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Existence and Uniqueness of Solution for Fractional Differential Equations with Integral Boundary Conditions [1]

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Abstract: This paper is devoted to the existence and uniqueness results of solutions for fractional differential equations with integral boundary conditions.

$$\begin{cases} {}^{C}D^{\alpha}x(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0, 1), \\ x(0) = \int_0^1 g_0(s, x(s))\mathrm{d}s, \\ x(1) = \int_0^1 g_1(s, x(s))\mathrm{d}s, \\ x^{(k)}(0) = 0, \;\; k = 2, 3, \cdots, [\alpha] - 1. \end{cases}$$

By means of the Banach contraction mapping principle, some new results on the existence and uniqueness are obtained. It is interesting to note that the sufficient conditions for the existence and uniqueness of solutions are dependent on the order α.

Keywords: Caputo derivative; fractional differential equations; integral boundary conditions; Banach contraction mapping principle; existence and uniqueness.

MSC: 34B15, 26A33.

1 Introduction

In this paper, we study the existence and uniqueness of solutions for the fractional differential equation with nonlocal boundary conditions.

$$\begin{cases} {}^{C}D^{\alpha}x(t) + f(t, x(t), x'(t)) = 0, \quad t \in (0, 1), \\ x(0) = \int_0^1 g_0(s, x(s))\mathrm{d}s, \\ x(1) = \int_0^1 g_1(s, x(s))\mathrm{d}s, \\ x^{(k)}(0) = 0, \;\; k = 2, 3, \cdots, [\alpha] - 1, \end{cases} \tag{1.1}$$

where ${}^{C}D^{\alpha}$ is the standard Caputo derivative, and $1 < \alpha \in \mathbb{R}$. $f \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and g_0, g_1 are given functions.

There are many applications of fractional differential equations in the fields of various sciences such as physics, mechanics, chemistry, engineering, etc. As a result, fractional differential equations have been of great interest. For details, see [1]–[4] and references therein.

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Recently, there are some papers which deal with the existence of the solutions of the initial value problem or the linear boundary values problems for fractional differential equations. In [5]–[6], the basic theory for the initial value problem of fractional functional differential equations involving Riemann-Liouville differential operators is discussed. The general existence and uniqueness results are proved by means of monotone iterative technique and the method of upper and lower solutions, see [7]–[8].

In [9], by using some fixed-point theorems on cone, Bai investigates the existence and multiplicity of positive solutions for nonlinear fractional differential equation with linear boundary conditions

$$\begin{cases} D^\alpha u(t) + f(t, u(t)) = 0, \quad t \in (0,1), \\ u(0) = u(1) = 0, \end{cases}$$

where $1 < \alpha \leq 2$ is a real number, D^α is the standard Riemann- Liouville differentiation.

In [10], the authors study the nonlinear fractional differential equation with linear boundary conditions

$$\begin{cases} D^\alpha u(t) = f(t, u(t)), \quad t \in (0,1), \\ u(0) = u(1) = u'(0) = u'(1) = 0, \end{cases}$$

where $3 < \alpha \leq 4$ is a real number, and D^α is the standard Riemann- Liouville differentiation. Some multiple positive solutions for singular and nonsingular boundary value problems are given.

In [11], the authors gave a unified approach for studying the existence of multiple positive solutions of nonlinear order differential equations of the form

$$u''(t) + g(t)f(t, u(t)) = 0, \quad t \in (0,1)$$

with integral boundary conditions of Riemann-Stieltjes type.

However, no contributions, as far as we know, on the researches for the existence and uniqueness of solutions for the fractional differential equations with integral boundary conditions have been discovered.

In this paper, we focus on the existence and uniqueness results for fractional differential equations with integral boundary conditions. By means of the famous Banach contraction mapping principle, we obtain some new results on the existence and uniqueness of the solutions. It is interesting to note that the sufficient conditions for the existence and uniqueness of solutions are dependent on the order α.

