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Oscillation of Complex High Order Linear Differential Equations with Coefficients of Finite Iterated Order *

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Abstract

In this paper, we investigate the growth of solutions of complex high order linear differential equations with entire or meromorphic coefficients of finite iterated order and we obtain some results which improve and extend some previous results of Z. X. Chen and L. Kinnunen.
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1. Definitions and Notations

In this paper, we assume that readers are familiar with the fundamental results and standard notations of the Nevanlinna's theory of meromorphic functions (see [6, 9]). In order to describe the growth of order of entire functions or meromorphic functions more precisely, we first introduce some notations about finite iterated order. Let us define inductively, for $r \in [0,\infty), \exp_1 r = e^r$ and $\exp_{i+1} r = \exp(\exp_i r),\ i \in N$. For all sufficiently large r, we define $\log_1 r = \log r$ and $\log_{i+1} r = \log(\log_i r),\ i \in N$. We also denote $\exp_0 r = r = \log_0 r$ and $\exp_{-1} r = \log_1 r$. Moreover we denote the linear measure and the logarithmic measure of a set $E \subset [1,+\infty)$ by $mE = \int_E dt$ and $m_l E = \int_E dt/t$ respectively. In the following, we recall some definitions of entire functions or meromorphic functions of finite iterated order (see [2, 3, 10, 12]).

Definition 1.1. The p-iterated order of a meromorphic function f is defined by

$$\sigma_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log r} \quad (p \in N). \tag{1.1}$$

Remark 1.1. If $p = 1$, the classical growth of order of f is defined by (see [6, 9])

$$\sigma(f) = \overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_2 M(r,f)}{\log r} \quad (p \in N).$$

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If $p = 2$, the hyper-order of f is defined by (see [13])

$$\sigma_2(f) = \overline{\lim_{r\to\infty}} \frac{\log_2 T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_3 M(r,f)}{\log r} \quad (p \in N).$$

If f is an entire function, then the p-iterated order of f is defined by

$$\sigma_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_{p+1} M(r,f)}{\log r} \quad (p \in N). \tag{1.2}$$

Definition 1.2. The finiteness degree of the order of a meromorphic function f is defined by

$$i(f) = \begin{cases} 0 & \text{for } f \text{ rational,} \\ \min\{p \in N : \sigma_p(f) < \infty\} & \text{for } f \text{ transcendental for which some} \\ & p \in N \text{ with } \sigma_p(f) < \infty \text{ exists ,} \\ \infty & \text{for } f \text{ with } \sigma_p(f) = \infty \text{ for all } p \in N. \end{cases} \tag{1.3}$$

Definition 1.3. The p-iterated lower order of a meromorphic function f is defined by

$$\mu_p(f) = \underline{\lim_{r\to\infty}} \frac{\log_p T(r,f)}{\log r} \quad (p \in N). \tag{1.4}$$

Definition 1.4. The p-iterated exponent of convergence of a-point of a meromorphic function f is defined by

$$\lambda_p(f,a) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r,a)}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_p N(r,a)}{\log r} \quad (p \in N). \tag{1.5}$$

If $a = 0$, the p-iterated exponent of convergence of zero-sequence of a meromorphic function f is defined by

$$\lambda_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p n(r,\frac{1}{f})}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_p N(r,\frac{1}{f})}{\log r} \quad (p \in N). \tag{1.6}$$

The p-iterated exponent of convergence of different zero-sequence of a meromorphic function f is defined by

$$\overline{\lambda}_p(f) = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{n}(r,\frac{1}{f})}{\log r} = \overline{\lim_{r\to\infty}} \frac{\log_p \overline{N}(r,\frac{1}{f})}{\log r} \quad (p \in N). \tag{1.7}$$

If $a=\infty$, the p-iterated exponent of convergence of pole-sequence of a meromorphic function f is defined by

$$\lambda_p\left(\frac{1}{f}\right)=\overline{\lim_{r\to\infty}}\frac{\log_p n(r,f)}{\log r}=\overline{\lim_{r\to\infty}}\frac{\log_p N(r,f)}{\log r}\quad (p\in N). \tag{1.8}$$

2. Introduction and Main results

Many authors have investigated complex oscillation properties of the high order linear differential equations

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_0(z)f=0 \tag{2.1}$$

and

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_0(z)f=F(z) \tag{2.2}$$

and obtained many results when the coefficients in (2.1) or (2.2) are entire functions or meromorphic functions of finite order (see[1, 2,3, 10, 11, 12]). When the coefficients in (2.1) or (2.2) are entire functions of finite iterated order, we have the following results.

