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Oscillatory and Asymptotic Behavior of Fourth order Quasilinear Difference Equations

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Abstract

The authors consider the fourth order quasilinear difference equation

$$\Delta^2\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right)+q_n|x_{n+3}|^{\beta-1}x_{n+3}=0,$$

where α and β are positive constants, and $\{p_n\}$ and $\{q_n\}$ are positive real sequences. They obtain sufficient conditions for oscillation of all solutions when $\sum\limits_{n=n_0}^{\infty}\left(\frac{n}{p_n}\right)^{\frac{1}{\alpha}}<\infty$ and $\sum\limits_{n=n_0}^{\infty}\left(\frac{n}{p_n^{\frac{1}{\alpha}}}\right)<\infty$. The results are illustrated with examples.

Keywords: Fourth order difference equation, nonoscillation, oscillation, asymptotic behavior.
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1 Introduction

In this paper, we are concerned with the fourth order quasilinear difference equation

$$\Delta^2\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right)+q_n|x_{n+3}|^{\beta-1}x_{n+3}=0 \tag{1}$$

where Δ is the forward difference operator defined by $\Delta x_n = x_{n+1}-x_n$, α and β are positive constants, $\{p_n\}$ and $\{q_n\}$ are positive real sequences defined for all $n\in\mathbb{N}(n_0)=\{n_0,n_0+1,...\}$, and n_0 a nonnegative integer.

By a solution of equation (1), we mean a real sequence $\{x_n\}$ that satisfies equation (1) for all $n\in\mathbb{N}(n_0)$. If any four consecutive values of $\{x_n\}$ are given, then a solution $\{x_n\}$ of equation (1) can be defined recursively. A nontrival solution of equation (1) is said to be nonoscillatory if it is either eventually positive or eventually negative, and it is oscillatory otherwise.

The oscillatory and asymptotic behavior of solutions of the nonlinear difference equation of the form (1) has been considered by Thandapani and Selvaraj

[11] and Thandapani,Pandian,Dhanasekaran, and Graef [10]. In [11], the equation (1) is discussed under the condition

$$\sum_{n=n_0}^{\infty}\left(\frac{n}{p_n}\right)^{\frac{1}{\alpha}}=\infty \quad \text{and} \sum_{n=n_0}^{\infty}\left(\frac{n}{p_n{}^{\frac{1}{\alpha}}}\right)=\infty, \tag{2}$$

and in [10] the authors obtain a more detailed analysis of the asymptotic behavior of nonoscillatory solutions of equation (1) under the same condition (2). In [9], the authors considered the equation (1) under the condition (2), or $\sum\limits_{n=n_0}^{\infty}\left(\frac{n}{p_n}\right)^{\frac{1}{\alpha}}<\infty$ and $\sum\limits_{n=n_0}^{\infty}\left(\frac{n}{p_n^{\frac{1}{\alpha}}}\right)=\infty$ or (5), and obtain condition for the existence of nonoscillatory solutions of equation (1).
For the case $\alpha=1$, equation (1) becomes

$$\Delta^2\left(p_n\Delta^2 x_n\right)+q_n\left|x_{n+3}\right|^{\beta-1}x_{n+3}=0. \tag{3}$$

The oscillatory and asymptotic behavior of solutions of equation (3) with

$$\sum_{n=n_0}^{\infty}\left(\frac{n}{p_n}\right)=\infty \quad \text{or} \sum_{n=n_0}^{\infty}\left(\frac{n}{p_n}\right)<\infty \tag{4}$$

was discussed by Yan and Liu [13], and Thandapani and Arockiasamy [12] respectively. Later, the results in [13] were extended to the more general equation

$$\Delta\left(a_n\Delta\left(b_n\Delta\left(c_n\Delta x_n\right)\right)\right)+q_n\left|x_n\right|^{\beta-1}x_n=0$$

by Graef and Thandapani [5].

The main objective here is to discuss the oscillatory behavior of solutions of equation (1) under the condition

$$\sum_{n=n_0}^{\infty}\left(\frac{n}{p_n}\right)^{\frac{1}{\alpha}}<\infty \text{ and} \sum_{n=n_0}^{\infty}\left(\frac{n}{p_n{}^{\frac{1}{\alpha}}}\right)<\infty. \tag{5}$$

In Section 2, we describe the classification of nonoscillatory solutions of equation (1), and in Section 3, we obtain sufficient conditions for the oscillation of all solutions of equation (1). Examples illustrating the results are presented in Section 4.

For related results on the oscillation of fourth order difference equation, we refer the reader to the monographs of Agarwal, Bohner, Grace, and O'Regan [1] as well as the references ([2], [3], [6], [7], [8]) given below.