2 Preliminaries

For the sake of clarity, we list the necessary definitions from fractional calculus theory here. These definitions can be found in the recent literature.

Definition 2.1[4] Let $\alpha > 0$, for a function $y : (0, +\infty) \to \mathbb{R}$. The the fractional integral of order α of y is defined by

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s,$$

provided the integral exists. The Caputo derivative of a function $y : (0, +\infty) \to R$ is given by

$${}^{C}D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{y^{(n)}(t)}{(t-s)^{\alpha+1-n}}\mathrm{d}s,$$

provided the right side is pointwise defined on $(0, +\infty)$, where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of the real number α. Γ denotes the Gamma function:

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-t}t^{\alpha-1}dt.$$

The Gamma function satisfies the following basic properties:
(1) For any $z \in \mathbb{R}$

$$\Gamma(z+1) = z\Gamma(z);$$

(2)For any $1 < \alpha \in \mathbb{R}$, then

$$\frac{\alpha+1}{\Gamma(\alpha+1)} = \frac{\alpha+1}{\alpha\Gamma(\alpha)} < \frac{2}{\Gamma(\alpha)}. \tag{2.1}$$

From Definition 2.1, we can obtain the following lemma.

Lemma 2.1 *Let $0 < n-1 < \alpha < n$. If we assume $y \in C^n(0,1) \cap L[0,1]$, the fractional differential equation*

$${}^{C}D^{\alpha}y(t) = 0$$

has a unique solution

$$y(t) = \sum_{k=0}^{n-2} \frac{y^{(k)}(0)}{k!}t^k.$$

Lemma 2.2 *The function $x \in C^n[0,1]$ is a solution of boundary value problem (1.1), if and only if $x \in C[0,1]$ is a solution of the following fractional integral*

$$\begin{aligned} x(t) = &\int_0^1 \big(tg_1(s,x(s)) + (1-t)g_0(s,x(s))\big)\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f(s,x(s),x'(s))\mathrm{d}s \\ &+ \frac{t}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,x(s),x'(s))\mathrm{d}s. \end{aligned} \tag{2.2}$$

That is, every solution of (1.1) is also a solution of (2.2) and vice versa.

Proof By ${}^{C}D^{\alpha}x(t) + f(t, x(t), x'(t)) = 0,\ t \in (0,1)$ and the boundary conditions $x''(0) = x'''(0) = \cdots = x^{(n-2)}(0) = 0$, we have

$$\begin{aligned} x(t) &= -I^{\alpha} f(t, x(t), x'(t)) + x(0) + x'(0)t + \frac{x''(0)}{2!}t^2 + \cdots + \frac{x^{(n-2)}(0)}{(n-2)!}t^{n-2} \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s)) \mathrm{d}s + x(0) + x'(0)t. \end{aligned} \tag{2.3}$$

Then

$$x(1) = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), x'(s)) \mathrm{d}s + x(0) + x'(0).$$

By the boundary value conditions

$$\begin{cases} x(0) = \int_0^1 g_0(s, x(s)) \mathrm{d}s, \\ x(1) = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), x'(s)) \mathrm{d}s + x(0) + x'(0) = \int_0^1 g_1(s, x(s)) \mathrm{d}s, \end{cases}$$

we have

$$\begin{cases} x(0) = \int_0^1 g_0(s, x(s)) \mathrm{d}s, \\ x'(0) = \int_0^1 \big(g_1(s, x(s)) - g_0(s, x(s))\big) \mathrm{d}s + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), x'(s)) \mathrm{d}s. \end{cases}$$

Thus,

$$\begin{aligned} x(t) = &\int_0^1 \big(t g_1(s, x(s)) + (1-t) g_0(s, x(s))\big) \mathrm{d}s - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s), x'(s)) \mathrm{d}s \\ &+ \frac{t}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s), x'(s)) \mathrm{d}s. \end{aligned}$$

Therefore, the proof is completed.