Theorem A [10]. Let $A_0(z),\cdots,A_{k-1}(z)$ be entire functions, if $i(A_j)\le p(j=0,\cdots,k-1,p\in N)$, then $\sigma_{p+1}(f)\le\max\{\sigma_p(A_j),j=0,\cdots,k-1\}$ holds for all solutions of (2.1).

Theorem B [10]. Let $A_0(z),\cdots,A_{k-1}(z)$ be entire functions and let $i(A_0)=p,p\in N$. If $i(A_j)<p$ or $\sigma_p(A_j)<\sigma_p(A_0)$ for all $j=1,\cdots,k-1$, then $i(f)=p+1$ and $\sigma_{p+1}(f)=\sigma_p(A_0)$ hold for all non-trivial solutions of (2.1).

Theorem C [1]. Let $A_0(z),\ldots,A_{k-1}(z)$ be entire functions, and let $i(A_0)=p$ $(p\in N)$. Assume that $\max\{\rho_p(A_j):j=1,2,\ldots,k-1\}\le\sigma_p(A_0)(>0)$ and $\max\{\tau_p(A_j):\sigma_p(A_j)=\sigma_p(A_0)\}<\tau_p(A_0)=\tau$ $(0<\tau<+\infty)$. Then every solution $f\not\equiv 0$ of (2.1) satisfies $i(f)=p+1$ and $\sigma_{p+1}(f)=\sigma_p(A_0)$.

When the coefficients in (2.1) or (2.2) are meromorphic functions of finite iterated order, we have also the following results.

Theorem D [3,11]. Let $A_0(z),\cdots,A_{k-1}(z),F\not\equiv 0$ be meromorphic functions, and let $f(z)$ be a meromorphic solution of (2.2) satisfying one of the following conditions:

(i) $\max\{i(F)=q,i(A_j)(j=0,\cdots,k-1)\}<i(f)=p+1(p\in N)$,

(ii) $b=\max\{\sigma_{p+1}(F),\sigma_{p+1}(A_j)(j=0,\ldots,k-1)\}<\sigma_{p+1}(f)$,

then $\overline{\lambda}_{p+1}(f)=\lambda_{p+1}(f)=\sigma_{p+1}(f)$.

Theorem E [11]. Let $A_j(z)(j=0,\cdots,k-1)$ be meromorphic functions of finite iterated order satisfying $\beta=\max\{\sigma_p(A_j),\lambda_p(\frac{1}{A_0}),j\neq 0\}<\sigma_p(A_0),(p\in N)$, or $i(A_j)<p$ for $j\neq 0$, if $f\not\equiv 0$ is a meromorphic solution of (2.1), then $\sigma_{p+1}(f)\geq\sigma_p(A_0)$.

Theorem F [11]. Let$A_j(z)(j=0,\cdots,k-1)$ be meromorphic functions of finite iterated order satisfying $\max\{\sigma_p(A_j),\lambda_p(\frac{1}{A_s}),\ j\neq s\}<\mu_p(A_s)\leq\sigma_p(A_s)<\infty, s\in\{0,1,\cdots,k-1\}(p\in N)$, or $i(A_j)<p$ for $j\neq s$, if $f\not\equiv 0$ is a meromorphic solution of (2.1) satisfying $\frac{N(r,f)}{\overline{N}(r,f)}<M$ (a positive constant), then $\sigma_{p+1}(f)\leq\sigma_p(A_s)$.

In this paper, we investigate the growth of solutions of high order linear differential equations (2.1) and (2.2) with entire or meromorphic coefficients of finite iterated order under certain conditions and obtain the following results which improve and extend the above results.

Theorem 2.1. Let $A_0(z),\cdots,A_{k-1}(z)$ be entire functions of finite iterated order satisfying $i(A_0)=p,\sigma_p(A_0)=\sigma_1$ and $\overline{\lim\limits_{r\to\infty}}\sum\limits_{j=1}^{k-1}m(r,A_j)/m(r,A_0)<1$, then every non-trivial solution $f(z)$ of (2.1) satisfies $\sigma_{p+1}(f)=\sigma_p(A_0)=\sigma_1$.

Corollary 2.1. Let $A_0(z),A_1(z),\cdots,A_{k-1}(z)$ be entire functions of finite iterated order satisfying $\overline{\lim\limits_{r\to\infty}}\sum\limits_{j=1}^{k-1}m(r,A_j)/m(r,A_0)<1$ and $A_0(z)$ be transcendental with $\sigma(A_0)<\infty$, then every non-trivial solution $f(z)$ of (2.1) satisfies $\sigma_2(f)=\sigma(A_0)$.