2 Classification of Nonoscillatory Solutions

In this section, we state and prove some basic results regarding the classification of nonoscillatory solutions of equation (1). Without any loss of generality, we restrict our attention to the set of positive solutions, because if $\{x_n\}$ satisfies (1), then so does $\{-x_n\}$.

Lemma 1. *If $\{x_n\}$ is an eventually positive solution of equation (1), then one of the following four cases holds for all sufficiently large n:*

$$\begin{aligned}
&(I) \quad \Delta x_n > 0, \quad \Delta^2 x_n > 0, \quad \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) > 0;\\
&(II) \quad \Delta x_n < 0, \quad \Delta^2 x_n > 0, \quad \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) > 0;\\
&(III) \quad \Delta x_n > 0, \quad \Delta^2 x_n < 0, \quad \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) > 0;\\
&(IV) \quad \Delta x_n > 0, \quad \Delta^2 x_n < 0, \quad \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) < 0.
\end{aligned}$$

Proof. From equation (1), we have $\Delta^2\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) < 0$ for all large n. It follows that $\{\Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right)\}$, $\{\Delta^2 x_n\}$ and $\{\Delta x_n\}$ are eventually monotonic and one-signed. Next, we consider the following eight cases:

$$\begin{aligned}
&(a) \quad \Delta x_n > 0, \quad \Delta^2 x_n > 0, \quad \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) > 0;\\
&(b) \quad \Delta x_n < 0, \quad \Delta^2 x_n > 0, \quad \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) > 0;\\
&(c) \quad \Delta x_n > 0, \quad \Delta^2 x_n < 0, \quad \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) > 0;\\
&(d) \quad \Delta x_n < 0, \quad \Delta^2 x_n < 0, \quad \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) > 0;\\
&(e) \quad \Delta x_n > 0, \quad \Delta^2 x_n > 0, \quad \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) < 0;\\
&(f) \quad \Delta x_n < 0, \quad \Delta^2 x_n > 0, \quad \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) < 0;\\
&(g) \quad \Delta x_n > 0, \quad \Delta^2 x_n < 0, \quad \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) < 0;\\
&(h) \quad \Delta x_n < 0, \quad \Delta^2 x_n < 0, \quad \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) < 0.
\end{aligned}$$

If $\Delta x_n < 0$ and $\Delta^2 x_n < 0$ eventually, then $\lim\limits_{n\to\infty} x_n = -\infty$, which contradicts the positivity of the solution $\{x_n\}$. Hence, the cases (d) and (h) cannot hold. Similarly, since $\Delta^2\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) < 0$, if $\Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) < 0$, then $\lim\limits_{n\to\infty}\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) = -\infty$, that is, $\Delta^2 x_n < 0$ for large n. This observation rule out the cases (e) and (f). This completes the proof of the lemma.

Remark 1. When $\alpha = 1$, then Lemma 1 reduces to Lemma 3 of [12].

Lemma 2. *Let $\{x_n\}$ be a positive solution of equation (1) of type (IV). Then there exists a positive number c such that the following inequality holds for all large n:*

$$x_{n+3} \ \geq x_n \ \geq c\, n\, \Delta x_n. \tag{6}$$

Proof. Let $\{x_n\}$ be a positive solution of equation (1) of type (IV). Since Δx_n is decreasing and positive, we have

$$x_n > x_n - x_N = \sum_{s=N}^{n-1} \Delta x_s \geq (n-N)\Delta x_n, \quad n \geq N.$$

Then, there is a constant $c > 0$, and a sufficiently large n, such that (6) holds. This completes the proof.

Lemma 3. *Let $\{x_n\}$ be a positive solution of equation (1) of type (II). Then there exists a positive number c such that the following inequalities hold for all large n:*

$$x_{n+3} \;\geq\; p_n{}^{\frac{1}{\alpha}} \;\psi_{n+3}\; \Delta^2 x_n, \tag{7}$$

$$x^{\alpha}_{n+3} \;\geq\; c\; n\; \psi^{\alpha}_{n+3}\; \Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right), \tag{8}$$

where $\psi_n = \sum\limits_{s=n}^{\infty} \frac{(s-n+1)}{p_s^{\frac{1}{\alpha}}}$.

Proof. The proof can be modeled as that of Lemma 4(ii) of Thandapani and Arockiasamy [12], and therefore the details are omitted.

