mathbb{T}$ and has a finite left-sided limit at each left-dense point of $\mathbb{T}$ and will be denoted by C_{rd}.

Fix $t \in \mathbb{T}^k$ and let $x : \mathbb{T} \to \mathbb{R}$. Define $x^{\Delta}(t)$ to be number (if it exists) with the property that given $\epsilon > 0$ there is a neighbourhood U of t with

$$\left|[x\,(\sigma\,(t)) - x\,(s)] - x^{\Delta}\,(t)\,[\sigma\,(t) - s]\right| \leq \epsilon\,|\sigma\,(t) - s|\,,$$

for all $s \in U$.

A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^{\Delta}\,(t) = f\,(t)$ holds for all $t \in I_{\mathbb{T}}$. In this case we define the integral of f by

$$\int_s^{\tau} f\,(\tau)\Delta\tau = F\,(t) - F\,(s)\,,$$

where $s, t \in \mathbb{T}$. The function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu\,(t)\,p\,(t) \neq 0$ for all $t \in I_{\mathbb{T}}$. We denote by $\Re$ the set of all regressive and rd-continuous functions and define the set of all regressive functions by

$$\Re^{+} = \{p \in \mathbb{R} : 1 + \mu\,(t)\,p\,(t) > 0 \;\; for\ all\ \, t \in \mathbb{T}\}\,.$$

For $p \in \Re^{+}$ we define (see [2]) the exponential function $e_p(., t_0)$ on time scale $\mathbb{T}$ as the unique solution to the scalar initial value problem

$$x^{\Delta}\,(t) = p\,(t)\,x, \;\; x\,(t_0) = 1.$$

If $p \in \Re^{+}$, then $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$. The exponential function $e_p(., t_0)$ is given by

$$e_p\,(t, t_0) = \begin{cases} \exp\left(\int\limits_{t_0}^{t} p\,(s)\,\Delta s\right) & for\ \ t \in \mathbb{T}, \;\; \mu > 0 \\ \exp\left(\int\limits_{t_0}^{t} \frac{log(1+\mu(s)p(s))}{\mu(s)}\Delta s\right) & for\ \ t \in \mathbb{T}, \;\; \mu > 0 \end{cases}$$

where log is a principle logarithm function. In order to allow a comparison of the results in the paper with the continuous case, we note that, if $\mathbb{T} = \mathbb{R}$, the exponential function is given by

$$e_p(t,s) = \exp\left(\int_s^t p(\tau)\,d\tau\right),\ e_\alpha(t,s) = \exp(\alpha(t-s)),\ e_\alpha(t,0) = \exp(\alpha t)$$

for $s,t \in \mathbb{R}$, where $\alpha \in \mathbb{R}$ is a constant and $p : \mathbb{R} \to \mathbb{R}$ is a continuous function. To compare with the discrete case, if $\mathbb{T} = \mathbb{Z}$ (the set of integers), the exponential function is given by

$$e_p(t,s) = \prod_{\tau=s}^{t-1}[1+p(\tau)],\ e_\alpha(t,s) = (1+\alpha)^{t-s},\ e_\alpha(t,0) = (1+\alpha)^t,$$

for $s,t \in \mathbb{Z}$ with $s < t$, where $\alpha \neq -1$ is a constant and $p : \mathbb{Z} \to \mathbb{R}$ is a sequence satisfying $p(t) \neq -1$ for all $t \in \mathbb{Z}$.

We denote by $\Omega(t,s)$ the class of functions $k : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ which are continuous at (t,t), where $t \in \mathbb{T}^k$ with $t > t_0$, $t_0 \in \mathbb{T}^k$ such that $k(t,.)$ is rd-continuous on $[t_0, \sigma(t)]$ and for each $\epsilon > 0$ there exists a neighbourhood U of t independent of $\tau \in [t_0, \sigma(t)]$, such that

$$\left|k(\sigma(t),\tau) - k(s,\tau) - k^\Delta(t,\tau)(\sigma(t)-s)\right| \leq \in |\sigma(t) - s|,$$

for all $s \in U$, where k^Δ denotes the delta derivative of k with respect to first variable.

We use following fundamental result proved in Bohner and Peterson [2] (see also [4,9]).

Lemma 2.1. *Let $k \in \Omega(t,s)$. Then*

$$g(t) = \int_{t_0}^t k(t,\tau)\,\Delta\tau, \tag{2.1}$$

for $t \in I_\mathbb{T}$ implies

$$g^\Delta(t) = \int_{t_0}^t k^\Delta(t,\tau)\,\Delta\tau + k(\sigma(t),t), \tag{2.2}$$

for $t \in I_\mathbb{T}$.

We also need the following special version of Theorem 3.10 given in [4].

Lemma 2.2. *Assume that $u, a \in C_{rd}$, $u \geq 0$, $a \geq 0$. Let $k(t,s) \in \Omega(t,s)$, $k(\sigma(t),t) \geq 0$ and $k^\Delta(t,s) \geq 0$ for $s,t \in \mathbb{T}$ with $s \leq t$. If*

$$u(t) \leq a(t) + \int_{t_0}^t k(t,\tau)\,u(\tau)\Delta\tau, \tag{2.3}$$

for all $t \in \mathbb{T}$ *then*

$$u(t) \leq a(t) + \int_{t_0}^{t} B(\tau)\, e_A(t, \sigma(\tau))\Delta\tau, \tag{2.4}$$

for all $t \in \mathbb{T}$ *where*

$$A(t) = k(\sigma(t), t) + \int_{t_0}^{t} k^{\Delta}(t, \tau)\Delta\tau, \tag{2.5}$$

$$B(t) = k(\sigma(t), t)\, a(t) + \int_{t_0}^{t} k^{\Delta}(t, \tau) a(\tau)\, \Delta\tau, \tag{2.6}$$

for $t \in \mathbb{T}$.