Theorem 2.2. Let $A_0(z),A_1(z),\cdots,A_{k-1}(z)$ be entire functions of finite iterated order satisfying $\max\{\sigma_p(A_j),j\neq 0\}\leq\mu_p(A_0)=\sigma_p(A_0)$, and $\lim\limits_{r\to\infty}\sum\limits_{j=1}^{k-1}m(r,A_j)/m(r,A_0)<1(r\notin E_1)$, where E_1 is a set of r of finite linear measure, then every non-trivial solution $f(z)$ of (2.1) satisfies $\sigma_{p+1}(f)=\mu_p(A_0)=\sigma_p(A_0)$.

Remark 2.1. In Theorems B-C and our Theorem 2.1, the authors investigated the growth of the solutions of (2.1) under the same case that the coefficient $A_0(z)$ in (2.1) grows faster than other coefficients $A_j(z)(j=1,\cdots,k-1)$ and obtain the same conclusion $\sigma_{p+1}(f)=\sigma_p(A_0)(p\in N)$. We have to note that the condition $\max\{\sigma_p(A_j),j=1,\cdots,k-1\}<\sigma_p(A_0)$ in Theorem B is stronger than our condition $\overline{\lim\limits_{r\to\infty}}\sum\limits_{j=1}^{k-1}m(r,A_j)/m(r,A_0)<1$ in Theorem 2.1. Thus, Theorem 2.1 is an improvement of Theorem B and Corollary 2.1 is an improvement of [5, p.121, Theorem 4]. In Theorem 2.1, if we replace the condition $\overline{\lim\limits_{r\to\infty}}\sum\limits_{j=1}^{k-1}m(r,A_j)/m(r,A_0)<1$ with

$\underline{\lim}_{r\to\infty} \sum_{j=1}^{k-1} m(r, A_j)/m(r, A_0) < 1$ and $\mu_p(A_0) = \sigma_p(A_0)$, and then we can get the same conclusion as Theorem 2.1, therefore Theorem 2.2 is a supplement of Theorem 2.1.

Theorem 2.3. Let $A_0(z), A_1(z), \cdots, A_{k-1}(z), F(z) \not\equiv 0$ be meromorphic functions. If $f(z)$ is a meromorphic solution of (2.2) satisfying $i(f) = p+1, \sigma_{p+1}(f) = \sigma_2$ and $\overline{\lim}_{r\to\infty} [\sum_{j=0}^{k-1} T(r, A_j) + T(r, F)]$ $/T(r, f) < 1$, then $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma_2$.

Remark 2.2. Theorem 2.3 is an improvement of Theorem D since the conditions in Theorem D are stronger than our condition in Theorem 2.3. Can we get the same conclusion when the coefficients in (2.1) are meromorphic functions? The following Theorem 2.4 give us an affirmative answer.

Theorem 2.4. Let $A_0(z), A_1(z), \cdots, A_{k-1}(z)$ be meromorphic functions of finite iterated order satisfying $i(A_0) = p$, $\delta(\infty, A_0) = \underline{\lim}_{r\to\infty} \frac{m(r,A_0)}{T(r,A_0)} > 0$ and $\overline{\lim}_{r\to\infty} \sum_{j=1}^{k-1} m(r, A_j)/m(r, A_0) < 1$, then every non-trivial solution $f(z)$ of (2.1) satisfies $\sigma_{p+1}(f) \geq \sigma_p(A_0)$.

Theorem 2.5. Let $A_0(z), A_1(z), \cdots, A_{k-1}(z), F(z)$ be meromorphic functions of finite iterated order satisfying $\max\{\sigma_p(A_j), \sigma_p(F), \lambda_p(\frac{1}{A_s}), j \neq s\} < \mu_p(A_s) \leq \sigma_p(A_s)$ or $i(A_j) < p \quad (j \neq s)$, if $f(z)$ is a meromorphic solution of (2.2) satisfying $\frac{N(r,f)}{\overline{N}(r,f)} \leq \exp_{p-1}\{r^b\}(b < \mu_p(A_s))$, then $\sigma_{p+1}(f) \leq \sigma_p(A_s)$.

Corollary 2.2. Let $A_0(z), A_1(z), \cdots, A_{k-1}(z)$ be meromorphic functions of finite iterated order satisfying $\max\{\sigma_p(A_j), \lambda_p(\frac{1}{A_0}), j \neq 0\} < \mu_p(A_0) \leq \sigma_p(A_0) < \infty$, if $f \not\equiv 0$ is a meromorphic solution of (2.1) satisfying $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p-1}\{r^b\}$ $(b < \mu_p(A_0))$, then $\sigma_{p+1}(f) = \sigma_p(A_0)$.