Lemma 4. *Let $\alpha \leq 1$. If $\{x_n\}$ is a positive solution of equation (1) of type (IV), then there exists a positive number c such that the following inequality holds for all large n:*

$$x_{n+3} \;\geq\; c\; n\; \left|\Delta\left(p_n\; |\Delta^2 x_n|^{\alpha-1}\; \Delta^2 x_n\right)\right|^{\frac{1}{\alpha}}\; \psi_{n+3}. \tag{9}$$

Proof. Since $\Delta(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n)$ is decreasing, we find that

$$\Delta(p_s|\Delta^2 x_s|^{\alpha-1}\Delta^2 x_s) \;\leq\; \Delta(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n), \quad s \geq n.$$

Summing the last inequality from n to s-1, and using the fact $\Delta^2 x_n < 0$, we obtain

$$\Delta^2 x_s \leq -|\Delta(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n)|^{\frac{1}{\alpha}} \left(\frac{s-n+1}{p_s}\right)^{\frac{1}{\alpha}}, \quad s \geq n.$$

Since the $\lim\limits_{n\to\infty} \Delta x_n = \eta \geq 0$ is finite, and noting that $\frac{1}{\alpha} \geq 1$, summing the last inequality from n to ∞, we have

$$\begin{aligned}
\Delta x_n \;&\geq\; |\Delta(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n)|^{\frac{1}{\alpha}} \sum_{s=n}^{\infty} \left(\frac{s-n+1}{p_s{}^{\frac{1}{\alpha}}}\right) \\
&=\; \left|\Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right)\right|^{\frac{1}{\alpha}} \;\psi_n \\
&\geq\; \left|\Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right)\right|^{\frac{1}{\alpha}} \;\psi_{n+3}.
\end{aligned}$$

Combining the last inequality and the inequality (6), we obtain (9). This completes the proof.

Lemma 5. *Let $\beta < 1 \leq \alpha$. Then the condition*

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{p_n} \sum_{s=n_0}^{n-1} (n-s)\, s^{\beta} q_s \right)^{\frac{1}{\alpha}} = \infty, \tag{10}$$

implies

$$\sum_{n=n_0}^{\infty} n^{\frac{\beta}{\alpha}}\ \psi_{n+1}^{\beta}\ q_n = \infty. \tag{11}$$

Proof. If (10) holds, then for any $N > n_0$, we have

$$\sum_{n=N}^{\infty} p_n{}^{\frac{-1}{\alpha}} \left(\sum_{s=N}^{n-1} (n-s)\, s^{\beta} q_s \right)^{\frac{1}{\alpha}} = \infty, \tag{12}$$

and choose $N > 1$, such that $\psi_n \leq 1$ for $n \geq N$. In view of (5), the equation (12) implies that

$$\lim_{n\to\infty} \sum_{s=N}^{n-1} (n-s)\, s^{\beta} q_s = \infty.$$

Since $\{n\}$ is increasing and $\lim\limits_{n\to\infty} n = \infty$, then by Stolz's theorem [4], we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{\sum\limits_{s=N}^{n-1} (n-s)\, s^{\beta} q_s}{n} &= \lim_{n\to\infty} \frac{\Delta\left(\sum\limits_{s=N}^{n-1} (n-s)\, s^{\beta} q_s\right)}{(\Delta n)} \\
&= \lim_{n\to\infty} \sum_{s=N}^{n} s^{\beta} q_s \in (0,\infty].
\end{aligned}$$

Therefore, there exists a constant $d > 0$, and an integer $N_1 > N$, such that

$$\sum_{s=N}^{n} (n-s)\, s^{\beta} q_s \geq d\, n, \quad \text{for every } n \geq N_1. \tag{13}$$

For all $n \geq N_1$, using summation by parts, we obtain

$$\sum_{s=N_1}^{n-1} p_s{}^{\frac{-1}{\alpha}} \left(\sum_{t=N}^{s-1} (s-t)\, t^{\beta} q_t \right)^{\frac{1}{\alpha}} = \sum_{s=N_1}^{n-1} \Delta^2 \psi_s \left(\sum_{t=N}^{s-1} (s-t)\, t^{\beta} q_t \right)^{\frac{1}{\alpha}}$$

$$= \left[\Delta\psi_s \left(\sum_{t=N}^{s-1} (s-t)\, t^{\beta} q_t \right)^{\frac{1}{\alpha}} \right]_{N_1}^{n} - \sum_{s=N_1}^{n-1} \Delta\psi_{s+1}\, \Delta \left(\sum_{t=N}^{s-1} (s-t)\, t^{\beta} q_t \right)^{\frac{1}{\alpha}}. \tag{14}$$

By Mean value theorem, we have

$$\Delta\left(\sum_{t=N}^{s-1}(s-t)\,t^{\beta}q_t\right)^{\frac{1}{\alpha}} \leq \frac{1}{\alpha}\left(\sum_{t=N}^{s}(s-t)\,t^{\beta}q_t\right)^{\frac{1}{\alpha}-1}\sum_{t=N}^{s}t^{\beta}q_t. \qquad (15)$$

