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Oscillation of solutions of some higher order linear differential equations*

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Abstract

In this paper, we deal with the order of growth and the hyper order of solutions of higher order linear differential equations

$$f^{(k)} + B_{k-1}f^{(k-1)} + \cdots + B_1 f' + B_0 f = F$$

where $B_j(z)\,(j = 0, 1, \ldots, k-1)$ and F are entire functions or polynomials. Some results are obtained which improve and extend previous results given by Z.-X. Chen, J. Wang, T.-B. Cao and C.-H. Li.
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1 Introduction and Main Results

We shall assume that reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions(see [11,14]). In addition, we will use the notation $\sigma(f)$ to denote the order of growth of entire function $f(z)$, $\sigma_2(f)$ to denote the hyper-order of $f(z)$, $\lambda(f)(\lambda_2(f))$ to denote the exponent(hyper-exponent) of convergence of the zero-sequence of $f(z)$ and $\overline{\lambda}(f)(\overline{\lambda}_2(f))$ to denote exponent(hyper-exponent) of convergence of distinct zero sequence of meromorphic function $f(z)$. We also define

$$\overline{\lambda}(f-\varphi) = \limsup_{r\to\infty} \frac{\log \overline{N}(r, \frac{1}{f-\varphi})}{\log r}, \quad \textbf{and} \quad \overline{\lambda}_2(f-\varphi) = \limsup_{r\to\infty} \frac{\log\log \overline{N}(r, \frac{1}{f-\varphi})}{\log r},$$

for any meromorphic function $\varphi(z)$.

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For a set $E \subset R^+$, let $m(E)$, respectively $m_l(E)$, denote the linear measure, respectively the logarithmic measure of E. By $\chi_E(t)$, we denote the characteristic function of E. Moreover, the upper logarithmic density and the lower logarithmic density of E are defined by

$$\overline{\log dens}(E) = \limsup_{r\to\infty} \frac{m_l(E\cap[1,r])}{\log r}, \qquad \underline{\log dens}(E) = \liminf_{r\to\infty} \frac{m_l(E\cap[1,r])}{\log r}.$$

Observe that E may have a different meaning at different occurrences in what follows.

We now recall some previous results concerning linear differential equations

$$f'' + e^{-z}f' + Q(z)f = 0, \tag{1}$$

where $Q(z)$ is an entire function of finite order. It is well known that each solution f of (1) is an entire function and that if f_1 and f_2 are any two linearly independent solutions of (1), then at least one of f_1, f_2 must have infinitely order(see [13, P167-168]). Hence, "most" solutions of (1) will have infinite order. But the equation (1) with $Q(z) = -(1+e^{-z})$ possesses a solution $f = e^z$ of finite order.

Thus a natural question is: what condition on $Q(z)$ will guarantee that every solution $f \not\equiv 0$ of (1) has infinite order? Many authors, such as Amemiya and Ozawa [1], Gundersen [10] and Langley [15], Frei [6], Ozawa [20] have studied the problem. They proved that when $Q(z)$ is a nonconstant polynomial or $Q(z)$ is a transcendental entire function with order $\sigma(Q) \neq 1$, then every solution $f \not\equiv 0$ of (1) has infinite order.

For the above question, some mathematicians investigated the second order linear differential equations and obtained many results (see REF.[2,3,5,6,10,15,16,20,24]). In 2002, Chen [3] considered the question: what condition on $Q(z)$ when $\sigma(Q) = 1$ will guarantee every nontrivial solution of (1) has infinite order? He proved the following result, which greatly extended and improved results of Frei, Ozawa, Langley and Gundersen.

Theorem A(see. [3]) *Let $A_j(z)(\not\equiv 0)(j = 0, 1)$ be an entire function with $\sigma(A_j) < 1$. Suppose a, b are complex constants such that $ab \neq 0$ and $a = cb(c > 1)$. Then every nontrivial solution f of*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = 0 \tag{2}$$

has infinite order.

Recently, some mathematicians investigate the non-homogeneous equations of second order and higher order linear equations such as Li and Wang [18], Cao [5], Wang and Laine [22] and proved that every solution of these equation has infinite order.

