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Existence, Uniqueness and Stability Results of Impulsive Stochastic Semilinear Neutral Functional Differential Equations with Infinite Delays

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Abstract

This article presents the results on existence, uniqueness and stability of mild solutions of impulsive stochastic semilinear neutral functional differential equations without a Lipschitz condition and with a Lipschitz condition. The results are obtained by using the method of successive approximations.
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1 Introduction

Neutral differential equations arise in many area of science and engineering and have received much attention in the last decades. The ordinary neutral differential equation is used extensively to study the theory of aeroelasticity [10] and lossless transmission lines (see [4] and the references therein). Partial neutral differential equations with delays are motivated from stabilization of lumped control systems and the theory of heat conduction in materials (see [7; 8] and the references therein). Hernandez and O'Regan [6] studied some partial neutral differential equations by assuming a temporal and spatial regularity type condition for the function $t \to g(t, x_t)$. In [15; 4], the authors studied several existence results of stochastic differential equations (SDEs) with unbounded delays.

Recently impulsive differential equations have been used to model problems (see[11; 19]). Considerable work in the field of fixed impulsive type equations may be found in [1; 7; 16] and the references therein. The study of impulsive stochastic differential equations (ISDEs) is a new area of research and few publications on that subject can be found. Jun Yang et al.[9], studied the stability analysis of ISDEs with delays. Zhigno Yang et al.[21], studied the exponential p- stability of ISDEs with delays. In [17; 18], R. Sakthivel and J. Luo studied

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the existence and asymptotic stability in p-th moment of mild solutions to ISDEs with and without infinite delays through fixed point theory. Motivated by [13; 14], we generalize the existence and uniqueness of the solution to impulsive stochastic partial neutral functional differential equations (ISNFDEs) under non-Lipschitz conditions and under Lipschitz conditions. Moreover, we study the stability through the continuous dependence on the initial values by means of a corollary of Bihari's inequality. Further, we refer [3; 5; 12; 20].

This paper is organized as follows. In Section 2, we recall briefly the notation, definitions, lemmas and preliminaries which are used throughout this paper. In Section 3, we study the existence and uniqueness of ISNFDEs by relaxing the linear growth conditions. In Section 4, we study stability through the continuous dependence on the initial values. Finally in Section 5, an example is given to illustrate our results.

2 Preliminaries

In this article, we will examine impulsive stochastic semilinear neutral functional differential equations of the form

$$\begin{aligned} d\big[x(t)+g(t,x_t)\big] &= \Big[A\big[x(t)+g(t,x_t)\big]+f(t,x_t)\Big]dt+a(t,x_t)dw(t),\ t\neq t_k,\ 0\leq t\leq T,\\ \Delta x(t_k) &= x(t_k^+)-x(t_k^-)=I_k(x(t_k)),\ t=t_k,\quad k=1,2,\dots m, \\ x(t) &= \varphi\in D^b_{B_0}((-\infty,0],X), \end{aligned} \tag{2.1}$$

where A is the infinitesimal generator of a strongly continuous semigroup of bounded linear operators $\{S(t),\ t\geq 0\}$ with $D(A)\subset X$.

Let X, Y be real separable Hilbert spaces and $L(Y,X)$ be the space of bounded linear operators mapping Y into X. For convenience, we shall use the same notations $\|.\|$ to denote the norms in X, Y and $L(Y,X)$ without any confusion. Let (Ω,B,P) be a complete probability space with an increasing right continuous family $\{B_t\}_{t\geq 0}$ of complete sub σ-algebra of B. Let $\{w(t):t\geq 0\}$ denote a Y-valued Wiener process defined on the probability space (Ω,B,P) with covariance operator Q, that is

$$E<w(t),x>_Y<w(s),y>_Y=(t\wedge s)<Qx,y>_Y,\quad \text{for all } x,y\in Y,$$

where Q is a positive, self-adjoint, trace class operator on Y. In particular, we denote by $w(t)$, $t\geq 0$, a Y- valued Q- Wiener process with respect to $\{B_t\}_{t\geq 0}$.