From the inequalities (14) and (15), using (13), and the fact that $\frac{1}{\alpha} \leq 1$ as well as that $\Delta\psi_n$ is a negative function , we obtain

$$\sum_{s=N_1}^{n-1} p_s^{\frac{-1}{\alpha}}\left(\sum_{t=N}^{s-1}(s-t)\,t^{\beta}q_t\right)^{\frac{1}{\alpha}} \leq d_1 - d_2\sum_{s=N_1}^{n-1}\Delta\psi_{s+1}\; s^{\frac{1}{\alpha}-1}\sum_{t=N}^{s}t^{\beta}q_t, n \geq N_1; \qquad (16)$$

where

$$d_1 = -\Delta\psi_{N_1}\left(\sum_{t=N}^{N_1-1}(N_1-t)\,t^{\beta}q_t\right)^{\frac{1}{\alpha}} > 0, \quad d_2 = \frac{1}{\alpha}d^{\frac{1}{\alpha}-1} > 0.$$

Now, from (16), using the fact that

$$0 \leq \frac{(1-\alpha)(\beta-1)}{\alpha} = \frac{\beta}{\alpha} - \left(\frac{1}{\alpha} - 1 + \beta\right)$$

we have

$$\begin{aligned}
\sum_{s=N_1}^{n-1} p_s^{\frac{-1}{\alpha}}\left(\sum_{t=N}^{s-1}(s-t)\,t^{\beta}q_t\right)^{\frac{1}{\alpha}} &\leq d_1 - d_2\sum_{s=N_1}^{n-1}\Delta\psi_{s+1}\left(\sum_{t=N}^{s}t^{\frac{\beta}{\alpha}}q_t\right)\\
&= d_1 + d_2\left[\left(-\psi_{s+1}\sum_{t=N}^{s-1}t^{\frac{\beta}{\alpha}}q_t\right)_{N_1}^{n} + \sum_{s=N_1}^{n-1}\psi_{s+1}\; s^{\frac{\beta}{\alpha}}\; q_s\right]\\
&\leq d_3 + d_2\sum_{s=N_1}^{n-1}\psi_{s+1}\; s^{\frac{\beta}{\alpha}}\; q_s\\
&\leq d_3 + d_2\sum_{s=N_1}^{n-1}s^{\frac{\beta}{\alpha}}\;\psi_{s+1}^{\beta}\; q_s, \quad n \geq N_1,
\end{aligned}$$

where $d_3 = d_1 + d_2\;\psi_{N_1+1}\sum_{t=N_1}^{N_1-1}t^{\frac{\beta}{\alpha}}\;q_t$. Letting $n \to \infty$, we conclude that (10) implies (11). The proof is now complete.

In the following, we state and prove some useful propositions that play an important role in proving the main results given in Section 3.

Proposition 1. *Let $\beta > \alpha$. If there exists a positive solution $\{x_n\}$ of the equation (1) of type (II), then*

$$\sum_{n=n_0}^{\infty} n\ q_n\ \psi_{n+3}^{\beta} < \infty. \tag{17}$$

Proof. Let $\{x_n\}$ be a positive solution of equation (1) of type (II). Then we have $x_n \sim c\psi_n$ as $n \to \infty$ $(0 < c < \infty)$. Otherwise if $\{x_n\}$ is of type (I) , (III), or (IV) then $\lim\limits_{n\to\infty} x_n = c_0 \in (0, \infty]$. Moreover $\lim\limits_{n\to\infty} \psi_n = 0$. Hence we have $\lim\limits_{n\to\infty} \frac{x_n}{\psi_n} = \infty$, a contradiction . Therefore there is an integer $N \geq n_0$ such that

$$d_4\psi_n \leq x_n \leq 2\ d_4\psi_n, \tag{18}$$

for all $n \geq N$. Now multiply the equation (1) by n, and summing from N to n-1, we obtain

$$n\ \Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right) - p_{n+1}\left(\Delta^2 x_{n+1}\right)^{\alpha} + \sum_{s=N}^{n-1} s\ q_s\ x_{s+3}^{\beta} = c, \tag{19}$$

where c is a constant. From (7) and (18), we have

$$p_n\left(\Delta^2 x_n\right)^{\alpha} \leq 2^{\alpha}\ d_4^{\alpha} \tag{20}$$

for $n \geq N$. From (19) and (20) and the fact that $\Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right) > 0$, we have

$$d_4^{\beta} \sum_{n=n_0}^{\infty} n\ q_n\ \psi_{n+3}^{\beta} < \infty,$$

which gives (17). This completes the proof.