□

3 Main results

Theorem 3.1 *We suppose that*

(H**$_1$**) *The functions $g_0, g_1 \in C([0,1] \times \mathbb{R}, \mathbb{R})$, there exist $m_0,\ m_1 \in C([0,1],\ [0, +\infty))$ and a constant $0 < \rho < 1$, such that*

$$\big|g_0(s,x) - g_0(s,y)\big| \le m_0(s)\big|x-y\big|,\ \ \big|g_1(s,x) - g_1(s,y)\big| \le m_1(s)\big|x-y\big|,\ \ for\ \ s \in [0,1],\ \ x,\, y \in \mathbb{R},$$

and

$$0 \le M_1 := \max\Big\{ \int_0^1 m_0(s) \mathrm{d}s,\ \int_0^1 m_1(s) \mathrm{d}s \Big\} < \rho,\ \ M_0 := \int_0^1 \big(m_0(s) + m_1(s)\big) \mathrm{d}s.$$

(H**$_2$**) *$f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exist constants $0 < k_2 < \frac{\rho - M_1}{M_0} k_1$, and $r_1, r_2 \ge 0$ with*

$$r_1 \le \frac{\big(k_1(\rho - M_1) - k_2 M_0\big)\Gamma(\alpha+1)}{k_1 + 2\alpha k_2}, \quad r_2 \le \frac{\rho k_2 \Gamma(\alpha+1)}{k_1 + 2k_2}$$

such that

$$\big|f(t,u_1,v_1)-f(t,u_2,v_2)\big|\le r_1\big|u_1-u_2\big|+r_2\big|v_1-v_2\big|,\quad for\ \ t\in[0,1],\ \ u_1,u_2,\ \ v_1,v_2\in\mathbb{R}.$$

Then the boundary value problem (1.1) has a unique solution.

Proof. Let $E=C^1[0,1]$ with the norm

$$||x||:=k_1\big|x(t)\big|_\infty+k_2\big|x'(t)\big|_\infty,\ \text{ where }\big|x\big|_\infty=\max_{t\in[0,1]}\big|x(t)\big|,\ \text{ and }\big|x'\big|_\infty=\max_{t\in[0,1]}\big|x'(t)\big|.$$

Then $(E,\ ||\cdot||)$ is a Banach space.

Consider the operator $F:\ E\to E$ defined by

$$\begin{aligned}(Fx)(t):=&\int_0^1\big(tg_1(s,x(s))+(1-t)g_0(s,x(s))\big)\mathrm{d}s-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}f(s,x(s),x'(s))\mathrm{d}s\\&+\frac{t}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,x(s),x'(s))\mathrm{d}s.\end{aligned}$$

It is easy to see that x is the solution of the boundary value problem (1.1) if and only if x is the fixed point of F. The mapping $F:E\to E$ is a continuous and compact operator on E.

In the following, we prove that F has a unique fixed point in E.

First of all, for any $x,y\in E$, we can get that

$$\begin{aligned}\big|(Fx)(t)-(Fy)(t)\big|=&\ \Big|\int_0^1\Big(t(g_1(s,x(s))-g_1(s,y(s)))+(1-t)(g_0(s,x(s))-g_0(s,y(s))\Big)\mathrm{d}s\\&-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\big(f(s,x(s),x'(s))-f(s,y(s),y'(s))\big)\mathrm{d}s\\&+\frac{t}{\Gamma(\alpha)}\int_0^t(1-s)^{\alpha-1}\big(f(s,x(s),x'(s))-f(s,y(s),y'(s))\big)\mathrm{d}s\\&+\frac{t}{\Gamma(\alpha)}\int_t^1(1-s)^{\alpha-1}\big(f(s,x(s),x'(s))-f(s,y(s),y'(s))\big)\mathrm{d}s\Big|\\\le&\int_0^1\Big(t\big|g_1(s,x(s))-g_1(s,y(s))\big|+(1-t)\big|g_0(s,x(s))-g_0(s,y(s))\big|\Big)\mathrm{d}s\\&+\frac{1}{\Gamma(\alpha)}\int_0^t\big|t(1-s)^{\alpha-1}-(t-s)^{\alpha-1}\big|\big|f(s,x(s),x'(s))-f(s,y(s),y'(s))\big|\mathrm{d}s\\&+\frac{t}{\Gamma(\alpha)}\int_t^1(1-s)^{\alpha-1}\big|f(s,x(s),x'(s))-f(s,y(s),y'(s))\big|\mathrm{d}s.\end{aligned}$$