In order to define stochastic integrals with respect to the Q- Wiener process $w(t)$, we introduce the subspace $Y_0=Q^{1/2}(Y)$ of Y which, endowed with the inner product $<u,v>_{Y_0}=<Q^{-1/2}u,Q^{-1/2}v>_Y$ is a Hilbert space. We assume that there exists a complete orthonormal system $\{e_i\}_{i\geq 1}$ in Y, a bounded sequence of nonnegative real numbers

λ_i such that $Qe_i = \lambda_i e_i, i = 1, 2, \dots,$ and a sequence $\{\beta_i\}_{i\geq 1}$ of independent Brownian motions such that

$$< w(t), e >= \sum_{n=1}^{\infty} \sqrt{\lambda_i} < e_i, e > \beta_i(t), \quad e \in Y,$$

and $B_t = B_t^w$, where B_t^w is the sigma algebra generated by $\{w(s) : 0 \leq s \leq t\}$. Let $L_2^0 = L_2(Y_0, X)$ denote the space of all Hilbert- Schmidt operators from Y_0 into X. It turns out to be a separable Hilbert space equipped with the norm $\|\mu\|^2_{L_2^0} = tr((\mu Q^{1/2})(\mu Q^{1/2})^*)$ for any $\mu \in L_2^0$. Clearly for any bounded operators $\mu \in L(Y, X)$ this norm reduces to $\|\mu\|^2_{L_2^0} = tr(\mu Q \mu^*)$.

We now make the system (2.1) precise: Let $A : X \to X$ be the infinitesimal generator of a strongly continuous semigroup $\{S(t),\ t \geq 0\}$ defined on X. Let $\Re^+ = [0, \infty)$ and let the functions $f : \Re^+ \times \hat{D} \to X$; $a : \Re^+ \times \hat{D} \to L(Y, X)$ be Borel measurable and let $g : \Re^+ \times \hat{D} \to X$ be continuous. Here $\hat{D} = D((-\infty, 0], X)$ denotes the family of all right piecewise continuous functions with left-hand limit φ from $(-\infty, 0]$ to X. The phase space $D((-\infty, 0], X)$ is assumed to be equipped with the norm $\|\varphi\|_t = \sup_{-\infty<\theta\leq 0} |\varphi(\theta)|$. We also assume that $D^b_{B_0}((-\infty, 0], X)$ denotes the family of all almost surely bounded, B_0-measurable, $\hat{D}$- valued random variables. Further, let $\mathcal{B}_T$ be a Banach space of all B_t-adapted processes $\varphi(t, w)$ which are almost surely continuous in t for fixed $w \in \Omega$ with norm defined for any $\varphi \in \mathcal{B}_T$ by

$$\|\varphi\|_{\mathcal{B}_T} = (\sup_{0\leq t\leq T} E\|\varphi\|_t^2)^{1/2}.$$

Furthermore, the fixed moments of time t_k satisfy $0 < t_1 < \dots < t_m < T$, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$, respectively. And $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$, represents the jump in the state x at time t_k with I_k determining the size of the jump.

Lemma 2.1.[2] *Let $T > 0$, $u_0 \geq 0$, and let $u(t), v(t)$ be continuous functions on $[0, T]$. Let $K : \Re^+ \to \Re^+$ be a concave continuous and nondecreasing function such that $K(r) > 0$ for $r > 0$. If*

$$u(t) \leq u_0 + \int_0^t v(s)K(u(s))ds \textit{ for all } \quad 0 \leq t \leq T,$$

then

$$u(t) \leq G^{-1}\Big(G(u_0) + \int_0^t v(s)ds\Big) \quad \textit{for all such } t \in [0, T] \textit{ that}$$
$$G(u_0) + \int_0^t v(s)ds \in Dom(G^{-1}),$$

where $G(r) = \int_1^r \frac{ds}{K(s)}$, $r \geq 0$ and G^{-1} is the inverse function of G. In particular, if $u_0 = 0$ and $\int_{0^+} \frac{ds}{K(s)} = \infty$, then $u(t) = 0$ for all $0 \leq t \leq T$.

In order to obtain the stability of solutions, we use the following extended Bihari's inequality

Lemma 2.2.[13] *Let the assumptions of Lemma* 2.1 *hold. If*

$$u(t) \le u_0 + \int_t^T v(s)K(u(s))ds \text{ for all } 0 \le t \le T,$$

then

$$u(t) \le G^{-1}\Big(G(u_0) + \int_t^T v(s)ds\Big) \quad \text{for all such } t \in [0,T] \text{ that}$$
$$G(u_0) + \int_t^T v(s)ds \in Dom(G^{-1}),$$

where $G(r) = \int_1^r \frac{ds}{K(s)}$, $r \ge 0$ *and* G^{-1} *is the inverse function of* G.