Remark 2.3. Theorem 2.4 is a supplement of Theorem E. Theorem 2.5 is an extension of Theorem F since Theorem F is a special case of Theorem 2.5 with $F(z) \equiv 0$.

3. Preliminary Lemmas

Lemma 3.1 [9]. Let $g : (0, +\infty) \longrightarrow R, h : (0, +\infty) \longrightarrow R$ be monotone increasing functions such that

(i) $g(r) \leq h(r)$ outside of an exceptional set E_2 of finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(\alpha r)$ for all $r > r_0$.

(ii) $g(r) \leq h(r)$ outside of an exceptional set E_2 of finite logarithmic measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) \leq h(r^\alpha)$ for all $r > r_0$.

Lemma 3.2 [7, 8, 9]. Let $f(z)$ be a transcendental entire function, and let z be a point with

$|z| = r$ at which $|f(z)| = M(r,f)$. Then for all $|z|$ outside a set E_3 of r of finite logarithmic measure, we have

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^n (1+o(1)) \quad (n \in N, r \notin E_3), \tag{3.1}$$

where $\nu_f(r)$ is the central index of f.

Lemma 3.3 [11]. Let $f(z) = \frac{g(z)}{d(z)}$, where $g(z), d(z)$ are entire functions of finite iterated order satisfying $i(d) < p$ or $\sigma_p(d) < \mu_p(g) \le \sigma_p(g) < \infty,\ p \in N$. Let z be a point with $|z| = r$ at which $|g(z)| = M(r,g)$ and $\nu_g(r)$ denotes the central-index of g, then the estimation

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^n (1+o(1)) \quad (n \in N). \tag{3.2}$$

holds for all $|z| = r$ outside a set E_4 of r of finite logarithmic measure.

Lemma 3.4 [3,11]. Let $f(z)$ be an entire function of finite iterated order satisfying $\sigma_p(f) = \sigma_3, \mu_q(f) = \mu, 0 < q \le p < \infty$, and let $\nu_f(r)$ be the central index of f, then we have

$$\overline{\lim_{r\to\infty}} \frac{\log_p \nu_f(r)}{\log r} = \sigma_3, \tag{3.3}$$

$$\lim_{r\to\infty} \frac{\log_q \nu_f(r)}{\log r} = \mu. \tag{3.4}$$

Lemma 3.5 [8]. Let $A_0(z), A_1(z), \cdots, A_{k-1}(z)$ be entire coefficients in (1.1), and at least one of them is transcendental. If $A_s(z)(0 \le s \le k-1)$ is the first one (according to the sequence of $A_0, A_1, \cdots, A_{k-1}$) satisfying $\varliminf_{r\to\infty} \sum_{j=s+1}^{n} m(r,A_j)/m(r,A_s) < 1 (r \notin E_5)$, where $E_5 \subset (1,+\infty)$ is a set having finite linear measure, then (1.1) possesses at most s linearly independent entire solutions satisfying $\overline{\lim_{r\to\infty}} \frac{\log T(r,f)}{m(r,A_p)} = 0 (r \notin E_5)$.

Lemma 3.6 [14]. Let $f(z)$ be an entire function of finite iterated order with $i(f) = p,\ p \in N$. Then there exist entire functions $\beta(z)$ and $D(z)$ such that

$$f(z) = \beta(z)e^{D(z)},$$

$$\sigma_p(f) = \max\{\sigma_p(\beta), \sigma_p(e^{D(z)})\}$$

and

$$\sigma_p(\beta) = \overline{\lim_{r\to\infty}} \frac{\log_p N(r, \frac{1}{f})}{\log r}.$$

Moreover, for any given $\varepsilon > 0$, then

$$\log|\beta(z)| \ge -\exp_{p-1}\{r^{\sigma_p(\beta)+\varepsilon}\} \quad (r \notin E_6), \tag{3.5}$$

where E_6 is a set of r of finite linear measure.