Proposition 2. *Let $\beta < \alpha$. If there exists a positive solution $\{x_n\}$ of the equation (1) of type (II), then*

$$\sum_{n=n_0}^{\infty} n^{\frac{\beta}{\alpha}}\ q_n\ \psi_{n+3}^{\beta} < \infty. \tag{21}$$

Proof. For a positive solution $\{x_n\}$ of equation (1), let us choose $N \geq n_0$ such that type(II) and (8) hold for all $n \geq N$. Denote by

$$A_n = \Delta\left(p_n\left(\Delta^2 x_n\right)^{\alpha}\right).$$

Then, from the equation (1), and using Mean value theorem, we have

$$\begin{aligned}
\Delta\left(A_n^{(1-\frac{\beta}{\alpha})}\right) &= \left(1-\frac{\beta}{\alpha}\right) t^{\frac{-\beta}{\alpha}}\ \Delta\left(A_n\right), \quad A_{n+1} < t < A_n, \\
&\le -\frac{(\alpha-\beta)}{\alpha}\ q_n\ x_{n+3}^{\beta}\ A_n^{\frac{-\beta}{\alpha}} \\
&\le -\frac{(\alpha-\beta)}{\alpha}\ c^{\frac{\beta}{\alpha}}\ n^{\frac{\beta}{\alpha}}\ \psi_{n+3}^{\beta}\ q_n,
\end{aligned}$$

where we have used the inequality (8). Summing the last inequality from N to n-1, we obtain

$$\frac{(\alpha-\beta)}{\alpha}\ c^{\frac{\beta}{\alpha}} \sum_{s=N}^{n-1} s^{\frac{\beta}{\alpha}}\ q_s\ \psi_{s+3}^{\beta} < A_N^{1-\frac{\beta}{\alpha}} < \infty,$$

which gives (21). This completes the proof.

Proposition 3. *Let $\alpha < 1 \le \beta$. If there exists a positive solution $\{x_n\}$ of the equation (1) of type (IV), then*

$$\sum_{n=n_0}^{\infty} n\ q_n\ \psi_{n+3}^{\beta} < \infty. \tag{22}$$

Proof. For a positive solution $\{x_n\}$ of equation (1), let us choose an integer $N \ge n_0$ such that type (IV) and (9) hold for all $n \ge N$. Denote by $A_n = \Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right)$. Then, from the equation (1), and using Mean value theorem, we have

$$\begin{aligned}
-\Delta\left(|A_n|^{1-\frac{\beta}{\alpha}}\right) &\ge -\frac{(\alpha-\beta)}{\alpha}\ |A_n|^{\frac{-\beta}{\alpha}}\ \left(-\Delta A_n\right) \\
&= -\frac{(\alpha-\beta)}{\alpha}\ |A_n|^{\frac{-\beta}{\alpha}}\ q_n\ x_{n+3}^{\beta}.
\end{aligned}$$

From the last inequality and (9), we see that $c_2 > 0$ exists such that

$$-\Delta\left(|A_n|^{1-\frac{\beta}{\alpha}}\right) \ge \frac{(\beta-\alpha)}{\alpha} |A_n|^{\frac{-\beta}{\alpha}}\ c_2^{\beta}\ n^{\beta}\ q_n\ |A_n|^{\frac{\beta}{\alpha}}\ \psi_{n+3}^{\beta}, \quad n \ge N.$$

Using the fact that $\beta \ge 1$, that is, $n^{\beta} \ge n$ for all $n \ge max\{1, N\} = N_1$, we have

$$-\Delta\left(|A_n|^{1-\frac{\beta}{\alpha}}\right) \ge \frac{(\beta-\alpha)}{\alpha}\ c_2^{\beta}\ n\ q_n\ \psi_{n+3}^{\beta}, \quad n \ge N_1.$$

Summing the last inequality from N_1 to $n-1$, we obtain

$$-|A_n|^{1-\frac{\beta}{\alpha}} + |A_{N_1}|^{1-\frac{\beta}{\alpha}} \ge \frac{(\beta-\alpha)}{\alpha}\ c_2^{\beta} \sum_{s=N_1}^{n-1} s\ q_s\ \psi_{s+3}^{\beta},$$

which gives (22). This completes the proof.

Proposition 4. *Let $\beta < 1 \le \alpha$. If there exists a positive solution $\{x_n\}$ of equation (1) of type (IV), then*

$$\sum_{n=n_0}^{\infty} p_n^{\frac{-1}{\alpha}} \left(\sum_{s=n_0}^{n-1} (n-s)\, s^{\beta} q_s \right)^{\frac{1}{\alpha}} < \infty. \tag{23}$$

Proof. Assume that for a positive solution $\{x_n\}$ of equation (1), type (IV) holds for all $n \ge N \in \mathbb{N}(n_0)$. By Lemma 2, there exists a constant $c_1 > 0$, and an integer $N_1 \ge N$, such that (6) holds for all $n \ge N_1$. Summing the equation (1) from N_1 to $n-1$, we see that

$$-\Delta \left(p_n |\Delta^2 x_n|^{\alpha-1} \Delta^2 x_n \right) \ge \sum_{s=N_1}^{n-1} q_s\, x_{s+3}^{\beta}, \quad n \ge N_1.$$