We notice that for $t\in[0,1]$ and $s\le t$,

$$-(1-s)^{\alpha-1}\le-(t-s)^{\alpha-1}\le t(1-s)^{\alpha-1}-(t-s)^{\alpha-1}\le(1-s)^{\alpha-1},$$

that is

$$|t(1-s)^{\alpha-1}-(t-s)^{\alpha-1}|\le(1-s)^{\alpha-1}.$$

Hence, we have

$$\begin{aligned}\big|(Fx)(t)-(Fy)(t)\big| \le &\int_0^1 \Big(t\big|g_1(s,x(s))-g_1(s,y(s))\big|+(1-t)\big|g_0(s,x(s))-g_0(s,y(s))\big|\Big)\mathrm{d}s\\ &+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\big|f(s,x(s),x'(s))-f(s,y(s),y'(s))\big|\mathrm{d}s\\ \le &\Big(t\int_0^1 m_1(s)\mathrm{d}s+(1-t)\int_0^1 m_0(s)\mathrm{d}s\Big)\big|x-y\big|_\infty\\ &+\frac{1}{\Gamma(\alpha)}\Big(r_1\big|x-y\big|_\infty+r_2\big|x'-y'\big|_\infty\Big)\int_0^1(1-s)^{\alpha-1}\mathrm{d}s\\ \le &M_1\big|x-y\big|_\infty+\frac{1}{\alpha\Gamma(\alpha)}(r_1\big|x-y\big|_\infty+r_2\big|x'-y'\big|_\infty)\\ = &\Big(M_1+\frac{r_1}{\Gamma(\alpha+1)}\Big)\big|x-y\big|_\infty+\frac{r_2}{\Gamma(\alpha+1)}\big|x'-y'\big|_\infty.\end{aligned}$$

Then

$$\big|Fx-Fy\big|_\infty \le (M_1+\frac{r_1}{\Gamma(\alpha+1)})\big|x-y\big|_\infty+\frac{r_2}{\Gamma(\alpha+1)}\big|x'-y'\big|_\infty. \tag{3.1}$$

Also, we have

$$\begin{aligned}(Fx)'(t)=&\int_0^1\big(g_1(s,x(s))-g_0(s,x(s))\big)\mathrm{d}s-\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}f(s,x(s),x'(s))\mathrm{d}s\\ &+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}f(s,x(s),x'(s))\mathrm{d}s.\end{aligned}$$

Then, we have

$$\begin{aligned}\big|(Fx)'(t)-(Fy)'(t)\big| \le &\int_0^1\big(\big|g_1(s,x(s))-g_1(s,y(s))\big|+\big|g_0(s,x(s))-g_0(s,y(s))\big|\big)\mathrm{d}s\\ &+\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\big|f(s,x(s),x'(s))-f(s,y(s),y'(s))\big|\mathrm{d}s\\ &+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\big|f(s,x(s),x'(s))-f(s,y(s),y'(s))\big|\mathrm{d}s\\ \le &\int_0^1\big(m_1(s)+m_0(s)\big)\mathrm{d}s\big|x-y\big|_\infty\\ &+\Big(\frac{1}{\Gamma(\alpha-1)}\int_0^t(t-s)^{\alpha-2}\mathrm{d}s+\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}\mathrm{d}s\Big)(r_1\big|x-y\big|_\infty+r_2\big|x'-y'\big|_\infty)\\ \le &M_0\big|x-y\big|_\infty+\Big(\frac{1}{(\alpha-1)\Gamma(\alpha-1)}+\frac{1}{\alpha\Gamma(\alpha)}\Big)(r_1\big|x-y\big|_\infty+r_2\big|x'-y'\big|_\infty)\\ = &\Big(M_0+\frac{r_1(\alpha+1)}{\Gamma(\alpha+1)}\Big)\big|x-y\big|_\infty+\frac{r_2(\alpha+1)}{\Gamma(\alpha+1)}\big|x'-y'\big|_\infty.\end{aligned}$$