Corollary 2.3.[13] *Let the assumptions of Lemma* 2.1 *hold and* $v(t) \ge 0$ *for* $t \in [0,T]$. *If for all* $\epsilon > 0$, *there exists* $t_1 \ge 0$ *such that for* $0 \le u_0 < \epsilon$, $\int_{t_1}^T v(s)ds \le \int_{u_0}^{\epsilon} \frac{ds}{K(s)}$ *holds. Then for every* $t \in [t_1, T]$, *the estimate* $u(t) \le \epsilon$ *holds.*

Lemma 2.4.[3] *For any* $r \ge 1$ *and for arbitrary* L_2^0*- valued predictable process* $\Phi(\cdot)$

$$\sup_{s\in[0,t]} E\|\int_0^s \Phi(u)dw(u)\|_X^{2r} = (r(2r-1))^r\Big(\int_0^t (E\|\Phi(s)\|_{L_2^0}^{2r})ds\Big)^r.$$

Definition 2.1. *A semigroup* $\{S(t),\ t \ge 0\}$ *is said to be uniformly bounded if* $\|S(t)\| \le M$ *for all* $t \ge 0$, *where* $M \ge 1$ *is some constant. If* $M < 1$, *then the semigroup is said to be a contraction semigroup.*

Definition 2.2. *A stochastic process* $\{x(t), t \in (-\infty, T]\}, (0 < T < \infty)$ *is called a mild solution of the equation* (2.1) *if*
(i) $x(t)$ *is* B_t*- adapted;*
(ii) $x(t)$ *satisfies the integral equation*

$$x(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0], \\ S(t)\big[\varphi(0) + g(0,\varphi)\big] - g(t,x_t) + \int_0^t S(t-s)f(s,x_s)ds & \\ + \int_0^t S(t-s)a(s,x_s)dw(s) + \sum\limits_{0<t_k<t} S(t-t_k)I_k(x(t_k)), & a.s\ t \in [0,T]. \end{cases} \tag{2.2}$$

3 Existence and uniqueness

In this section, we discuss the existence and uniqueness of mild solutions of the system (2.1). We use the following hypotheses to prove our results.

Hypotheses:

(H_1) : A is the infinitesimal generator of a strongly continuous semigroup $S(t)$, whose domain $D(A)$ is dense in X.

(H_2) : For each $x, y \in \hat{D}$ and for all $t \in [0,T]$, such that,

$$\|f(t,x_t) - f(t,y_t)\|^2 \vee \|a(t,x_t) - a(t,y_t)\|^2 \leq K(\|x-y\|_t^2),$$

where $K(\cdot)$ is a concave non-decreasing function from $\Re^+$ to $\Re^+$, such that $K(0) = 0$, $K(u) > 0$, for $u > 0$ and $\int_{0^+} \frac{du}{K(u)} = \infty$.

(H_3) : Assuming that there exists a positive number L_g such that $L_g < \frac{1}{10}$, for any $x, y \in \hat{D}$ and for $t \in [0,T]$, we have

$$\|g(t,x_t) - g(t,y_t)\|^2 \leq L_g \ \|x-y\|_t^2,$$

(H_4) : The function $I_k \in C(X,X)$ and there exists some constant h_k such that

$$\|I_k(x(t_k)) - I_k(y(t_k))\|^2 \quad \leq \quad h_k \,\|x-y\|_t^2\,, \quad \text{for each } x, y \in \hat{D},\ k = 1, 2\ldots, m.$$

(H_5) : For all $t \in [0,T]$, it follows that $f(t,0), g(t,0), a(t,0), I_k(0) \in L^2$, for $k = 1, 2\ldots, m$ such that

$$\|f(t,0)\|^2 \vee \|g(t,0)\|^2 \vee \|a(t,0)\|^2 \vee \|I_k(0)\|^2 \leq \kappa_0,$$

where $\kappa_0 > 0$ is a constant.

Let us now introduce the successive approximations to equation (2.2) as follows

$$x^n(t) = \begin{cases} \varphi(t), & t \in (-\infty, 0],\ \text{for } n = 0, 1, 2, \ldots, \\[2ex] S(t)\big[\varphi(0) + g(0,\varphi)\big] - g(t, x_t^n) + \displaystyle\int_0^t S(t-s) f(s, x_s^{n-1}) ds \\ + \int_0^t S(t-s) a(s, x_s^{n-1}) dw(s) + \displaystyle\sum_{0<t_k<t} S(t-t_k) I_k(x^{n-1}(t_k)), \\ \qquad\qquad\qquad\qquad a.s \quad t \in [0,T],\ \text{for } n = 1, 2, \ldots. \end{cases} \tag{3.1}$$

$$x^0(t) \quad = \quad S(t)\varphi(0), \qquad\qquad t \in [0,T], \quad \text{for } n = 0, \tag{3.2}$$

with an arbitrary non-negative initial approximation $x^0 \in \mathcal{B}_{\mathcal{T}}$.

Theorem 3.1. *Let the assumptions $(H_1) - (H_5)$ hold, then the system (2.1) has unique mild solution $x(t)$ in $\mathcal{B}_{\mathcal{T}}$ and*

$$E\{\sup_{0\leq t\leq T} \|x^n(t) - x(t)\|^2\} \to 0 \quad \text{as } n \to \infty$$

where $\{x^n(t)\}_{n\geq 1}$ are the successive approximations (3.1).