Summing the last inequality from N_1 to n-1, we have

$$-\Delta^2 x_n \ge \frac{1}{p_n^{\frac{1}{\alpha}}} \left(\sum_{s=N_1}^{n-1} (n-s)\ q_s\ x_{s+3}^{\beta} \right)^{\frac{1}{\alpha}}, \quad n \ge N_1.$$

From the last inequality, we see that

$$\begin{aligned} -\Delta \left((\Delta x_n)^{1-\beta} \right) &\ge (1-\beta)(\Delta x_n)^{-\beta}(-\Delta^2 x_n) \\ &\ge (1-\beta)(\Delta x_n)^{-\beta} \frac{1}{p_n^{\frac{1}{\alpha}}} \left(\sum_{s=N_1}^{n-1} (n-s)\ q_s\ x_{s+3}^{\beta} \right)^{\frac{1}{\alpha}}, \quad n \ge N_1. \end{aligned}$$

Since $\{\Delta x_n\}$ is decreasing, that is, $(\Delta x_n)^{-\beta} \ge (\Delta x_s)^{-\beta}$ for $n \ge s$, we have

$$\begin{aligned} -\Delta \left((\Delta x_n)^{1-\beta} \right) &\ge (1-\beta) \frac{1}{p_n^{\frac{1}{\alpha}}} \left(\sum_{s=N_1}^{n-1} (n-s)\ q_s\ (\Delta x_s)^{-\beta\alpha}\ x_{s+3}^{\beta} \right)^{\frac{1}{\alpha}} \\ &= (1-\beta) \frac{1}{p_n^{\frac{1}{\alpha}}} \left(\sum_{s=N_1}^{n-1} (n-s)\ q_s\ (s\Delta x_s)^{-\beta\alpha}\ s^{\beta\alpha}\ x_{s+3}^{\beta} \right)^{\frac{1}{\alpha}}, \quad n \ge N_1. \end{aligned}$$

Now using (6), we obtain

$$-\Delta \left((\Delta x_n)^{1-\beta} \right) \ge (1-\beta)\ \frac{c_1^{\beta}}{p_n^{\frac{1}{\alpha}}} \left(\sum_{s=N_1}^{n-1} (n-s)\ q_s\ x_{s+3}^{-\beta\alpha}\ s^{\beta\alpha}\ x_{s+3}^{\beta} \right)^{\frac{1}{\alpha}}, \quad n \ge N_1.$$

Since $\Delta x_{n+3} \le \Delta x_{N_1}, \quad n \ge N_1$, there exists some constant $c > 0$, and an integer $N_2 \ge N_1$, such that $x_{n+3} \le c\ n$ for $n \ge N_2$. Therefore, the fact that

$\beta(1-\alpha) \le 0$ implies $x_{n+3}^{\beta(1-\alpha)} \ge c^{\beta(1-\alpha)} n^{\beta(1-\alpha)}$ for $n \ge N_2$. Hence we have

$$-\Delta\left((\Delta x_n)^{1-\beta}\right) \ge (1-\beta)c^{\frac{\beta(1-\alpha)}{\alpha}} \frac{c_1^\beta}{p_n^{\frac{1}{\alpha}}} \left(\sum_{s=N_2}^{n-1} (n-s)\ q_s\ s^{\beta(1-\alpha)}\ s^{\beta\alpha}\right)^{\frac{1}{\alpha}},\quad n \ge N_2.$$

Summing the last inequality from N_2 to ∞, we obtain

$$(\Delta x_{N_2})^{1-\beta} \ge K \sum_{n=N_2}^{\infty} \frac{1}{p_n^{\frac{1}{\alpha}}} \left(\sum_{s=N_2}^{n-1} (n-s)\ q_s\ s^\beta\right)^{\frac{1}{\alpha}},$$

where $K = (1-\beta)c^{\frac{\beta(1-\alpha)}{\alpha}}\ c_1^\beta > 0$, which gives (23). This completes the proof.

3 Oscillation Theorems

In this section, we state and prove criteria for the oscillation of all solutions of equation (1).

Theorem 1. *Let $\beta \ge 1 > \alpha$. If*

$$\sum_{n=n_0}^{\infty} n q_n \psi_{n+3}^\beta = \infty, \tag{24}$$

then every solution of equation (1) is oscillatory.

Proof. Assume, to contrary, that $\{x_n\}$ is a positive solution of equation(1). Then $\{x_n\}$ falls into one of the four types (I)-(IV) mentioned in Lemma 1. Therefore it is enough to show that in each case, we are led to a contradiction to (24).