Lemma 3.7. Let $f(z)$ be a meromorphic function of finite iterated order with $i(f)=p,\ p\in N$. Then there exist entire functions $\pi_1(z),\pi_2(z)$ and $D(z)$ such that

$$f(z)=\frac{\pi_1(z)e^{D(z)}}{\pi_2(z)}, \tag{3.6}$$

and

$$\sigma_p(f)=\max\{\sigma_p(\pi_1),\sigma_p(\pi_2),\sigma_p(e^{D(z)})\}. \tag{3.7}$$

Moreover, for any given $\varepsilon>0$, we have

$$\exp\{-\exp_{p-1}\{r^{\sigma_p(f)+\varepsilon}\}\}\le|f(z)|\le\exp_p\{r^{\sigma_p(f)+\varepsilon}\}\quad(r\notin E_7), \tag{3.8}$$

where E_7 is a set of r of finite linear measure.
Proof. By Lemma 3.6, it is easy to see that (3.6) and (3.7) hold. Set $f(z)=\frac{\pi_1(z)e^{D(z)}}{\pi_2(z)}$, where $\pi_1(z),\pi_2(z)$ are the canonical products formed with the zeros and poles of $f(z)$ respectively. Since $\max\{\sigma_p(\pi_1),\sigma_p(\pi_2),\sigma_p(e^{D(z)})\}=\sigma_p(f)$ and by Lemma 3.6, for sufficiently large $|z|=r$, we have

$$|\pi_1(z)|\le\exp_p\{r^{\sigma_p(\pi_1)+\frac{\varepsilon}{2}}\}\le\exp_p\{r^{\sigma_p(f)+\frac{\varepsilon}{2}}\},\quad |e^{D(z)}|\le\exp_p\{r^{\sigma_p(f)+\frac{\varepsilon}{2}}\} \tag{3.9}$$

$$|\pi_2(z)|\le\exp_p\{r^{\sigma_p(\pi_2)+\frac{\varepsilon}{2}}\}\le\exp_p\{r^{\sigma_p(f)+\frac{\varepsilon}{2}}\}, \tag{3.10}$$

$$|\pi_1(z)|\ge\exp\{-\exp_{p-1}\{r^{\sigma_p(\pi_1)+\frac{\varepsilon}{2}}\}\}\ge\exp\{-\exp_{p-1}\{r^{\sigma_p(f)+\frac{\varepsilon}{2}}\}\}\quad(r\notin E_7), \tag{3.11}$$

$$|\pi_2(z)|\ge\exp\{-\exp_{p-1}\{r^{\sigma_p(\pi_2)+\frac{\varepsilon}{2}}\}\}\ge\exp\{-\exp_{p-1}\{r^{\sigma_p(f)+\frac{\varepsilon}{2}}\}\}\quad(r\notin E_7), \tag{3.12}$$

where E_7 is a set of r of finite linear measure. Since $\sigma_{p-1}(D)=\sigma_p(e^{D(z)})\le\sigma_p(f)$ and $|e^{D(z)}|\ge e^{-|D(z)|}$, for sufficiently large $|z|=r$, we have

$$|e^{D(z)}|\ge e^{-|D(z)|}\ge\exp\{-\exp_{p-1}\{r^{\sigma_p(f)+\frac{\varepsilon}{2}}\}\}. \tag{3.13}$$

By (3.9)-(3.13), we can easily obtain (3.8). Thus, we complete the proof of Lemma 3.7.

Remark 3.1. Lemma 3.7 is an improvement of Lemma 3.6 and extends the conclusion of [4, p.84, Lemma 4].

Lemma 3.8. Let $f(z)$ be a meromorphic function of finite iterated order satisfying $i(f)=p$, then there exists a set $E_8\subset(1,+\infty)$ having infinite logarithmic measure such that for all $r\in E_8$, we have

$$\lim_{r\to\infty}\frac{\log_p T(r,f)}{\log r}=\sigma_p(f).$$

Proof. By Definition 1.1, there exists a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ and satisfying $(1+\frac{1}{n})r_n<r_{n+1}$ such that

$$\lim_{r_n\to\infty}\frac{\log_p T(r_n,f)}{\log r_n}=\sigma_p(f). \tag{3.14}$$

There exists an n_1 such that for $n \geq n_1$ and for any $r \in [r_n, (1+\frac{1}{n})r_n]$, we have

$$\frac{\log_p T(r_n, f)}{\log{(1+\frac{1}{n})r_n}} \leq \frac{\log_p T(r, f)}{\log r} \leq \frac{\log_p T((1+\frac{1}{n})r_n, f)}{\log r_n}. \tag{3.15}$$

Set $E_8 = \bigcup\limits_{n=n_1}^{\infty} [r_n, (1+\frac{1}{n})r_n]$, for any $r \in E_8$, by (3.15), we have

$$\lim_{r\to\infty} \frac{\log_p T(r, f)}{\log r} = \lim_{r_n\to\infty} \frac{\log_p T(r_n, f)}{\log r_n} = \sigma_p(f),$$

and $m_l E_8 = \sum\limits_{n=n_1}^{\infty} \int_{r_n}^{(1+\frac{1}{n})r_n} \frac{dt}{t} = \sum\limits_{n=n_1}^{\infty} \log(1+\frac{1}{n}) = \infty$.