Proof : Let $x^0 \in \mathcal{B}_\mathcal{T}$ be a fixed initial approximation to (3.1). To begin with under assumptions (H_1) - (H_5), $Q_i > 0, i = 1, \ldots, 7$, are some constants, we observe that $\|S(t)\| \leq M$ for some $M \geq 1$ and all $t \in [0, T]$. Then for any $n \geq 1$, we have,

$$\begin{aligned}
E\left\|x^n(t)\right\|^2 \leq\ & 5M^2 E\left\|\varphi(0) + g(0, \varphi)\right\|^2 \\
& + 10E\big[\|g(t, x_t^n) - g(t, 0)\|^2 + \|g(t, 0)\|^2\big] \\
& + 10TM^2 E\int_0^t \big[\|f(s, x_s^{n-1}) - f(s, 0)\|^2 + \|f(s, 0)\|^2\big]ds \\
& + 10M^2 E\int_0^t \big[\|a(s, x_s^{n-1}) - a(s, 0)\|^2 + \|a(s, 0)\|^2\big]ds \\
& + 10M^2 mE\sum_{k=1}^m \big[\|I_k(x^{n-1}(t_k)) - I_k(0)\|^2 + \|I_k(0)\|^2\big].
\end{aligned}$$

Thus,

$$\begin{aligned}
E\left\|x^n\right\|_t^2 \leq\ & \frac{Q_1}{1 - 10L_g} + \frac{10M^2(T+1)}{1 - 10L_g} E\int_0^t K(\|x^{n-1}\|_s^2)ds \\
& + \frac{10M^2 m\sum_{k=1}^m h_k}{1 - 10L_g}\Big\{E\|x^{n-1}\|_t^2\Big\},
\end{aligned}$$

where, $Q_1 = 10M^2\big(E\|\varphi(0)\|^2 + L_g E\|\varphi\|_0^2\big) + 10\big(1 + M^2T(T+1) + M^2 m\sum_{k=1}^m h_k\big)\kappa_0$.

Given that $K(\cdot)$ is concave and $K(0) = 0$, we can find a pair of positive constants a and b such that

$$K(u) \leq a + bu, \text{ for all } \ u \geq 0.$$

Then we have,

$$\begin{aligned}
E\left\|x^n\right\|_t^2 \leq\ & Q_2 + \frac{10M^2(T+1)b}{1 - 10L_g}\int_0^t E\|x^{n-1}\|_s^2 ds \\
& + \frac{10M^2 m\sum_{k=1}^m h_k}{1 - 10L_g}\{E\left\|x^{n-1}\right\|_t^2\}, \quad n = 1, 2, \ldots
\end{aligned} \tag{3.3}$$

where, $Q_2 = \frac{Q_1}{1-10L_g} + \frac{10M^2(T+1)Ta}{1-10L_g}$,
since

$$E\left\|x^0(t)\right\|^2 \leq M^2 E\left\|\varphi(0)\right\|^2 = Q_3 < \infty. \tag{3.4}$$

Thus $E\left\|x^n\right\|_t^2 < \infty$ for all $n = 1, 2, \ldots$ and $t \in [0, T]$. This proves the boundedness of $\{x^n\}$.
Let us next show that $\{x^n\}$ is Cauchy in $\mathcal{B}_\mathcal{T}$. For this consider,

$$\begin{aligned}
E\left\|x^{n+1}(t) - x^n(t)\right\|^2 \leq\ & 4L_g E\|x^{n+1} - x^n\|_t^2 \\
& + 4M^2(T+1)\int_0^t K(E\|x^n - x^{n-1}\|_s^2)ds \\
& + 4M^2 m\sum_{k=1}^m h_k E\|x^n - x^{n-1}\|_t^2.
\end{aligned}$$

Thus,

$$E\left\|x^{n+1}-x^{n}\right\|_{t}^{2} \leq \frac{4M^{2}(T+1)}{1-4L_{g}}\int_{0}^{t}K(E\|x^{n}-x^{n-1}\|_{s}^{2})ds + \frac{4M^{2}m\sum_{k=1}^{m}h_{k}}{1-4L_{g}}E\|x^{n}-x^{n-1}\|_{t}^{2}. \tag{3.5}$$

Set

$$\Psi_{n}(t)=\sup_{t\in[0,T]}E\|x^{n+1}-x^{n}\|_{t}^{2}. \tag{3.6}$$

Then, we have in the view of (3.5),

$$\Psi_{n}(t) \leq \frac{4M^{2}(T+1)}{1-4L_{g}}\int_{0}^{t}K(\Psi_{n-1}(s))ds + \frac{4M^{2}m\sum_{k=1}^{m}h_{k}}{1-4L_{g}}\ \Psi_{n-1}(t), \quad 0\leq t\leq T. \tag{3.7}$$