Case (I): Let $\{x_n\}$ be a positive solution of equation (1) of type (I) for all $n \ge N$. Since $\Delta x_n \ge \Delta x_N,\ \ n \ge N$, there exists some constant $c > 0$, and $N_1 \ge N$, such that $x_n \ge cn$ for $n \ge N_1$. Summing the equation (1) from N_1 to ∞ and using the last inequality we have

$$c^\beta \sum_{n=N_1}^{\infty} n^\beta q_n \le \sum_{n=N_1}^{\infty} x_n^\beta q_n \le \Delta\left(p_{N_1}|\Delta^2 x_{N_1}|^{\alpha-1}\Delta^2 x_{N_1}\right).$$

Hence, we conclude that

$$\sum_{n=N_1}^{\infty} n^\beta q_n < \infty. \tag{25}$$

On the other hand, from (5),there exists an integer$N_2 \geq N_1$, such that $\psi_{n+3} \leq 1$ for $n \geq N_2$. The fact that $\beta \geq 1$, implies $n^\beta \geq n$ for $n \geq N_3 = max\{N_2, 1\}$, and consequently, we have

$$\sum_{n=N_3}^{\infty} n\ q_n\ \psi_{n+3}^{\beta} \leq \sum_{n=N_3}^{\infty} n\ q_n\ \leq \sum_{n=N_3}^{\infty} n^\beta q_n < \infty. \tag{26}$$

by (25), a contradiction to (24).

Case(II): If there exists a positive solution $\{x_n\}$ of type(II), then from Proposition 1, we see that condition (24) fails to hold.

Case(III): Let $\{x_n\}$ be a positive solution of equation (1) of type (III) for all $n \geq N$. Multiplying equation (1) by n, and summing the resulting equation from N to n-1, using summation by parts, we have

$$\sum_{s=N}^{n-1} s\ q_s\ x_{s+3}^{\beta} = c_3 - n\Delta\left(p_n|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) + p_{n+1}|\Delta^2 x_{n+1}|^{\alpha-1}\Delta^2 x_{n+1} < c_3,$$

where $c_3 = N\Delta\left(p_N|\Delta^2 x_N|^{\alpha-1}\Delta^2 x_N\right) - p_{N+1}|\Delta^2 x_{N+1}|^{\alpha-1}\Delta^2 x_{N+1} > 0$ is constant. Hence, we conclude that

$$\sum_{n=n_0}^{\infty} n\ q_n\ x_{n+3}^{\beta} < \infty.$$

Since $\{x_n\}$ is increasing, this implies that

$$\sum_{n=n_0}^{\infty} n\ q_n < \infty. \tag{27}$$

Combining (26) and (27) we are led to contradiction with the assumption (24).

Case (IV): If there exists a positive solution $\{x_n\}$ of type (IV), then from Proposition 3, we conclude that (24) is not satisfied. This completes the proof of the theorem.

Theorem 2. *Let $\beta < 1 \leq \alpha$. If*

$$\sum_{n=n_0}^{\infty}\left(\frac{1}{p_n}\sum_{s=n_0}^{n-1}(n-s)s^\beta q_s\right)^{\frac{1}{\alpha}} = \infty, \tag{28}$$

then every solution of equation (1) is oscillatory .

Proof. Assume, to contrary, that $\{x_n\}$ is a positive solution of equation (1). Then $\{x_n\}$ falls into one of the four types (I)-(IV) mentioned in Lemma 1. Therefore it is enough to show that in each case, we are led to a contradiction to (28).

Case (I): Let $\{x_n\}$ be a positive solution of equation (1) of type (I) for all $n \geq N$. As in the proof of Theorem 1, we can obtain (25). Then

$$\sum_{s=n_0}^{n-1}(n-s)s^{\beta}q_s \leq (n-n_0)\sum_{s=n_0}^{n-1}s^{\beta}q_s \leq Q\;n \quad \text{for} \quad n \geq N,$$

where $Q = \sum\limits_{n=n_0}^{\infty} n^{\beta}q_n$. Using the last inequality, we see that

$$\sum_{n=N}^{\infty}\frac{1}{p_n^{\frac{1}{\alpha}}}\left[\sum_{s=n_0}^{n-1}(n-s)s^{\beta}q_s\right]^{\frac{1}{\alpha}} \leq Q^{\frac{1}{\alpha}}\sum_{n=N}^{\infty}\left(\frac{n}{p_n}\right)^{\frac{1}{\alpha}}. \tag{29}$$

In view of (5), (29) implies that (28) fails to hold.

Case(II): If there exists a positive solution $\{x_n\}$ of type (II), then from Proposition 2, we conclude that (28) is not satisfied.

Case (III): Let $\{x_n\}$ be a positive solution of equation (1) of type (III) for all $n \geq N$. As in the proof of Theorem 1, we can obtain (27). Using (27) and the fact that $\beta < 1$, implies $n^{\beta} < n$ for all $n \geq max\{1, n_0\} = N_1$, we get (25). Now, following the same arguments as in case (I), using (25), we are led to a contradiction to (28).