3. Existence and Uniqueness

In what follows, we assume that the functions $g, g^{\Delta} : I_{\mathbb{T}} \to \mathbb{R}^n$ and for $\tau \leq t$, $f, f^{\Delta} : I_{\mathbb{T}}^2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ are rd-continuous. By a solution of equation (1.1) we mean a rd-continuous function $x(t)$ for $t \in \mathbb{T}$ which is delta differentiable with respect to t and satisfies the equation (1.1). For every rd-continuous function $x(t)$ together with its delta derivative $x^{\Delta}(t)$, we denote by $|x(t)|_1 = |x(t)| + \left|x^{\Delta}(t)\right|$. For $t \in \mathbb{T}$ the notation $a(t) = O(b(t))$ for $t \to t_0$ we mean that there exists a constant $k \geq 0$ such that $\left|\frac{a(t)}{b(t)}\right| \leq k$ on some right hand neighbourhood of the point t_0, we denote by G the space of all rd-continuous functions $x(t)$ whose delta derivative $x^{\Delta}(t)$ exist, which fulfill the condition

$$|x(t)|_1 = O(e_{\lambda}(t, t_0)), \tag{3.1}$$

where λ is a positive constant. In the space G we define the norm

$$|x|_G = \sup\left\{|x(t)|_1\, e_{\lambda}(t, t_0) : t \in I_{\mathbb{T}}\right\}. \tag{3.2}$$

It is easy to see that G with the norm defined in (3.2) is a Banach space. The condition (3.1) implies that there is a constant $N \geq 0$ such that

$$|x(t)|_1 \leq N e_{\lambda}(t, t_0).$$

Using this fact in (3.2) we observe that

$$|x|_G \leq N. \tag{3.3}$$

Our main result in this section is given in the following theorem.

Theorem 3.1. *Assume that*
(i) the function f *in equation* (1.1) *and its delta derivative with respect to* t *are rd-continuous and satisfy the conditions*

$$|f(t, \tau, u, v) - f(t, \tau, \overline{u}, \overline{v})| \leq h_1(t, \tau)\left[|u - \overline{u}| + |v - \overline{v}|\right], \tag{3.4}$$

$$\left|f^{\Delta}(t, \tau, u, v) - f^{\Delta}(t, \tau, \overline{u}, \overline{v})\right| \leq h_2(t, \tau)\left[|u - \overline{u}| + |v - \overline{v}|\right], \tag{3.5}$$

where for $i = 1, 2$ and $t_0 \le \tau \le t$, $h_i(t,\tau) \in \Omega(t,\tau)$.
(ii) there exists a nonnegative constant α such that $\alpha < 1$ and

$$h_1(\sigma(t),\tau)\, e_\lambda(t,t_0) + \int_{t_0}^{t} [h_1(t,\tau) + h_2(t,\tau)] e_\lambda(\tau,t_0)\,\Delta\tau \le \alpha e_\lambda(t,t_0), \tag{3.6}$$

for $t \in I_{\mathbb{T}}$, and λ is given as in (3.1).
(iii) there exists a nonnegative constant β such that

$$|g(t)|_1 + |f(\sigma(t),t,0,0)| + \int_{t_0}^{t} \left[|f(t,\tau,0,0)| + \left|f^\Delta(t,\tau,0,0)\right|\right]\Delta\tau \le \beta e_\lambda(t,t_0), \tag{3.7}$$

where g, g^Δ, f, f^Δ are as in equation (1.1) and λ is given as in (3.1). Then Equation (1.1) has a unique solution $x(t)$ in G on $I_{\mathbb{T}}$.

Proof. Let $x : I_{\mathbb{T}} \to \mathbb{R}^n$ be rd-continuous and define the operator S by

$$(Sx)(t) = g(t) + \int_{t_0}^{t} f\left(t,\tau,x(\tau),x^\Delta(\tau)\right)\Delta\tau. \tag{3.8}$$

By taking delta derivative on both sides of (3.8) (see Lemma 1), we get

$$(Sx)^\Delta(t) = g^\Delta(t) + f\left(\sigma(t),t,x(t),x^\Delta(t)\right) + \int_{t_0}^{t} f^\Delta\left(t,\tau,x(\tau),x^\Delta(\tau)\right)\Delta\tau. \tag{3.9}$$