In 2008, Li and Wang [18] investigated the non-homogeneous equation related to (1) in the case when $Q(z) = h(z)e^{bz}$, where $h(z)$ is a transcendental entire function of order $\sigma(h) < \frac{1}{2}$, and b is a real constant and obtained the following results.

Theorem B(see. [18]) *If $Q(z) = h(z)e^{bz}$, where $h(z)$ is a transcendental entire function of order $\sigma(h) < \frac{1}{2}$, and b is a real constant. Then all nontrivial solutions f of equation*

$$f'' + e^{-z}f' + Q(z)f = H(z)$$

satisfies $\sigma(f) = \overline{\lambda}(f - z) = \infty$, provided that $\sigma(H) < 1$.

In 2008, Wang and Laine [22] investigated the non-homogeneous equation related to (2) and obtained the following result.

Theorem C(see. [22, Theorem 1.1]) *Suppose that* $A_j \not\equiv 0 (j = 0, 1)$, H *are entire functions of order less than one, and the complex constants* a, b *satisfy* $ab \neq 0$ *and* $a \neq b$. *Then every nontrivial solution* f *of equation*

$$f'' + A_1(z)e^{az}f' + A_0(z)e^{bz}f = H(z)$$

is of infinite order.

For equation (2), Li and Huang [17], Tu and Yi [21], Chen and Shon [4] and Gan and Sun [7] investigated the higher order homogeneous and non-homogeneous linear differential equations and obtained many results. In 2009, Chen and Xu [23] investigated the higher order non-homogeneous linear differential equations and obtained the following result.

Theorem D(see. [23, Theorem 1.5]) *Let* $k \geq 2$, $s \in \{1, \dots, k-1\}$, $h_0 \not\equiv 0, h_1, \dots, h_{k-1}$ *be meromorphic functions and* $\sigma = \max\{\sigma(h_j) : j = 1, \dots, k-1\} < n$; $P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ *and* $Q(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0$ *be two nonconstant polynomials, where* $a_i, b_i (i = 0, 1, 2, \dots, n)$ *with* $a_n \neq 0, b_n \neq 0$; $F \not\equiv 0$ *be an meromorphic function of finite order. Suppose all poles of* f *are of uniformly bounded multiplicity and if at least one of the following statements hold*

1. If $a_n = b_n$, and $\deg(P - Q) = m \geq 1, \sigma < m$;

2. If $a_n = cb_n$ with $c > 1$, and $\deg(P - Q) = m > 1, \sigma < m$;

3. If $\sigma < \sigma(h_0) < 1/2, a_n = cb_n$ with $c \geq 1$ and $P(z) - cQ(z)$ is a constant,

then all solutions f *of non-homogeneous linear differential equation*

$$f^{(k)} + h_{k-1}f^{(k-1)} + \cdots + h_s e^{P(z)}f^{(s)} + \cdots + h_1 f' + h_0 e^{Q(z)}f = F, \tag{3}$$

with at most one exceptional solution f_0 *of finite order, satisfy*

$$\lambda(f) = \overline{\lambda}(f) = \sigma(f) = \infty, \qquad \lambda_2(f) = \overline{\lambda}_2(f) = \sigma_2(f).$$

Furthermore, if such an exceptional solution f_0 *of finite order of (1.3) exists, then we have*

$$\sigma(f_0) \leq \max\{n, \sigma(F), \overline{\lambda}(f_0)\}.$$

We find that there is an exceptional possible solution with finite order for equation (3). It is natural to ask the following question: what condition on the coefficients of equation

$$f^{(k)} + B_{k-1}(z)f^{(k-1)} + \cdots + B_1(z)f' + B_0(z)f = F \tag{4}$$

when $F \not\equiv 0$ will guarantee every nontrivial solution has infinite order?

The main purpose of this paper is to study the above problem and the relation between small functions and solutions of higher order linear differential equation related to (4). We will prove the following results.