Choose $T_1\in[0,T)$ such that

$$C_{1}\int_{0}^{t}K\Big(\Psi_{n-1}(s)\Big)ds \leq C_{1}\int_{0}^{t}\Psi_{n-1}(s)ds, \qquad n=1,2,\ldots \ \text{ for all } \ 0\leq t\leq T_{1}.$$

Moreover,

$$\begin{aligned}\left\|x^{1}(t)-x^{0}(t)\right\|^{2} &= \|S(t)g(0,\varphi)-\big[g(t,x_{t}^{1})-g(t,x_{t}^{0})\big]-g(t,x_{t}^{0})\\ &\quad+\int_{0}^{t}S(t-s)f(s,x_{s}^{0})ds+\int_{0}^{t}S(t-s)a(s,x_{s}^{0})dw(s)\\ &\quad+\sum_{0<t_{k}<t}S(t-t_{k})I_{k}(x^{0}(t_{k}))\|^{2}.\end{aligned}$$

Then, we get

$$\begin{aligned}E\left\|x^{1}-x^{0}\right\|_{t}^{2} &\leq Q_{4}+\frac{12L_{g}+12M^{2}m\sum_{k=1}^{m}h_{k}}{1-6L_{g}}E\|x^{0}\|_{t}^{2}\\ &\quad+\frac{12M^{2}(T+1)}{1-6L_{g}}\int_{0}^{t}K(E\ \|x^{0}\|_{s}^{2})ds.\end{aligned}$$

If we take the supremum over t, and use (3.4), we get

$$\begin{aligned}\Psi_{0}(t)=\sup_{t\in[0,T]}E\left\|x^{1}-x^{0}\right\|_{t}^{2} &\leq Q_{5}+\frac{12M^{2}(T+1)}{1-6L_{g}}\int_{0}^{t}K(Q_{3})ds\\ &\leq Q_{6}.\end{aligned} \tag{3.8}$$

Now, for $n=1$ in (3.7) we get

$$\Psi_{1}(t) \leq C_{1}\int_{0}^{t}K(\Psi_{0}(s))ds+C_{2}\Psi_{0}(t), \quad 0\leq t\leq T_{1}$$

where $C_1 = \frac{4M^2(T_1+1)}{1-4L_g}$ and $C_2 = \frac{4M^2 m \sum_{k=1}^{m} h_k}{1-4L_g}$.

Therefore,

$$\begin{aligned}\Psi_1(t) &\leq C_1 \int_0^t \Psi_0(s)ds + C_2\Psi_0(t) \\ &\leq C_1 \int_0^t Q_6 \; ds + C_2 Q_6 \\ &\leq (C_1 + C_2)T_1 Q_6.\end{aligned}$$

Now, for $n = 2$ in (3.7), we get

$$\begin{aligned}\Psi_2(t) &\leq C_1 \int_0^t K(\Psi_1(s))ds + C_2\Psi_1(t) \\ &\leq C_1 \int_0^t (C_1 + C_2)s \; Q_6 \; ds + C_2(C_1 + C_2)T_1 Q_6 \\ &\leq (C_1 + C_2)^2 \frac{T_1^2}{2!} \; Q_6.\end{aligned}$$

Thus by applying mathematical induction in (3.7) and using the above work we get

$$\Psi_n(t) \leq \frac{(C_1 + C_2)^n T_1^n}{n!} \; Q_6. \;\; n \geq 0, \quad t \in [0, T_1].$$

Note that for any $m > n \geq 0$, we have,

$$\begin{aligned}\sup_{t\in[0,T_1]} E\,\|x^m(t) - x^n(t)\|^2 &\leq \sum_{r=n}^{+\infty} \sup_{t\in[0,T_1]} E\,\|x^{r+1} - x^r\|_t^2 \\ &\leq \sum_{r=n}^{+\infty} \frac{(C_1 + C_2)^r T_1^r}{r!} \; Q_6 \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned} \tag{3.9}$$

This shows that $\{x^n\}$ is Cauchy in $\mathcal{B}_{\mathcal{T}}$. Then the standard Borel- Cantelli lemma argument can be used to show that $x^n(t) \to x(t)$ uniformly in t on $[0, T_1]$. By iteration, the existence of solution of (2.1) on $[0, T]$ can be obtained.