We show that Sx maps G into itself. Evidently Sx and $(Sx)^\Delta$ are rd-continuous on $\mathbb{T}$. We first verify that (3.1) is satisfied. From (3.8) and (3.9), we have

$$\begin{aligned}
|(Sx)(t)|_1 &= |(Sx)(t)| + \left|(Sx)^\Delta(t)\right| \\
&\le |g(t)| + \int_{t_0}^{t} \left|f\left(t,\tau,x(\tau),x^\Delta(\tau)\right) - f(t,\tau,0,0) + f(t,\tau,0,0)\right|\Delta\tau \\
&\quad + \left|g^\Delta(t)\right| + \left|f\left(\sigma(\tau),\tau,x(t),x^\Delta(t)\right) - f(\sigma(t),t,0,0) + f(\sigma(t),t,0,0)\right| \\
&\quad + \int_{t_0}^{t} \left|f^\Delta\left(t,\tau,x(\tau),x^\Delta(\tau)\right) - f^\Delta(t,\tau,0,0) + f^\Delta(t,\tau,0,0)\right|\Delta\tau \\
&\le |g(t)|_1 + \int_{t_0}^{t} |f(t,\tau,0,0)|\Delta\tau + \int_{t_0}^{t} h_1(t,\tau)\,|x(t)|_1\,\Delta\tau \\
&\quad + |f(\sigma(t),t,0,0)| + \int_{t_0}^{t} \left|f^\Delta(t,\tau,0,0)\right|\Delta\tau
\end{aligned}$$

$$
\begin{aligned}
&+ h_1\left(\sigma\left(t\right),\tau\right)\left|x\left(t\right)\right|_1 + \int\limits_{t_0}^{t} h_2\left(t,\tau\right)\left|x\left(\tau\right)\right|_1 \Delta\tau \\
&\leq \beta e_\lambda\left(t,t_0\right) + h_1\left(\sigma\left(t\right),\tau\right)\left|x\left(t\right)\right|_1 + \int\limits_{t_0}^{t} h_1\left(t,\tau\right)\left|x\left(\tau\right)\right|_1 \Delta\tau \\
&+ \int\limits_{t_0}^{t} h_2\left(t,\tau\right)\left|x\left(\tau\right)\right|_1 \Delta\tau \\
&\leq \beta e_\lambda\left(t,t_0\right) + \left|x\right|_G \left\{h_1\left(\sigma\left(t\right),t\right) e_\lambda\left(t,t_0\right)\right. \\
&\left.+ \int\limits_{t_0}^{t} [h_1(t,\tau)+h_2(t,\tau)] e_\lambda\left(\tau,t_0\right)\Delta\tau\right\} \\
&\leq \left[\beta + N\alpha\right] e_\lambda\left(t,t_0\right).
\end{aligned}
\tag{3.10}
$$

From (3.10) it follows that $Sx \in G$. This proves that S maps G into itself.

Now we verify that S is a contraction map. Let $x, y \in C\left(I_{\mathbb{T}}, \mathbb{R}^n\right)$. From $(3.9), (3.10)$ and using the hypotheses, we have

$$
\begin{aligned}
\left|(Sx)(t)-(Sy)(t)\right|_1 &= \left|\left(Sx\right)\left(t\right)-\left(Sy\right)\left(t\right)\right| + \left|\left(Sx\right)^{\Delta}\left(t\right)-\left(Sy\right)^{\Delta}\left(t\right)\right| \\
&\leq \int\limits_{t_0}^{t} \left|f\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right) - f\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right| \Delta\tau \\
&+ \left|f\left(\sigma\left(t\right),t,x(t),x^{\Delta}(t)\right) - f\left(\sigma\left(t\right),t,y(t),y^{\Delta}(t)\right)\right| \\
&+ \int\limits_{t_0}^{t} \left|f^{\Delta}\left(t,\tau,x\left(\tau\right),x^{\Delta}\left(\tau\right)\right) - f^{\Delta}\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\right| \Delta\tau \\
&\leq \int\limits_{t_0}^{t} h_1\left(t,\tau\right)\left|x\left(\tau\right)-y(\tau)\right|_1 \Delta\tau + h_1\left(\sigma\left(t\right),t\right)\left|x\left(t\right)-y(t)\right|_1 \\
&+ \int\limits_{t_0}^{t} h_2\left(t,\tau\right)\left|x\left(t\right)-y\left(t\right)\right|_1 \Delta\tau \\
&\leq \left|x-y\right|_G \left\{h_1\left(\sigma\left(t\right),t\right) e_\lambda\left(t,t_0\right)\right. \\
&\left.+ \int\limits_{t_0}^{t} [h_1\left(t,\tau\right)+h_2\left(t,\tau\right)] e_\lambda\left(\tau,t_0\right)\Delta\tau\right\} \\
&\leq \left|x-y\right|_G \alpha e_\lambda\left(t,t_0\right).
\end{aligned}
\tag{3.11}
$$

From (3.11) we obtain

$$
\left|Sx-Sy\right|_G \leq \alpha \left|x-y\right|_1.
$$

Since $\alpha < 1$, it follows from the Banach fixed point theorem that S has a unique fixed point in G. The fixed point of S is however a solution of equation (1.1). The proof is complete. $\square$

Remark 1. If we choose $f(t, \tau, x, u) = E(t, \tau, x) - u$, then by simple calculation it is easy to observe that the equation (1.1) reduces to the equation of the following form

$$y(t) = h(t) + \int_{t_0}^{t} k(t, \tau, y(\tau))\, \Delta\tau, \tag{3.12}$$

which is recently studied in [6] by Kulik and Tisdell, using Banach and Schafer fixed point theorems (see also [7]). In [1] the authors have studied the continuous case of a variant of equation (1.1) by using Pervo's fixed point theorem and Successive approximations.

The following theorem deals with the uniqueness of solution of equation (1.1) whose proof is based on the application of the inequality given in Lemma 2.2.