Theorem 1.1 *Let $P(z)$ and $Q(z)$ be a nonconstant polynomials as above, for some complex numbers $a_i, b_i, (i = 0, 1, \ldots, n)$ with $a_n b_n \neq 0$ and $a_n \neq b_n$. Suppose that $h_{i-1}(2 \leq i \leq k-1)$ are polynomials of degree no more $n-1$ in z, $A_j(z) \not\equiv 0$ $(j = 0, 1)$ and $H(z)$ are entire functions satisfying $\sigma := \max\{\sigma(A_j), j = 0, 1\} < n$ and $\sigma(H) < n$, and $\varphi(z)$ is an entire function of finite order. Then every nontrivial solution f of equation*

$$f^{(k)} + h_{k-1} f^{(k-1)} + \cdots + h_2 f'' + A_1 e^{P(z)} f' + A_0 e^{Q(z)} f = H \tag{5}$$

satisfies $\sigma(f) = \infty$, $\sigma(f) = \lambda(f) = \overline{\lambda}(f) = \overline{\lambda}(f - \varphi) = \infty$ and $\sigma_2(f) = \lambda_2(f) = \overline{\lambda}_2(f) = \overline{\lambda}_2(f - \varphi) \leq n$.

Remark 1.1 *We can see that the conclusions of Theorem 1.1 improve Theorem D and extend Theorem B and Theorem C.*

Theorem 1.2 *Suppose that $A_j(z) \not\equiv 0, D_j(z)(j = 0, 1)$, and $H(z)$ are entire functions satisfying $\sigma(A_j) < n, \sigma(D_j) < n(j = 0, 1)$, and $\sigma(H) < n$, and $P(z), Q(z)$, $h_{i-1}(2 \leq i \leq k-1)$ are as in Theorem 1.1 satisfying $a_n b_n \neq 0$ and $a_n b_n < 0$. Then every nontrivial solution f of equation*

$$f^{(k)} + h_{k-1} f^{(k-1)} + \cdots + h_2 f'' + (A_1 e^{P(z)} + D_1) f' + (A_0 e^{Q(z)} + D_0) f = H \tag{6}$$

is of infinite order.

Remark 1.2 *From Theorem 1.1 and Theorem 1.2, we give an answer to the above question.*

2 Some Lemmas

To prove the theorems, we need the following lemmas:

Lemma 2.1 (see. [24, Lemma 1.10]) *Let $f_1(z)$ and $f_2(z)$ be nonconstant meromorphic functions in the complex plane and c_1, c_2, c_3 be nonzero constants. If $c_1 f_1 + c_2 f_2 \equiv c_3$, then*

$$T(r, f_1) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}(r, f_1) + S(r, f_1).$$

Lemma 2.2 (see. [3,19]) *Suppose that $P(z) = (\alpha + \beta i) z^n + \cdots$ (α, β are real numbers, $|\alpha| + |\beta| \neq 0$) is a polynomial with degree $n \geq 1$, that $A(z)(\not\equiv 0)$ is an entire function with $\sigma(A) < n$. Set $g(z) = A(z) e^{P(z)}, z = re^{i\theta}, \delta(P, \theta) = \alpha \cos n\theta - \beta \sin n\theta$. Then for any given $\varepsilon > 0$, there exists a set $H_1 \subset [0, 2\pi)$ that has the linear measure zero, such that for any $\theta \in [0, 2\pi) \backslash (H_1 \cup H_2)$, there is $R > 0$ such that for $|z| = r > R$, we have:*

(i) If $\delta(P, \theta) > 0$, then

$$\exp\{(1 - \varepsilon)\delta(P, \theta) r^n\} < |g(re^{i\theta})| < \exp\{(1 + \varepsilon)\delta(P, \theta) r^n\};$$

(ii) If $\delta(P, \theta) < 0$, then

$$\exp\{(1 + \varepsilon)\delta(P, \theta) r^n\} < |g(re^{i\theta})| < \exp\{(1 - \varepsilon)\delta(P, \theta) r^n\},$$

where $H_2 = \{\theta \in [0, 2\pi); \delta(P, \theta) = 0\}$ is a finite set.