Now, we prove the uniqueness of the solution (2.2). Let $x_1, x_2 \in \mathcal{B}_{\mathcal{T}}$ be two solutions to (2.2) on some interval $(-\infty, T]$. Then, for $t \in (-\infty, 0]$, the uniqueness is obvious and for $0 \leq t \leq T$, we have

$$\begin{aligned}E\,\|x_1(t) - x_2(t)\|^2 &\leq 4\Big[L_g + M^2 m \sum_{k=1}^{m} h_k\Big] E\|x_1 - x_2\|_t^2 \\ &\quad +4M^2(T+1) \int_0^t K(E\|x_1 - x_2\|_s^2)ds.\end{aligned}$$

Thus,

$$E\,\|x_1 - x_2\|_t^2 \leq \frac{4M^2(T+1)}{1 - Q_7} \int_0^t K(E\|x_1 - x_2\|_s^2)ds,$$

where, $Q_7 = 4\Big[L_g + M^2 m \sum_{k=1}^{m} h_k\Big]$.

Thus, Bihari's inequality yields that

$$\sup_{t\in[0,T]} E\,\|x_1 - x_2\|_t^2 \;=\; 0, \qquad 0 \le t \le T.$$

Thus, $x_1(t) = x_2(t)$, for all $0 \le t \le T$. Therefore, for all $-\infty < t \le T$, $x_1(t) = x_2(t)$ a.s. This completes the proof. □

4 Stability

In this section, we study the stability through the continuous dependence on initial values.

Definition 4.1. *A mild solution $x(t)$ of the system (2.1) with initial value ϕ is said to be stable in the mean square if for all $\epsilon > 0$, there exists $\delta > 0$ such that*

$$E\|x(t) - \hat{x}(t)\|^2 \le \epsilon \quad \textit{whenever} \quad E\|\phi - \hat{\phi}\|^2 < \delta, \; \textit{for all } t \in [0,T] \tag{4.1}$$

where $\hat{x}(t)$ is another mild solution of the system (2.1) with initial value $\hat{\phi}$.

Theorem 4.1. *Let $x(t)$ and $y(t)$ be mild solutions of the system (2.1) with initial values φ_1 and φ_2 respectively. If the assumptions of Theorem 3.1 are satisfied, then the mild solution of the system (2.1) is stable in the mean square.*

Proof: By the assumptions, $x(t)$ and $y(t)$ are two mild solutions of equations (2.1) with initial values φ_1 and φ_2, respectively, so that for $0 \le t \le T$ we have

$$\begin{aligned} x(t) - y(t) = \;& S(t)\Big(\big[\varphi_1(0) - \varphi_2(0)\big] + \big[g(0,\varphi_1) - g(0,\varphi_2)\big]\Big) - \big[g(t,x_t) - g(t,y_t)\big] \\ & + \int_0^t S(t-s)\big[f(s,x_s) - f(s,y_s)\big]ds + \int_0^t S(t-s)\big[a(s,x_s) - a(s,y_s)\big]dw(s) \\ & + \sum_{0<t_k<t} S(t-t_k)\big[I_k(x(t_k)) - I_k(y(t_k))\big]. \end{aligned}$$

So, estimating as before, we get

$$\begin{aligned} E\|x(t) - y(t)\|^2 \;\le\; & 6M^2\big(1+L_g\big)E\|\varphi_1 - \varphi_2\|^2 \\ & + 6\big(L_g + M^2 m \sum_{k=1}^{m} h_k\big)E\|x - y\|_t^2 \\ & + 6M^2(T+1)\int_0^t K(E\|x-y\|_s^2)ds, \end{aligned}$$

Thus,

$$\begin{aligned} E\|x - y\|_t^2 \;\le\; & \frac{6M^2\big(1+L_g\big)}{1 - 6\big(L_g + M^2 m \sum_{k=1}^{m} h_k\big)}E\|\varphi_1 - \varphi_2\|^2 \\ & + \frac{6M^2(T+1)}{1 - 6\big(L_g + M^2 m \sum_{k=1}^{m} h_k\big)}\int_0^t K(E\|x-y\|_s^2)ds. \end{aligned}$$

Let $K_1(u) = \frac{6M^2(T+1)}{1-6\left(L_g+M^2m\sum_{k=1}^m h_k\right)}K(u)$ where K is a concave increasing function from $\Re^+$ to $\Re^+$ such that $K(0) = 0$, $K(u) > 0$ for $u > 0$ and $\int_{0^+}\frac{du}{K(u)} = +\infty$. So, $K_1(u)$ is a concave function from $\Re^+$ to $\Re^+$ such that $K_1(0) = 0$, $K_1(u) \geq K(u)$, for $0 \leq u \leq 1$ and $\int_{0^+}\frac{du}{K_1(u)} = +\infty$. Now for any $\epsilon > 0$, $\epsilon_1 \triangleq \frac{1}{2}\,\epsilon$, we have $\lim_{s\to 0}\int_s^{\epsilon_1}\dfrac{du}{K_1(u)} = \infty$. Thus, there is a positive constant $\delta < \epsilon_1$, such that $\int_\delta^{\epsilon_1}\frac{du}{K_1(u)} \geq T$.