Theorem 3.2. *Assume that the function f in equation* (1.1) *and its delta derivative with respect to t satisfy the conditions* (3.4) *and* (3.5). *Further, assume that $h_1(t,s), h_2(t,s) \in \Omega(\sigma, s)$ and $h_1(\sigma(t), t) \le c$ where $c < 1$ is a constant. Then the equation* (1.1) *has at most one solution on $I_{\mathbb{T}}$.*

Proof. Let $x(t)$ and $y(t)$ be two solutions of (1.1), then from the hypotheses, we have

$$\begin{aligned}
&|x(t) - y(t)| + \left|x^{\Delta}(t) - y^{\Delta}(t)\right| \\
&\le \left|\int_{t_0}^{t} \left|f\left(t, \tau, x(\tau), x^{\Delta}(\tau)\right)\Delta\tau - f\left(t, \tau, y(\tau), y^{\Delta}(\tau)\right)\Delta\tau\right|\right. \\
&+ \left|f\left(\sigma(t), t, x(t), x^{\Delta}(t)\right) - f\left(\sigma(t), t, y(t), y^{\Delta}(t)\right)\right| \\
&+ \left|\int_{t_0}^{t} f^{\Delta}\left(t, \tau, x(\tau), x^{\Delta}(\tau)\right)\Delta\tau - \int_{t_0}^{t} f^{\Delta}\left(t, \tau, y(\tau), y^{\Delta}(\tau)\right)\Delta\tau\right| \\
&\le \int_{t_0}^{t} h_1(t, \tau)\left[|x(\tau) - y(\tau)| + \left|x^{\Delta}(\tau) - y^{\Delta}(\tau)\right|\right]\Delta\tau \\
&+ h_1(\sigma(t), t)\left[|x(t) - y(t)| + \left|x^{\Delta}(t) - y^{\Delta}(t)\right|\right] \\
&+ \int_{t_0}^{t} h_2(t, \tau)\left[|x(t) - y(\tau)| + \left|x^{\Delta}(t) - y^{\Delta}(\tau)\right|\right]\Delta\tau \\
&\le c\left[|x(t) - y(t)| + \left|x^{\Delta}(t) - y^{\Delta}(t)\right|\right] \\
&+ \int_{t_0}^{t} \left[h_1(t, \tau) + h_2(t, \tau)\right]\left[|x(\tau) - y(\tau)| + \left|x^{\Delta}(\tau) - y^{\Delta}(\tau)\right|\right]\Delta\tau.
\end{aligned} \tag{3.13}$$

From (3.13) we have

$$\begin{aligned}&|x(t)-y(t)|+\left|x^{\Delta}(t)-y^{\Delta}(t)\right|\\&\leq \frac{1}{1-c}\int_{t_0}^{t}\left[h_1(t,\tau)+h_2(t,\tau)\right]\left[|x(t)-y(t)|+\left|x^{\Delta}(\tau)-y^{\Delta}(\tau)\right|\right]\Delta\tau.\end{aligned} \tag{3.14}$$

Now a suitable application of lemma 2.2 (when $a(t)=0$) to (3.14) yields

$$|x(t)-y(t)|+\left|x^{\Delta}(t)-y^{\Delta}(t)\right|\leq 0,$$

and hence $x(t)=y(t)$. Thus there is at most one solution to equation (1.1) on $I_{\mathbb{T}}$. □

4. Properties of Solutions

In this section we study some basic properties of solutions of equation (1.1) under some suitable conditions on the functions involved therein.

First we present the following theorem concerning the estimate on the solution of equation (1.1).

Theorem 4.1. *Assume that the functions g, f in equation* (1.1) *and their delta derivatives with respect to t satisfy*

$$|g(t)|+\left|g^{\Delta}(t)\right|\leq r(t), \tag{4.1}$$

$$|f(t,s,u,v)|\leq h_1(t,s)\left[|u|+|v|\right], \tag{4.2}$$

$$\left|f^{\Delta}(t,s,u,v)\right|\leq h_2(t,s)\left[|u|+|v|\right], \tag{4.3}$$

where $r: I_T\to R_+=[0,\infty)$ is rd-continuous and $h_i\in\Omega(t,s)$ for $i=1,2$. Let $\overline{k}(t,s)=h_1(t,s)+h_2(t,s)$ and assume that $h_1(\sigma(t),t)\leq c$, where $c<1$ is a constant. If $x(t), t\in I_{\mathbb{T}}$ is any solution of equation (1.1) *then*

$$|x(t)|+\left|x^{\Delta}(t)\right|\leq \frac{r(t)}{1-c}+\int_{t_0}^{t}B_1(\tau)e_{A_1}(t,\sigma(t))\,\Delta\tau, \tag{4.4}$$

for $t\in I_{\mathbb{T}}$, where $A_1(t)$ and $B_1(t)$ are defined respectively by the right hand sides of (2.5) *and* (2.6) *by replacing $k(t,\tau)$ by $\frac{\overline{k}(t,\tau)}{1-c}$ and $a(t)$ by $\frac{r(t)}{1-c}$.*

Proof. Using the fact that $x(t)$ is a solution of equation (1.1) and hypotheses, we have

$$\begin{aligned}|x(t)|+\left|x^{\Delta}(t)\right|&\leq |g(t)|+\int_{t_0}^{t}\left|f\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right)\right|\Delta\tau\\&\quad+\left|g^{\Delta}(t)\right|+\left|f\left(\sigma(t),t,x(t),x^{\Delta}(t)\right)\right|+\int_{t_0}^{t}\left|f^{\Delta}\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right)\right|\Delta\tau\\&\leq r(t)+\int_{t_0}^{t}h_1(t,\tau)\left[|x(\tau)|+\left|x^{\Delta}(\tau)\right|\right]\Delta\tau\end{aligned}$$

$$+ h_1(\sigma(t), t)\left[|x(t)| + \left|x^{\Delta}(t)\right|\right] + \int_{t_0}^{t} h_2(t,\tau)\left[|x(\tau)| + \left|x^{\Delta}(\tau)\right|\right]\Delta\tau$$

$$\leq r(t) + c\left[|x(t)| + \left|x^{\Delta}(t)\right|\right] + \int_{t_0}^{t} \overline{k}(t,\tau)\left[|x(\tau)| + \left|x^{\Delta}(\tau)\right|\right]\Delta\tau. \tag{4.5}$$