Lemma 2.3 (see. [9]) *Let $f(z)$ be a transcendental meromorphic function of finite order $\sigma(f) = \sigma < \infty$, and let $\varepsilon > 0$ be a given constant. Then there exists a set $H \subset (1, \infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| \notin H \cup [0, 1]$ and for all $k, j, 0 \le j < k$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \le |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Similarly, there exists a set $E \subset [0, 2\pi)$ of linear measure zero such that for all $z = re^{i\theta}$ with $|z|$ sufficiently large and $\theta \in [0, 2\pi) \setminus E$, and for all $k, j, 0 \le j < k$, we have

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \le |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Lemma 2.4 (see. [22, Lemma 2.4]) *Let $f(z)$ be an entire function of finite order σ, and $M(r, f) = f(re^{i\theta_r})$ for every r. Given $\zeta > 0$ and $0 < C(\sigma, \zeta) < 1$, there exists a constant $0 < l_0 < \frac{1}{2}$ and a set E_ζ of lower logarithmic density greater than $1 - \zeta$ such that*

$$e^{-5\pi} M(r, f)^{1-C(\sigma,\zeta)} \le |f(re^{i\theta})|$$

for all $r \in E_\zeta$ large enough and all θ such that $|\theta - \theta_r| \le l_0$.

Lemma 2.5 (see. [8,12]) *Let $f(z)$ be a transcendental entire function, $\nu_f(r)$ be the central index of $f(z)$ and δ be a constant satisfying $0 < \delta < \frac{1}{8}$. Suppose z lying in the circle $|z| = r$ satisfies $|f(z)| > M(r, f)\nu_f(r)^{-\frac{1}{8}+\delta}$. Then except a set of r with finite logarithmic measure, we have*

$$\frac{f^{(j)}(z)}{f(z)} = \left\{ \frac{\nu_f(r)}{z} \right\}^j (1 + \eta_j(z)),$$

where $\eta_j(z) = O(\nu_f(r)^{-\frac{1}{8}+\delta}), j \in N$.

Lemma 2.6 (see. [22, Lemma 2.5]) *Let $f(z)$ and $g(z)$ be two nonconstant entire functions with $\sigma(g) < \sigma(f) < +\infty$. Given ε with $0 < 4\varepsilon < \sigma(f) - \sigma(g)$ and $0 < \delta < \frac{1}{8}$, there exists a set E with $\overline{\log dens}(E) > 0$ and a positive constant r_0 such that*

$$\left| \frac{g(z)}{f(z)} \right| \le \exp\{-r^{\sigma(f)-2\varepsilon}\}$$

for all z such that $r \in E$ is sufficiently large and that $|f(z)| \ge M(r, f)\nu_f(r)^{-\frac{1}{8}+\delta}$.

Lemma 2.7 (see. [12]) *Let $f(z)$ be an entire function of finite order $\sigma(f) = \sigma < \infty$, and let $\nu_f(r)$ be the central index of f. Then for any $\varepsilon(> 0)$, we have*

$$\limsup_{r\to\infty} \frac{\log \nu_f(r)}{\log r} = \sigma.$$

Lemma 2.8 (see. [16]) *Let $f(z)$ be an entire function of infinite order. Denote $M(r,f) = \max\{|f(z)| : |z| = r\}$, then for any sufficiently large number $\lambda > 0$, and any $r \in E \subset (1,\infty)$*

$$M(r,f) > c_1 \exp\{c_2 r^{\lambda}\},$$

where $m_l E = \infty$ and c_1, c_2 are positive constants.

Lemma 2.9 *Suppose $B_0, B_1, \ldots, B_{k-1}$ and $F(\not\equiv 0)$ are all entire functions of finite order and let $\varrho := \max\{\sigma(B_j), \sigma(F), j = 0, 1, \ldots, k-1\}$, $k \geq 2$. Then every solution f of infinite order of equation*

$$f^{(k)} + B_{k-1} f^{(k-1)} + \cdots + B_0 f = F$$

satisfies $\sigma_2(f) \leq \varrho$.