From Corollary 2.4, let

$$\begin{aligned}
u_0 &= \frac{6M^2\big(1+L_g\big)}{1-6\big(L_g+M^2m\sum_{k=1}^m h_k\big)}E\|\varphi_1-\varphi_2\|^2,\\
u(t) &= E\|x-y\|_t^2,\quad v(t)=1,
\end{aligned}$$

so that when $u_0 \leq \delta \leq \epsilon_1$ we have

$$\int_{u_0}^{\epsilon_1}\frac{du}{K_1(u)} \geq \int_\delta^{\epsilon_1}\frac{du}{K_1(u)} \geq T = \int_0^T v(s)ds.$$

Hence, for any $t \in [0,T]$, the estimate $u(t) \leq \epsilon_1$ holds. This completes the proof. $\square$

Remark 4.1.

If $m = 0$ in (2.1)*, then the system behaves as stochastic partial neutral functional differential equations with infinite delays of the form*

$$\begin{cases}
d\big[x(t)+g(t,x_t)\big] = \Big[A\big[x(t)+g(t,x_t)\big]+f(t,x_t)\Big]dt + a(t,x_t)dw(t),\ 0\leq t\leq T,\\
x(t) = \varphi\in D^b_{B_0}((-\infty,0],X).
\end{cases}\tag{4.2}$$

By applying Theorem 3.1 *under the hypotheses* $(H_1)-(H_3), (H_5)$ *the system* (4.2) *guarantees the existence and uniqueness of the mild solution.*

Remark 4.2.

If the system (4.2) *satisfies the Remark* 4.1*, then by Theorem* 4.1*, the mild solution of the system* (4.2) *is stable in the mean square.*

5 An example

We conclude this work with an example of the form

$$\begin{aligned}
& d\Big[u(t,x)+\int_0^\pi b(y,x)u(tsint,y)dy\Big]\\
= \quad & \Big[\frac{\partial^2}{\partial x^2}\Big[u(t,x)+\int_0^\pi b(y,x)u(tsint,y)dy\Big]+H(t,u(tsint,x))\Big]dt\\
& + \quad \sigma\, G(t,u(tsint,x))d\beta(t),\qquad t\neq t_k,\ 0\leq t\leq T,\ 0\leq x\leq\pi
\end{aligned}\tag{5.1}$$

together with the initial conditions

$$\begin{aligned} u(t_k^+) - u(t_k^-) &= (1+b_k)u(x(t_k)), \qquad t = t_k, \ k = 1,2,\dots m, && (5.2)\\ u(t,0) &= u(t,\pi) = 0 && (5.3)\\ u(t,x) &= \Phi(t,x), \ \ 0 \le x \le \pi, \quad -\infty < t \le 0. && (5.4) \end{aligned}$$

Let $X = L^2([0,\pi])$ and $Y = R^1$, the real number σ is the magnitude of continuous noise, $\beta(t)$ is a standard one dimension Brownian motion, $\Phi \in D^b_{B_0}((-\infty,0],X)$, $b_k \ge 0$ for $k = 1,2,\dots,m$ and $\sum_{k=1}^m b_k < \infty$.

Define A an operator on X by $Au = \frac{\partial^2 u}{\partial x^2}$ with the domain

$$D(A) = \Big\{ u \in X \,\Big|\, u \text{ and } \frac{\partial u}{\partial x} \text{ are absolutely continuous, } \frac{\partial^2 u}{\partial x^2} \in X, \ u(0) = u(\pi) = 0 \Big\}.$$

It is well known that A generates a strongly continuous semigroup $S(t)$ which is compact, analytic and self adjoint. Moreover, the operator A can be expressed as

$$Au = \sum_{n=1}^{\infty} n^2 < u, u_n > u_n, \quad u \in D(A),$$

where $u_n(\zeta) = (\frac{2}{\pi})^{\frac{1}{2}} \sin(n\zeta)$, $n = 1,2,\dots$, is the orthonormal set of eigenvectors of A, and

$$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} < u, u_n > u_n, \quad u \in X.$$