From (4.5), we obtain

$$|x(t)| + \left|x^{\Delta}(t)\right| \leq \frac{r(t)}{1-c} + \frac{1}{1-c}\int_{t_0}^{t} \overline{k}(t,\tau)\left[|x(\tau)| + \left|x^{\Delta}(\tau)\right|\right]\Delta\tau. \tag{4.6}$$

Now an application of Lemma 2.2 to (4.6) yields (4.4). □

Remark 2. We note that the estimate obtained in (4.4) gives not only bounds on solutions of equation (1.1) but also bounds on their delta derivatives. If the estimate on the right hand side of (4.4) is bounded then the solution of equation (1.1) and its delta derivative are bounded.

Next we obtain the estimate on the solution of equation (1.1) assuming that the function f and its delta derivative with respect to t satisfy Lipscitz type conditions.

Theorem 4.2. *Assume that the function f and its delta derivative with respect to t satisfy the conditions* (3.4) *and* (3.5). *Let $h_i(t,s)$, $\overline{k}(t,s)$, $h_1(\sigma(t),t)$ and c be as in Theorem 4.1 and*

$$\alpha(t) = \left|f\left(\sigma(t), t, g(t), g^{\Delta}(t)\right)\right| + \int_{t_0}^{t}\left|f\left(t,\tau,g(\tau),g^{\Delta}(\tau)\right)\right|\Delta\tau$$

$$+ \int_{t_0}^{t}\left|f^{\Delta}\left(t,s,g(\tau),g^{\Delta}(\tau)\right)\right|\Delta\tau,$$

where g is defined as in equation (1.1). *If $x(t), t \in I_{\mathbb{T}}$ is any solution of equation* (1.1) *then*

$$|x(t) - g(t)| + \left|x^{\Delta}(t) - g^{\Delta}(t)\right| \leq \frac{\alpha(t)}{1-c} + \int_{t_0}^{t} B_2(\tau)e_{A_1}(\tau,\sigma(\tau))\Delta\tau, \tag{4.7}$$

for $t \in I_{\mathbb{T}}$ where $A_1(t)$ is as in Theorem 4.1 *and $B_2(t)$ is defined by the right hand side of* (2.5) *replacing $a(t)$ by $\frac{\alpha(t)}{1-c}$.*

Proof. Using the fact that $x(t)$ is a solution of (1.1) and hypotheses, we observe that

$$|x(t) - g(t)| + \left|x^{\Delta}(t) - g^{\Delta}(t)\right|$$

$$\leq \int_{t_0}^{t}\left|f\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right) - f\left(t,\tau,g(\tau),g^{\Delta}(\tau)\right)\right|\Delta\tau + \int_{t_0}^{t}\left|f\left(t,\tau,g(\tau),g^{\Delta}(\tau)\right)\right|\Delta\tau$$

$$+\left|f\left(\sigma(t),t,x(t),x^{\Delta}(t)\right)-f\left(\sigma(t),t,g(t),g^{\Delta}(t)\right)\right|+\left|f\left(\sigma(t),t,g(t),g^{\Delta}(t)\right)\right|$$

$$+\int_{t_0}^{t}\left|f^{\Delta}\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right)-f^{\Delta}\left(t,\tau,g(\tau),g^{\Delta}(\tau)\right)\right|+\int_{t_0}^{t}\left|f^{\Delta}\left(t,\tau,g(\tau),g^{\Delta}(\tau)\right)\right|\Delta\tau$$

$$\leq\int_{t_0}^{t}\left|f\left(t,\tau,g(\tau),g^{\Delta}(\tau)\right)\right|\Delta\tau+\int_{t_0}^{t}h_1(t,\tau)\left[|x(\tau)-g(\tau)|+\left|x^{\Delta}(\tau)-g^{\Delta}(\tau)\right|\right]\Delta\tau$$

$$+\left|f\left(\sigma(t),t,g(t),g^{\Delta}(t)\right)\right|+\int_{t_0}^{t}\left|f^{\Delta}\left(t,\tau,g(\tau),g^{\Delta}(\tau)\right)\right|\Delta\tau$$

$$+h_1(\sigma(t),t)\left[|x(t)-g(t)|+\left|x^{\Delta}(t)-g^{\Delta}(t)\right|\right]$$

$$+\int_{t_0}^{t}h_2(t,\tau)\left[|x(\tau)-g(\tau)|+\left|x^{\Delta}(\tau)-g^{\Delta}(\tau)\right|\right]\Delta\tau$$

$$\leq c\left[|x(t)-g(t)|+\left|x^{\Delta}(t)-g^{\Delta}(t)\right|\right]+\left|f\left(\sigma(t),t,g(t),g^{\Delta}(t)\right)\right|$$