4. Proofs of Theorems.

Proof of Theorem 2.1. We divide the proof into two parts: $(i)\sigma_{p+1}(f) \geq \sigma_1$, $(ii)\sigma_{p+1}(f) \leq \sigma_1$.

(i) By (2.1), we get

$$-A_0 = \frac{f^{(k)}(z)}{f(z)} + A_{k-1}\frac{f^{(k-1)}(z)}{f(z)} + \cdots + A_1\frac{f'(z)}{f(z)}. \tag{4.1}$$

By the lemma of the logarithmic derivative and (4.1), we have

$$m(r, A_0) \leq \sum_{j=1}^{k-1} m(r, A_j) + O\{\log(rT(r, f))\} \quad (r \notin E), \tag{4.2}$$

where E is a set of r of finite linear measure, not necessarily the same at each occurrence. Suppose that

$$\overline{\lim_{r\to\infty}} \sum_{j=1}^{k-1} m(r, A_j)/m(r, A_0) = \alpha < \beta_1 < 1,$$

then for sufficiently large r, we have

$$\sum_{j=1}^{k-1} m(r, A_j) < \beta_1 m(r, A_0). \tag{4.3}$$

By (4.2) and (4.3), we have

$$(1-\beta_1)m(r, A_0) \leq O\{\log(rT(r, f))\} \quad (r \notin E). \tag{4.4}$$

By $\sigma_p(A_0) = \sigma_1$ and Lemma 3.8, there exists a set $E_8 \subset (1, +\infty)$ having infinite logarithmic measure such that for all z satisfying $|z| = r \in E_8 \backslash E$ and for any $\varepsilon(> 0)$, we have

$$(1-\beta_1)\exp_{p-1}\{r^{\sigma_1-\varepsilon}\} \leq (1-\beta_1)m(r, A_0) \leq O\{\log(rT(r, f))\}. \tag{4.5}$$

By (4.5), we have $\sigma_{p+1}(f) \geq \sigma_p(A_0) = \sigma_1$.

(ii) By (2.1), we get

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |A_{k-1}| \left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_1| \left|\frac{f'(z)}{f(z)}\right| + |A_0|. \tag{4.6}$$

By Lemma 3.2 and (4.6), for all z satisfying $|z| = r \notin E_3$ and $|f(z)| = M(r, f)$, we have

$$\left|\left(\frac{v_f(r)}{z}\right)^k (1 + o(1))\right| \leq (|A_{k-1}| + |A_{k-2}| + \cdots + |A_0|) \left|\left(\frac{v_f(r)}{z}\right)^{k-1} (1 + o(1))\right| \quad (r \notin E_3), \tag{4.7}$$

where E_3 is a set of r of finite logarithmic linear measure. By (4.3) and $\sigma_p(A_0) = \sigma_1$, it is easy to see that $\sigma_p(A_j) \leq \sigma_1 (j = 1, \cdots, k-1)$. By Lemma 3.7 and (4.7), there exists a set $E_7 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin (E_3 \bigcup E_7)$, we have

$$\left|\left(\frac{v_f(r)}{z}\right)^k (1 + o(1))\right| \leq k \exp_p\{r^{\sigma_1 + \varepsilon}\} \left|\left(\frac{v_f(r)}{z}\right)^{k-1} (1 + o(1))\right|. \tag{4.8}$$

By Lemma 3.1, Lemma 3.4 and (4.8), we have $\sigma_{p+1}(f) \leq \sigma_1$.

From (i) and (ii), we have that every non-trivial solution $f(z)$ of (2.1) satisfies $\sigma_{p+1}(f) = \sigma_p(A_0) = \sigma_1$.