We assume that the following condition hold:
(i): The function b is measurable and

$$\int_0^{\pi} \int_0^{\pi} b^2(y,x)dydx < \infty.$$

(ii): Let the function $\frac{\partial}{\partial t} b(y,x)$ be measurable, let $b(y,0) = b(y,\pi) = 0$, and let

$$L_g = \Big[\int_0^{\pi} \int_0^{\pi} \big(\frac{\partial}{\partial t} b(y,x)\big)^2 dydx \Big]^{\frac{1}{2}} < \infty.$$

Assuming that conditions (i) and (ii) are verified, then the problem $(5.1) - (5.4)$ can be modeled as the abstract impulsive stochastic semilinear neutral functional differential equation of the form (2.1), as follows

$$\begin{aligned} g(t,x_t) &= \int_0^{\pi} b(y,x)u(tsint,y)dy, \quad f(t,x_t) = H(t,u(tsint,x)),\\ a(t,x_t) &= \sigma\ G(t,u(tsint,x)) \text{ and } I_k(x(t_k)) = (1+b_k)u(x(t_k)) \text{ for } k = 1,2,\dots m. \end{aligned}$$

The next results are consequences of Theorem 3.1 and Theorem 4.1, respectively.

Proposition 5.1. *If* $(H_1) - (H_5)$ *hold, then there exists a unique mild solution* u *for the system* $(5.1) - (5.4)$.

Proposition 5.2. *If all the hypotheses of Proposition* 5.1 *hold, then the mild solution u for the system* $(5.1) - (5.4)$ *is stable in the mean square.*

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References

[1] A. Anguraj, M. Mallika Arjunan and E. Hernández, *Existence results for an impulsive partial neutral functional differential equations with state-dependent delay*, Appl. Anal., 86(7)(2007) 861-872.

[2] I. Bihari, *A generalization of a lemma of Bellman and its application to uniqueness problem of differential equations*, Acta Math. Acad. Sci. Hungar., 7(1956)71-94.

[3] G. Da Prato and J. Zabczyk, *Stochastic Equations in Infinite Dimensions*, Cambridge University Press, Cambridge: 1992.

[4] T. E. Govindan, *An existence result for the Cauchy problem for stochastic systems with heredity* , Diff. Integral Eq., 15(2002) 103-113.

[5] T. E. Govindan, *Stability of mild solutions of stochastic evolution equations with variable delay*, Stochastic Anal. Appl. : 21(2003)1059-1077.

[6] E. Hernández and Donald O'Regan, *Existence results for abstract neutral functional differential equations*, Proc. Amer. Math. Soc., (to appear).

[7] E. Hernández, M. Rabello and H. R. Henriquez, *Existence of solutions for impulsive partial neutral functional differential equations*, J. Math. Anal. Appl., 331(2007)1135-1158.

[8] E. Hernández and H. R. Henriquez, *Existence results for partial neutral functional differential equations with unbounded delay*, J. Math. Anal. Appl., 221(1998)452-475.

[9] Jun Yang, Shouming Zhong and Wenpin Luo, *Mean square stability analysis of impulsive stochastic differential equations with delays*, J. Comput. Appl. Math., 216(2008)474-483.

[10] V. B. Kolmanovskii and V. R. Nosov, *Stability of neutral-type functional differential equations*, Nonlinear Analysis TMA, 6(1982) 873-910.

[11] V. Lakshmikantham, D. D. Bainov and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[12] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[13] Y. Ren and N. Xia, *Existence, uniqueness and stability of the solutions to neutral stochastic functional differential equations with infinite delay*, Appl. Math. Comput., 210(2009)72-79.

[14] Y. Ren, S. Lu and N. Xia, *Remarks on the existence and uniqueness of the solutions to stochastic functional differential equations with infinite delay*, J. Comput. Appl. Math., 220(2008)364-372.

[15] A. E. Rodkina, *On existence and uniqueness of solution of stochastic differential equations with heredity*, Stochastics, 12(1984)187-200.

[16] Yu.v.Rogovchenko, *Impusive evolution systems: main results and new trends*, Dynamics Contin. Diser.Impulsive Sys., 3(1997)57-88.

[17] R. Sakthivel and J. Luo. *Asymptotic stability of impulsive stochastic partial differential equations with infinite delays*, J. Math. Anal. Appl. (2009, in press).

[18] R. Sakthivel and J. Luo. *Asymptotic stability of nonlinear impulsive stochastic differential equations*, Statist. Probab. Lett. (2009, in press).

[19] A. M. Samoilenko and N. A Perestyuk., *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[20] S. J. Wu, X. L. Guo and S. Q. Lin, *Existence and uniqueness of solutions to random impulsive differential systems*, Acta Math. Appl. Sin., 22(4)(2006)595-600.

[21] Zhiguo Yang, Daoyi Xu and Li Xiang, *Exponential p-stability of impulsive stochastic differential equations with delays*, Physics Letter A: 356(2006)129-137.

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