By (2.1), we have

$$\big|(Fx)'(t)-(Fy)'(t)\big| \le \Big(M_0+\frac{2r_1}{\Gamma(\alpha)}\Big)\big|x-y\big|_\infty+\frac{2r_2}{\Gamma(\alpha)}\big|x'-y'\big|_\infty.$$

So

$$\big|(Fx)'-(Fy)'\big|_\infty \le (M_0+\frac{2r_1}{\Gamma(\alpha)})\big|x-y\big|_\infty+\frac{2r_2}{\Gamma(\alpha)}\big|x'-y'\big|_\infty. \tag{3.2}$$

Therefore, by (3.1) and (3.2), we can obtain that

$$\begin{aligned} ||Fx - Fy|| &= k_1 \big|Fx - Fy\big|_\infty + k_2 \big|(Fx)' - (Fy)'\big|_\infty \\ &\leq \Big(k_1(M_1 + \frac{r_1}{\alpha\Gamma(\alpha)}) + k_2(M_0 + \frac{2r_1}{\Gamma(\alpha)})\Big)\big|x - y\big|_\infty + \Big(\frac{k_1 r_2}{\Gamma(\alpha+1)} + \frac{2k_2 r_2}{\Gamma(\alpha+1)}\Big)\big|x' - y'\big|_\infty \\ &\leq \rho\big(k_1\big|x - y\big|_\infty + k_2\big|x' - y'\big|_\infty\big) \\ &= \rho||x - y||. \end{aligned}$$

Then

$$||Fx - Fy|| \leq \rho||x - y||,$$

which implies that F is a contraction mapping.

By means of the Banach contraction mapping principle, F has a unique fixed point which is a unique solution of the boundary value problems (1.1).

□

In the following, we establish sufficient conditions for the existence and uniqueness of positive solutions for the boundary value problems.

Theorem 3.2 *Suppose that the conditions* (H_1) *and* (H_2) *in Theorem 3.1 are satisfied. Moreover,*

$$f(t, u, v) \geq 0,\ for\ (t, u, v) \in [0, 1] \times [0, +\infty) \times \mathbb{R}$$

and

$$g_0(s, x),\ \ g_1(s, x) \geq 0\ \ for\ \ (s, x) \in [0, 1] \times [0, +\infty).$$

Then the boundary value problem (1.1) has a unique positive solution.

Proof. Since the conditions (H_1) and (H_2) in Theorem 3.1 are satisfied, by Theorem 3.1, the boundary value problem (1.1) has a unique solution, which we denote x. And

$$\begin{aligned} x(t) := &\int_0^1 \big(tg_1(s, x(s)) + (1 - t)g_0(s, x(s))\big)\mathrm{d}s - \frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1} f(s, x(s), x'(s))\mathrm{d}s \\ &+ \frac{t}{\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1} f(s, x(s), x'(s))\mathrm{d}s. \end{aligned}$$

We denote

$$q(t) := -\frac{1}{\Gamma(\alpha)}\int_0^t (t - s)^{\alpha-1} f(s, x(s), x'(s))\mathrm{d}s + \frac{t}{\Gamma(\alpha)}\int_0^1 (1 - s)^{\alpha-1} f(s, x(s), x'(s))\mathrm{d}s,$$

then

$$x(t) = \int_0^1 \big(tg_1(s, x(s)) + (1 - t)g_0(s, x(s))\big)\mathrm{d}s + q(t).$$

Since g_0, g_1 are nonnegative, then the first term

$$\int_0^1 \big(tg_1(s,x(s)) + (1-t)g_0(s,x(s))\big)\mathrm{d}s \geq 0, \ \text{ for } \ x \in P.$$

In order to determine the sign of $q(t)$, we have the following two cases to be discussed.