$$+\int_{t_0}^{t}\left|f\left(t,\tau,g\left(\tau\right),g^{\Delta}(\tau)\right)\right|\Delta\tau+\int_{t_0}^{t}f^{\Delta}\left(t,\tau,g\left(\tau\right),g^{\Delta}(\tau)\right)\Delta\tau$$

$$+\int_{t_0}^{t}\left[|h_1\left(t,\tau\right)+h_1\left(t,\tau\right)|\right]\left[|x(\tau)-g(\tau)|+\left|x^{\Delta}(\tau)-g^{\Delta}(\tau)\right|\right]\Delta\tau$$

$$=c\left[|x(t)-g(t)|+\left|x^{\Delta}(t)-g^{\Delta}(\tau)\right|\right]+\alpha(t)$$

$$+\int_{t_0}^{t}\left[|h_1\left(t,\tau\right)+h_1\left(t,\tau\right)|\right]\left[|x(\tau)-g(\tau)|+\left|x^{\Delta}(\tau)-g^{\Delta}(\tau)\right|\right]\Delta\tau. \tag{4.8}$$

From (4.8), we observe that

$$\left[|x(t)-g(t)|+\left|x^{\Delta}(t)-g^{\Delta}(t)\right|\right]$$

$$\leq\frac{\alpha(t)}{1-c}+\frac{1}{1-c}\int_{t_0}^{t}\overline{k}(t,\tau)\left[|x(\tau)-g(\tau)|+\left|x^{\Delta}(\tau)-g^{\Delta}(\tau)\right|\right]\Delta\tau. \tag{4.9}$$

Now an application of Lemma 2.2 to (4.9) yields (4.7). □

Consider the equation (1.1) and the corresponding Volterra integral equation

$$y(t)=H(t)+\int_{t_0}^{t}F\left(t,\tau,y\left(\tau\right),y^{\Delta}\left(\tau\right)\right)\Delta\tau, \tag{4.10}$$

where H,F are given functions and y is the unknown function, $H:I_{\mathbb{T}}\to\mathbb{R}^n$, for $\tau\leq t$, $F:I_{\mathbb{T}}^2\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ and the functions $H(t)$ and $F(t,\tau,u,v)$ are rd-continuous and delta differentiable with respect to t.

The following theorem deals with continuous dependence of solutions of equation (1.1) on functions involved therein.

Theorem 4.3. *Assume that the function f in equation* (1.1) *and its delta derivative with respect to t satisfy the conditions* (3.4) *and* (3.5). *Let, for i=1,2, $h_i\,(t,s)$, $\overline{k}(t,s)$, $h_1\,(\sigma(t),t)$ and c be as in Theorem 4.2. Let $x(t)$ and $y(t)$, $t \in I_{\mathbb{T}}$ be solutions of equations* (1.1) *and* (4.10) *respectively. Suppose that*

$$|g(t)-H(t)| + \int\limits_{t_0}^{t} \left| f\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right) - F\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right)\right| \Delta\tau \le r_1(t), \quad (4.11)$$

$$\begin{aligned}&\left|g^{\Delta}(t)-H^{\Delta}(t)\right| + \left|f\left(\sigma(t),t,y(t),y^{\Delta}(t)\right) - F\left(\sigma(t),t,y(t),y^{\Delta}(t)\right)\right| \\ &+ \int\limits_{t_0}^{t} \left|f^{\Delta}\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right) - F\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right)\right| \Delta\tau \le r_2(t), \end{aligned} \quad (4.12)$$

where g, f and H, F are functions involved in equations (1.1) *and* (4.10) *and $r_1(t), r_2(t) : I_T \to R_+$ are rd-continuous. Then the solution $x(t), t \in I_{\mathbb{T}}$ of equation* (1.1) *depends continuously on the functions involved on the right hand side of equation* (1.1).

Proof. As $x(t)$ and $y(t)$ are solutions of equations (1.1) and (4.10), we have

$$\begin{aligned}&|x(t)-y(t)| \\ &\le |g(t)-H(t)| + \int\limits_{t_0}^{t} \left|f\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right) - f\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right)\right| \Delta\tau \\ &+ \int\limits_{t_0}^{t} \left|f\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right) - F\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right)\right| \Delta\tau \\ &\le r_1(t) + \int\limits_{t_0}^{t} h_1(t,\tau)\left[|x(\tau)-y(\tau)| + \left|x^{\Delta}(\tau)-y^{\Delta}(\tau)\right|\right]\Delta\tau, \end{aligned} \quad (4.13)$$

and

$$\begin{aligned}&\left|x^{\Delta}(t)-y^{\Delta}\,(t)\right| \\ &\le \left|g^{\Delta}(t)-H^{\Delta}(t)\right| + \left|f\left(\sigma(t),t,x(t),x^{\Delta}(\tau)\right) - f\left(\sigma(t),t,y(t),y^{\Delta}(\tau)\right)\right| \\ &+ \left|f\left(\sigma(t),t,y(t),y^{\Delta}(\tau)\right) - F\left(\sigma(t),t,y(t),y^{\Delta}(\tau)\right)\right| \\ &+ \int\limits_{t_0}^{t} \left|f^{\Delta}\left(t,\tau,x(\tau),x^{\Delta}(\tau)\right) - f^{\Delta}\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right)\right|\Delta\tau \\ &+ \int\limits_{t_0}^{t} \left|f^{\Delta}\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right) - F^{\Delta}\left(t,\tau,y(\tau),y^{\Delta}(\tau)\right)\right|\Delta\tau \end{aligned}$$

$$\le r_2(t) + h_1(\sigma(t), t)\left[|x(t) - y(t)| + \left|x^{\Delta}(t) - y^{\Delta}(t)\right|\right]$$
$$+ \int_{t_0}^{t} h_2(t, \tau)\left[|x(\tau) - y(\tau)| + \left|x^{\Delta}(\tau) - y^{\Delta}(\tau)\right|\right]\Delta\tau. \tag{4.14}$$