Proof of Theorem 2.2. By Lemma 3.5, we obtain that every linearly independent solution f of (2.1) satisfying $\overline{\lim\limits_{r\to\infty}} \frac{\log T(r,f)}{m(r,A_0)} > 0 (r \notin E_1)$. This means that every solution $f \not\equiv 0$ of (2.1) satisfying $\overline{\lim\limits_{r\to\infty}} \frac{\log T(r,f)}{m(r,A_0)} > 0 (r \notin E_1)$, then there exist $\delta > 0$ and a sequence $\{r_n\}_{n=1}^{\infty}$ tending to ∞ such that for sufficiently large $r_n \notin E_1$ and for every solution $f \not\equiv 0$ of (2.1), we have

$$\log T(r_n, f) > \delta m(r_n, A_0). \tag{4.9}$$

Since $\mu_p(A_0) = \sigma_p(A_0)$ and by (4.9), we have

$$\sigma_{p+1}(f) \geq \mu_p(A_0) = \sigma_p(A_0). \tag{4.10}$$

On the other hand, by Theorem A, we have that every solution $f \not\equiv 0$ of (2.1) satisfying

$$\sigma_{p+1}(f) \leq \max\{\sigma_p(A_j), j = 0, \cdots, k-1\} = \sigma_p(A_0). \tag{4.11}$$

By (4.10) and (4.11), we have $\sigma_{p+1}(f) = \mu_p(A_0) = \sigma_p(A_0)$.

Proof of Theorem 2.3. From (2.2), we get

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}(z)}{f(z)} + A_{k-1}\frac{f^{(k-1)}(z)}{f(z)} + \cdots + A_1\frac{f'(z)}{f(z)} + A_0\right). \tag{4.12}$$

It is easy to see that if f has a zero at z_0 of order $\alpha(> k)$, and $A_0, \cdots, A_{k-1}$ are analytic at z_0, then F must have a zero at z_0 of order $\alpha - k$, hence

$$n\left(r, \frac{1}{f}\right) \leq k\overline{n}\left(r, \frac{1}{f}\right) + n\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} n(r, A_j), \tag{4.13}$$

and

$$N\left(r, \frac{1}{f}\right) \leq k\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{F}\right) + \sum_{j=0}^{k-1} N(r, A_j). \tag{4.14}$$

By the lemma of the logarithmic derivative and (4.12), we have

$$m(r, \frac{1}{f}) \leq m(r, \frac{1}{F}) + \sum_{j=0}^{k-1} m(r, A_j) + O\{\log(rT(r, f))\} \quad (r \notin E). \tag{4.15}$$

By (4.14) and (4.15), we get

$$T(r, f) = T(r, \frac{1}{f}) + o(1) \leq k\overline{N}(r, \frac{1}{f}) + T(r, F) + \sum_{j=0}^{k-1} T(r, A_j) + O\{\log(rT(r, f))\} \quad (r \notin E). \tag{4.16}$$

Suppose that

$$\overline{\lim_{r\to\infty}} \frac{\sum_{j=0}^{k-1} T(r, A_j) + T(r, F)}{T(r, f)} = \delta < c < 1, \tag{4.17}$$

by (4.17), for sufficiently large r and for any given $\varepsilon(0 < \varepsilon < c - \delta)$, we have

$$\sum_{j=0}^{k-1} T(r, A_j) + T(r, F) \leq (\delta + \varepsilon)T(r, f) < cT(r, f). \tag{4.18}$$

Substituting (4.18) into (4.16), we get

$$T(r, f) \leq k\overline{N}(r, \frac{1}{f}) + cT(r, f) + \varepsilon T(r, f) \quad (r \notin E), \tag{4.19}$$

by (4.19), we get

$$T(r, f) \leq \frac{k}{1 - c - \varepsilon}\overline{N}(r, \frac{1}{f}) \leq \frac{2k}{1 - c}\overline{N}(r, \frac{1}{f}) \quad (r \notin E), \tag{4.20}$$

by Lemma 3.1 and (4.20), we have $\overline{\lambda}_{p+1}(f) \geq \sigma_{p+1}(f) = \sigma_2$. Hence

$$\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma_2.$$

Proof of Theorem 2.4. Suppose that $f = \frac{g(z)}{d(z)}$ is a non-trivial meromorphic solution of (2.1), by (4.1)-(4.4) in the proof of Theorem 2.1, there exists a constant $\beta_1 < 1$ such that for sufficiently large r, we have

$$(1-\beta_1)m(r, A_0) \leq O\{\log(rT(r, f))\} \quad (r \notin E). \tag{4.21}$$

Since $i(A_0) = p$ and by Lemma 3.8, we have

$$\lim_{r\to\infty} \frac{\log_p T(r, A_0)}{\log r} = \sigma_p(A_0) \quad (r \in E_8), \tag{4.22}$$

where E_8 is a set of r of infinite logarithmic linear measure. Since $\delta(\infty, A_0) = \underline{\lim}_{r\to\infty} \frac{m(r,A_0)}{T(r,A_0)} > 0$, we have

$$\lim_{r\to\infty} \frac{\log_p m(r, A_0)}{\log r} = \sigma_p(A_0) \quad (r \in E_8). \tag{4.23}$$

By (4.21) and (4.23), we get $\sigma_{p+1}(f) \geq \sigma_p(A_0)$.