Proof: We rewrite the equation as

$$\frac{f^{(k)}}{f} = \frac{F}{f} - \sum_{j=1}^{k-1} B_j \frac{f^{(j)}}{f} - B_0.$$

Since $\varrho := \max\{\sigma(B_j), \sigma(F), j = 0, 1, \ldots, k-1\}$, by virtue of [2], for any positive number $\varepsilon(0 < \varepsilon < \sigma(F)+1)$ and $r \notin [0,1] \cup E_1$, we have

$$|B_j(z)| \leq \exp\{r^{\varrho+\varepsilon}\}, \quad |F(z)| \leq \exp\{r^{\sigma(F)+\varepsilon}\}, j = 0, 1, \ldots, k-1.$$

By Lemma 2.5, there exists a set $E_2 \subset (1, +\infty)$ satisfying $m_l E_2 < \infty$, taking z satisfying

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1 + o(1)) \quad (j = 0, 1, \ldots, k).$$

Since $\sigma(f) = \infty$, from Lemma 2.8 there exists $|z| = r \in H_1 \setminus ([0,1] \cup E_1 \cup E_2)$ satisfying $|f(z)| = M(r,f)$, for $\lambda > 2\sigma(F) + 1$, we have

$$\left(\frac{\nu_f(r)}{|z|}\right)^k (1+o(1)) \leq \frac{1}{c_1} \exp\{r^{\sigma(F)+\varepsilon} - c_2 r^{\lambda}\} + \exp\{r^{\varrho+\varepsilon}\} \left(\sum_{j=1}^{k-1} \left(\frac{\nu_f(r)}{|z|}\right)^j (1 + o(1)) + 1\right).$$

Thus, we have

$$\limsup_{r\to\infty, r\in H_1 \setminus ([0,1]\cup E_1 \cup E_2)} \frac{\log\log \nu_f(r)}{\log r} \leq \varrho + \varepsilon.$$

By the definition of hyper-order, we can get $\sigma_2(f) \leq \varrho$.
Therefore, we complete the proof of this lemma. □

3 The Proof of Theorem 1.1

Proof: **The growth of solutions** We first point out that $\sigma(f) \geq n$.

We rewrite (5) as

$$A_1 e^{P(z)} f' + A_0 e^{Q(z)} f = H - \left(f^{(k)} + h_{k-1} f^{(k-1)} + \cdots + h_2 f'' \right). \tag{7}$$

If $H - \left(f^{(k)} + h_{k-1} f^{(k-1)} + \cdots + h_2 f'' \right) \equiv 0$, by $a_n \neq b_n$, we have

$$f = K \exp\left\{ \int \frac{A_0}{A_1} e^{Q(z)-P(z)} dz \right\},$$

where K is a nonzero constant. If $H - \left(f^{(k)} + h_{k-1} f^{(k-1)} + \cdots + h_2 f'' \right) \not\equiv 0$, rewrite (7) as

$$\frac{A_1 e^{P(z)} f' + A_0 e^{Q(z)} f}{H - \left(f^{(k)} + h_{k-1} f^{(k-1)} + \cdots + h_2 f'' \right)} \equiv 1.$$

Suppose $\sigma(f) < n$, then by Lemma 2.1, we can get $T(r, e^{P(z)}) = S(r, e^{P(z)})$, which is a contradiction.

By Lemma 2.5, for any given $0 < \delta < \frac{1}{8}$, there exists a set E_1 of finite logarithmic measure such that

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z} \right)^j (1 + o(1)), \quad j = 1, 2, \ldots, k, \tag{8}$$

where $|f(z)| \geq M(r, f) \nu_f(r)^{-\frac{1}{8}+\delta}, r \notin E_1$. Furthermore, from the definition of the central index, we know that $\nu_f(r) \to \infty$ as $r \to \infty$. By Lemma 2.7, we have

$$\nu_f(r) \leq r^{\sigma(f)+1}, \tag{9}$$

for all r sufficiently large. By Lemma 2.3, we have

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq |z|^{j(\sigma(f)-1+\varepsilon)}, \quad j = 1, 2, \ldots, k, \tag{10}$$