Now from (4.13), (4.14) and using assumption that $h_1(\sigma(t), t) \le c$, we observe that

$$|x(t) - y(t)| + \left|x^{\Delta}(t) - y^{\Delta}(t)\right|$$
$$\le \frac{r_1(t) + r_2(t)}{1 - c} + \frac{1}{1 - c}\int_{t_0}^{t} \overline{k}(t, \tau)\left[|x(\tau) - y(\tau)| + \left|x^{\Delta}(\tau) - y^{\Delta}(\tau)\right|\right]\Delta\tau. \tag{4.15}$$

Now an application of Lemma 2.2 to (4.15) yields

$$|x(t) - y(t)| + \left|x^{\Delta}(t) - y^{\Delta}(t)\right| \le \frac{r_1(t) + r_2(t)}{1 - c} + \int_{t_0}^{t} B_3(\tau) e_{A_1}(t, \sigma(t))\Delta\tau, \tag{4.16}$$

for $t \in I_{\mathbb{T}}$, where $A_1(t)$ is as defined in Theorem 4.1 and $B_3(t)$ is defined by right hand side of equation (2.6) replacing $a(t)$ by $\frac{r_1(t)+r_2(t)}{1-c}$. From (4.16) it follows that the solution of equation (1.1) depends rd-continuously on the functions involved on the right side of equation (1.1). □

We next consider the Volterra integral equations on time scales of the forms

$$z(t) = g(t) + \int_{t_0}^{t} f\left(t, \tau, z(\tau), z^{\Delta}(\tau), \mu\right)\Delta\tau, \tag{4.17}$$

$$z(t) = g(t) + \int_{t_0}^{t} f\left(t, \tau, z(\tau), z^{\Delta}(\tau), \mu_0\right)\Delta\tau, \tag{4.18}$$

where g, f are given functions and z is the unknown function to be found and μ, μ_0 are real parameters.
The following theorem deals with the dependency of solutions of equations (4.17) and (4.18) on parameters.

Theorem 4.4. *Assume that the function f in* (4.17),(4.18) *and its delta derivative with respect to t satisfy the conditions*

$$|f(t, \tau, u, v, \mu) - f(t, \tau, \overline{u}, \overline{v}, \mu)| \le h_1(t, \tau)\left[|u - \overline{u}| + |v - \overline{v}|\right], \tag{4.19}$$
$$|f(t, \tau, u, v, \mu) - f(t, \tau, u, v, \mu_0)| \le e_1(t, \tau)\left|\mu - \mu_0\right|, \tag{4.20}$$
$$\left|f^{\Delta}(t, \tau, u, v, \mu) - f^{\Delta}(t, \tau, \overline{u}, \overline{v}, \mu)\right| \le h_2(t, \tau)\left[|u - \overline{u}| + |v - \overline{v}|\right], \tag{4.21}$$
$$\left|f^{\Delta}(t, \tau, u, v, \mu) - f^{\Delta}(t, \tau, u, v, \mu_0)\right| \le e_2(t, \tau)\left|\mu - \mu_0\right|, \tag{4.22}$$

where $h_1, h_2, e_1, e_2 \in \Omega(t, \tau)$. Let, for $i = 1, 2$, $h_i(t, \tau)$, $\overline{k}(t, \tau)$, $h_1(\sigma(t), \tau)$ and c be as in Theorem 4.2 *and*

$$\beta(t) = e_1(\sigma(t), t) + \int_{t_0}^{t} \left[e_1(t, \tau) + e_2(t, \tau)\right]\Delta\tau.$$

Let $z_1(t)$ and $z_2(t)$ be the solutions of equation (4.17) *and* (4.18) *respectively, then*

$$|z_1(t) - z_2(t)| + \left|z_1^{\Delta}(t) - z_2^{\Delta}(t)\right| \leq \frac{|\mu - \mu_0|}{1-c}\beta(t) + \int_{t_0}^{t} B_4(\tau) e_{A_1}(t, \sigma(\tau))\Delta\tau, \quad (4.23)$$

where $A_1(t)$ is as in Theorem 4.1 *and $B_4(t)$ is defined by right hand side of* (2.6) *replacing $a(t)$ by $\frac{|\mu-\mu_0|}{1-c}\beta(t)$.*