Proof of Theorem 2.5. Suppose that $f = \frac{g(z)}{d(z)}$ is a meromorphic solution of (2.2), then by (2.2), we get

$$-A_s = \frac{f^{(k)}}{f^{(s)}} + \cdots + A_{s+1}\frac{f^{(s+1)}}{f^{(s)}} + A_{s-1}\frac{f^{(s-1)}}{f^{(s)}} + \cdots + A_0\frac{f}{f^{(s)}} - \frac{F}{f^{(s)}}. \tag{4.24}$$

By (4.24), we have

$$T(r, A_s) \leq MT(r, f) + \sum_{j\neq s} T(r, A_j) + O\{\log(rT(r, f))\}, \quad (r \notin E) \tag{4.25}$$

where $M > 0$ is a constant, not necessarily the same at each occurrence. By $\mu_p(A_s) > \max\{\sigma_p(A_j)(j \neq s), \sigma_p(F)\}$ and (4.25), we get $\mu_p(f) \geq \mu_p(A_s)$. Since the poles of f must be the poles of $A_j(j = 0, \cdots, k-1)$ and F, we have

$$\overline{\lambda}_p\left(\frac{1}{f}\right) = \overline{\lambda}_p(d) \leq \max\left\{\lambda_p\left(\frac{1}{A_j}\right), \lambda_p\left(\frac{1}{F}\right), (j = 0, \cdots, k-1)\right\} < \mu_p(A_s), \tag{4.26}$$

and by $\frac{N(r,f)}{\overline{N}(r,f)} < \exp_{p-1}\{r^b\}$, where $b < \mu_p(A_s)$, we have $N(r, f) < \overline{N}(r, f)\exp_{p-1}\{r^b\}$. Hence

$$\lambda_p(d) = \lambda_p\left(\frac{1}{f}\right) \leq \max\{\overline{\lambda}_p\left(\frac{1}{f}\right), b\} < \mu_p(A_s). \tag{4.27}$$

By (4.27) and $\mu_p(f) \geq \mu_p(A_s)$, we have $\mu_p(g) = \mu_p(f) \geq \mu_p(A_s)$. By (2.2) again, we have

$$\left|\frac{f^{(k)}(z)}{f(z)}\right| \leq |A_{k-1}|\left|\frac{f^{(k-1)}(z)}{f(z)}\right| + \cdots + |A_s|\left|\frac{f^{(s)}(z)}{f(z)}\right| + \cdots + |A_0| + \left|\frac{F(z)}{f(z)}\right|. \tag{4.28}$$

By Lemma 3.3, there exists a set E_4 having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_4$ and $|g(z)| = M(r,g)$, we have

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^j (1 + o(1)) \quad (j = 1, \cdots, k). \tag{4.29}$$

Meanwhile for all z satisfying $|z| = r \notin E_4$ and $|g(z)| = M(r,g) > 1$, we have

$$\left|\frac{F(z)}{f(z)}\right| = \left|\frac{F(z) \cdot d(z)}{g(z)}\right| \le M \cdot \exp_p\{r^{\sigma_p(A_s)}\}. \tag{4.30}$$

By Lemma 3.7,there exists a set E_7 having finite linear measure such that for all z satisfying $|z| = r \notin E_7$ and for any given $\varepsilon > 0$, we have

$$|A_j(z)| \le \exp_p\{r^{\sigma_p(A_s)+\varepsilon}\} \qquad (j = 0, \cdots, k-1). \tag{4.31}$$

Substituting (4.29)-(4.31) into (4.28), we get

$$\left(\frac{\nu_f(r)}{z}\right)^k (1 + o(1)) \le k \left(\frac{\nu_f(r)}{z}\right)^{k-1} (1 + o(1)) \exp_p\{r^{\sigma_p(A_s)+\varepsilon}\}. \tag{4.32}$$

Since ε is arbitrary, by (4.32) and Lemma 3.4, we have $\sigma_{p+1}(f) \le \sigma_p(A_s)$.

Proof of Corollary 2.2. By the proof of Theorem 2.5, we obtain that every meromorphic solution $f \not\equiv 0$ of (2.1) satisfies $\sigma_{p+1}(f) \le \sigma_p(A_0)$. On the other hand, by Theorem E, we get $\sigma_{p+1}(f) \ge \sigma_p(A_0)$, hence every meromorphic solution $f \not\equiv 0$ of (2.1) satisfies $\sigma_{p+1}(f) = \sigma_p(A_0)$.

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