1) For $\alpha > 2$, obviously $q(0) = q(1) = 0$, and

$$q'(t) = -\frac{1}{\Gamma(\alpha-1)}\int_0^t (t-s)^{\alpha-2} f(s,x(s),x'(s))\mathrm{d}s + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1} f(s,x(s),x'(s))\mathrm{d}s;$$

$$q''(t) = -\frac{1}{\Gamma(\alpha-2)}\int_0^t (t-s)^{\alpha-3} f(s,x(s),x'(s))\mathrm{d}s \leq 0, \ \ t \in [0,1].$$

Then we get that

$$q(t) \geq 0, \ \ t \in [0,1].$$

Therefore,

$$x(t) \geq 0, \ \ t \in [0,1].$$

2) For $1 < \alpha \leq 2$, we have

$$q(t) = \frac{1}{\Gamma(\alpha)}\int_0^1 G(t,s) f(s,x(s),x'(s))\mathrm{d}s,$$

where

$$G(t,s) = \begin{cases} t(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s \leq t \leq 1, \\ t(1-s)^{\alpha-1}, & 0 \leq t < s \leq 1. \end{cases}$$

Then $q(t) \geq 0$ if $G(t,s) \geq 0, \ \ (t,s) \in [0,1] \times [0,1]$.

First, it is easy to see $G(t,s) = t(1-s)^{\alpha-1} \geq 0$, for $(t,s) \in D_1 = \big\{(t,s)\big|0 \leq t < s \leq 1\big\}$.

Second, we consider the case that $(t,s) \in D_2 = \big\{(t,s)\big|0 \leq s \leq t \leq 1\big\}$.

Because $G(t,s) \in C(D_2)$, then $G(t,s)$ has a minimum in D_2, i.e. there exists $(t_0,s_0) \in D_2$ with $G(t_0,s_0) = \min_{(t,s)\in D_2} G(t,s)$. But by using calculus methods, we conclude that $G(t,s)$ does not have a minimum in $\big\{(t,s)\big|0 < s < t < 1\big\}$. So

$$(t_0,s_0) \in \big\{(t,s)\big|0 \leq s \leq 1, \ \ t = 1)\big\} \cup \big\{(t,s)\big|s = 0, \ \ 0 \leq t \leq 1)\big\} \cup \big\{(t,s)\big|0 \leq t \leq 1, \ \ s = t\big\}.$$

For $0 \leq s \leq 1, \ \ t = 1$, $G(t,s) = G(1,s) = 0$, for $s = 0, \ \ 0 \leq t \leq 1$, $G(t,s) = G(t,0) = t - t^{\alpha-1} \geq 0$ and for $0 \leq t \leq 1, \ \ s = t$, $G(t,s) = G(t,t) = t(1-t^{\alpha-1}) \geq 0$. So

$$G(t,s) \geq 0, \ \ (t,s) \in D_2, \ \ \text{for } \ x \in P.$$

Since $f(t,x(t),x'(t)) \geq 0$ for $t \in [0,1]$, $x \in P$, we get

$$q(t) \geq 0, \ \ (t,s) \in D_2, \ \ \text{for } \ x \in P.$$

Then for $\alpha > 1$, we have

$$x(t) \geq 0, \ \ t \in [0,1],$$

which implies that the boundary value problem (1.1) has a unique positive solution.

$\square$

Remark According to the view of theorems, we see that the boundary value problem has only the trivial solution $x(t) \equiv 0$ for $t \in [0,1]$, if and only if $f(t,0,0) \equiv 0$, and $\int_0^1 g_0(s,0)\mathrm{d}s = \int_0^1 g_1(s,0)\mathrm{d}s = 0$.

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