Case (IV): If there exists a positive solution $\{x_n\}$ of type (IV), then from Proposition 4, we find that condition (28) fails to hold. This completes the proof of the theorem.

4 Example

In this section, we present some examples to illustrate the results given in the previous section.

Example 1. Consider the equation

$$\Delta^2\left(n^{\mu}|\Delta^2 x_n|^{\alpha-1}\Delta^2 x_n\right) + n^{-\lambda}|x_{n+3}|^{\beta-1}x_{n+3} = 0, \quad n \in \mathbb{N}(n_0). \tag{30}$$

Let $\beta \geq 1 > \alpha$ and $\mu > 1+\alpha$. The assumption $\mu > 1+\alpha$ ensures that (5) is satisfied for the function $p_n = n^{\mu}$. Then by using Theorem 1, we obtain that the equation (1) is oscillatory if (24) is satisfied .

Since $\psi_n = \sum\limits_{s=n}^{\infty} \frac{(s-n+1)}{p_s^{\frac{1}{\alpha}}} \sim n^{2-\frac{\mu}{\alpha}},\ n \to \infty$,it follows that

$$\sum_{n=n_0}^{\infty} n\ q_n\ \psi_{n+3}^{\beta} \sim \sum_{n=n_0}^{\infty} n^{1-\lambda+\beta(2-\frac{\mu}{\alpha})},\ n \to \infty.$$

Thus, the condition (24) holds if $2-\lambda+\beta(2-\frac{\mu}{\alpha}) > 0$. Therefore the equation (30) is oscillatory, if $\lambda < 2+\beta(2-\frac{\mu}{\alpha})$. Using the assumptions $\beta \geq 1 > \alpha$ and $\mu > 1+\alpha$, we have $\lambda < 2+(1-\frac{1}{\alpha})\beta < 2$.

Let $\beta < 1 \leq \alpha$. The assumption $\mu > 2\alpha$ ensures that (5) is satisfied.Then by using Theorem 2, we obtain that the equation (30) is oscillatory, if condition (28) is satisfied. For $q_n = n^{-\lambda}$, it is easy to see that

$$\sum_{s=n_0}^{n-1} \sum_{t=n_0}^{s-1} t^{\beta} q_t \sim n^{\beta-\lambda+2}, \quad n \to \infty,$$

and so ,

$$\sum_{n=n_0}^{\infty} \left(\frac{1}{p_n} \sum_{s=n_0}^{n-1} (n-s) s^{\beta} q_s \right)^{\frac{1}{\alpha}} \sim \sum_{n=n_0}^{\infty} n^{\frac{\beta-\lambda+2-\mu}{\alpha}}, \quad n \to \infty.$$

Then the condition (28) holds if $\lambda < \alpha+\beta-\mu+2$. Thus, the equation (30) is oscillatory if $\lambda < \alpha+\beta-\mu+2$. Using the assumptions $\beta < 1 \leq \alpha$ and $\mu > 2\alpha$, we obtain that $\lambda < \beta-\alpha+2 < 2$.

Example 2. Consider the difference equation

$$\Delta^2 \left(2^n (\Delta^2 x_n)^{\frac{1}{3}} \right) + 2^{3n+9} 9^{\frac{2}{3}} x_{n+3} = 0, \quad n \geq 1. \tag{31}$$

Here $p_n = 2^n, q_n = 2^{3n+9} 9^{\frac{2}{3}}, \alpha = \frac{1}{3}, \beta = 1$. It is easy to see that all conditions of Theorem 1 are satisfied and hence every solution of equation (31) is oscillatory. In fact $\{x_n\} = \left\{ \frac{(-1)^n}{2^{3n}} \right\}$ is one such solution of equation (31).

Example 3. Consider the difference equation

$$\Delta^2 \left(n^7 (\Delta^2 x_n)^3 \right) + 64 \left((n+2)^7 + 2(n+1)^7 + n^7 \right) x_{n+3}^{\frac{1}{3}} = 0, \quad n \geq 1. \tag{32}$$

Here $p_n = n^7, q_n = 64 \left((n+2)^7 + 2(n+1)^7 + n^7 \right), \alpha = 3,\ \beta = \frac{1}{3}$.
It is easy to see that all conditions of Theorem 2 are satisfied and hence all solutions of equation (32) are oscillatory. In fact $\{x_n\} = \{(-1)^n\}$ is one such solution of equation (32).
We conclude this paper with the following remark.

Remark 2. The results obtained in this paper gives a partial answer to the problem given in [10]. Further the results reduces to that of in [12] when $\alpha = 1$.

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