Proof. Let $w(t) = z_1(t) - z_2(t)$, where $z_1(t)$ and $z_2(t)$ are respectively solutions of equations (4.17) and (4.18), then

$$\begin{aligned}
|w(t)| &\leq \int_{t_0}^{t} \left|f\left(t, \tau, z_1(\tau), z_1^{\Delta}(\tau), \mu\right) - f\left(t, \tau, z_2(t), z_2^{\Delta}(t), \mu\right)\right|\Delta\tau \\
&+ \int_{t_0}^{t} \left|f\left(t, \tau, z_2(\tau), z_2^{\Delta}(\tau), \mu\right) - f\left(t, \tau, z_2(\tau), z_2^{\Delta}(\tau), \mu_0\right)\right|\Delta\tau \\
&\leq \int_{t_0}^{t} h_1(t, \tau)\left[|w(\tau)| + \left|w^{\Delta}(\tau)\right|\right]\Delta\tau + \int_{t_0}^{t} e_1(t, \tau)\,|\mu - \mu_0|\Delta\tau, \qquad (4.24)
\end{aligned}$$

and

$$\begin{aligned}
\left|w^{\Delta}(t)\right| &\leq \left|f\left(\sigma(t), t, z_1(t), z_1^{\Delta}(t), \mu\right) - f\left(\sigma(t), t, z_2(t), z_2^{\Delta}(t), \mu\right)\right| \\
&+ \left|f\left(\sigma(t), t, z_2(t), z_2^{\Delta}(t), \mu\right) - f\left(\sigma(t), t, z_2(t), z_2^{\Delta}(t), \mu_0\right)\right| \\
&+ \int_{t_0}^{t} \left|f^{\Delta}\left(t, \tau, z_1(\tau), z_1^{\Delta}(\tau), \mu\right) - f^{\Delta}\left(t, \tau, z_2(\tau), z_2^{\Delta}(\tau), \mu\right)\right|\Delta\tau \\
&+ \int_{t_0}^{t} \left|f^{\Delta}\left(t, \tau, z_2(\tau), z_2^{\Delta}(\tau), \mu\right) - f^{\Delta}\left(t, \tau, z_2(\tau), z_2^{\Delta}(\tau), \mu_0\right)\right|\Delta\tau \\
&\leq h_1(\sigma(t), t)\left[|w(t)| + \left|w^{\Delta}(t)\right|\right] + e_1(\sigma(t), t)\,|\mu - \mu_0| \\
&+ \int_{t_0}^{t} h_2(t, \tau)\left[|w(\tau)| + \left|w^{\Delta}(\tau)\right|\right]\Delta\tau + \int_{t_0}^{t} e_2(t, \tau)\,|\mu - \mu_0|\Delta\tau. \qquad (4.25)
\end{aligned}$$

From (4.24), (4.25) and using the assumption $h_1(\sigma(t), t) \leq c$, it is easy to observe that

$$|w(t)| + \left|w^{\Delta}(t)\right| \leq \frac{|\mu - \mu_0|}{1-c}\beta(t) + \frac{1}{1-c}\int_{t_0}^{t} \overline{k}(t, \tau)\left[|w(\tau)| + \left|w^{\Delta}(\tau)\right|\right]\Delta\tau. \quad (4.26)$$

Now an application of Lemma 2.2 to (4.26) yields (4.23), which shows the dependency of solutions of (4.17), (4.18) on parameters. □

Remark 3. Recently, in [6,7] the authors have studied some basic qualitative properties of solutions of dynamic equations on time scales by using Banach and Schafer fixed point theorems. Indeed, a particular feature of our approach is that it present conditions under which we can offer simple, unified and concise proofs of some of the important qualitative properties of solutions of equation (1.1), about which we believe almost nothing seems to be known.

References

[1] A. M. Bica, V. A. Caus and S. E. Muresan, Application of a trapezoid inequality to neutral Fredholm integro-differential equations in Banach Spaces, *J. Inequal. Pure. Appl. Math.*, 7(5)(2006), Art 173.

[2] M. Bohner and A. Peterson, Dynamic equations on time scales, *Birkhauser Boston/Berlin*, 2001.

[3] M. Bohner and A. Peterson, Advances in dynamic equations on time scales, *Birkhauser Boston/Berlin*, 2003.

[4] E. A. Bohner, M. Bohner and F. Akin, Pachpatte inequalities on time scales, *J. Inequal. Pure. Appl. Math.*, 6(1)(2005), Art 6.

[5] S. Hilger, Analysis on measure chains - A unified approch to continuous and discrete calculus, *Results Math*, 18:18–56, 1990.

[6] T. Kulik and C. C. Tisdell, Volterra integral equations on time scales: Basic qualitative and quantitative results with applications to initial value problems on unbounded domains, *Int. J. Difference Equ.*, Vol. 3 No. 1 (2008), pp. 103-133.

[7] C. C. Tisdell and A. Zaidi, Basic qualitative and quantitative results for solutions to nonlinear, dynamic equations on time scales with an application to economic modelling, *Nonlinear Anal.*, 68 (2008), No. 11, 3504–3524.

[8] D. B. Pachpatte, Properties of solutions to nonlinear dynamic integral equations on time scales, *Electronic Journal of Differential Equations*, Vol. 2008(2008), No. 136, pp. 1–8.

[9] D. B. Pachpatte, Explicit estimates on integral inequalities with time scale, *J.Inequal. Pure and Appl.Math*, 6(1)(2005), Art 143.

[10] B. G. Pachpatte, On Volterra and Fredholm type integrodifferential equations, *Tamsui Oxford Journal of Mathematical Sciences*, Vol. 24, No. 3, Aug 2008.

[11] B. G. Pachpatte, On certain Volterra-Fredholm type integral equations, *J.Inequal. Pure and Appl.Math*, Vol. 9(2008), Art 116.

[12] B. G. Pachpatte, On Fredholm type integrodifferential equations, *Tamkang Journal of Mathematics*, Vol. 39, No. 1, 85-94, Spring 2008.

(Received August 15, 2009)

Deepak B. Pachpatte
Department of Mathematics, Dr. B.A.M. University, Aurangabad, Maharashtra 431004, India
E-mail address: pachpatte@gmail.com