for all z satisfying $|z| = r \notin E_2$ where $m_l(E_2) < \infty$, and ε is any given constant with $0 < 4\varepsilon < \min\{1, n - \sigma(H), n - \sigma, n - t\}$, where $t = \max\{t_j = \deg(h_i(z)), 2 \leq i \leq k-1\}$. By Lemma 2.6, there is a set E_3 with $\zeta = \overline{\log dens} E_3 > 0$ such that

$$\frac{\nu_f(r)^{\frac{1}{8}-\delta} |H(z)|}{M(r, f)} \leq \exp\{-r^{n-2\varepsilon}\}, \tag{11}$$

when $r \in E_3$ is large enough. We may take θ_p such that $M(r, f) = |f(re^{i\theta_p})|$ for every p. By Lemma 2.4, given a constant $0 < C < 1$, there exists a constant l_0 and a set E_4 with $1 - \frac{\zeta}{2} \leq \underline{\log dens}(E_4)$ such that

$$e^{-5\pi} M(r, f)^{1-C} \leq |f(re^{i\theta})| \tag{12}$$

for all $r \in E_4$ and $|\theta - \theta_p| \leq l_0$. Since the characteristic functions of E_3 and E_4 satisfy the relation

$$\chi_{E_3 \cap E_4}(t) = \chi_{E_3}(t) + \chi_{E_4}(t) - \chi_{E_3 \cup E_4}(t).$$

Then $\overline{\log dens}(E_3 \cup E_4) \leq 1$. Thus, we can get

$$\frac{\zeta}{2} \leq \overline{\log dens} E_3 + \underline{\log dens}(E_4) - \overline{\log dens}(E_3 \cup E_4) \leq \overline{\log dens}(E_3 \cap E_4).$$

Since $m_l(E_1 \cup E_2) < \infty$, we have $\overline{\log dens}((E_3 \cap E_4) \setminus (E_1 \cup E_2)) > 0$. Thus, there exists a sequence of points $z_q = r_q e^{i\theta_q}$ with $r_q \uparrow \infty$ and

$$|f(z_q)| = M(r_q, f), \quad r_q \in (E_3 \cap E_4) \setminus (E_1 \cup E_2).$$

Passing to a sequence of $\{\theta_q\}$, we may assume that $\lim_{q \to \infty} \theta_q = \theta_0$ in this paper.

We now take the three cases as follows into consideration.

Case 1. $\delta(P, \theta_0) > 0$. From the continuity of $\delta(P, \theta)$, we have

$$\frac{1}{3}\delta(P, \theta_0) < \delta(P, \theta_q) < \frac{4}{3}\delta(P, \theta_0) \tag{13}$$

for sufficiently large q. By Lemma 2.2, we can get

$$\exp\left\{\frac{1-\varepsilon}{3}\delta(P, \theta_0) r_q^n\right\} < |A_1(z_q) e^{P(z_q)}| < \exp\left\{\frac{4(1+\varepsilon)}{3}\delta(P, \theta_0) r_q^n\right\} \tag{14}$$

for all q sufficiently large. From (7) we can get

$$\begin{aligned} \left|\frac{f'(z_q)}{f(z_q)} + \frac{A_0(z_q)}{A_1(z_q)} e^{Q(z_q) - P(z_q)}\right| \leq & \left|\frac{e^{-P(z_q)}}{A_1(z_q)}\right| \Bigg(\left|\frac{f^{(k)}(z_q)}{f(z_q)}\right| + \sum_{j=2}^{k-1} \left|h_j \frac{f^{(j)}(z_q)}{f(z_q)}\right| \\ & + \frac{|H(z_q)|}{M(r_q, f)}\Bigg) \end{aligned} \tag{15}$$

We divide the proof in Case 1 in three subcases in the following.

Subcase 1.1. We first assume that θ_0 satisfies $\xi := \delta(Q - P, \theta_0) > 0$. From the continuity of $\delta(Q - P, \theta_0)$ and Lemma 2.2, we have

$$\frac{1}{3}\delta(Q - P, \theta_0) \leq \delta(Q - P, \theta_q) \leq \frac{4}{3}\delta(Q - P, \theta_0) \tag{16}$$

for sufficiently large q. Similar to (14), we have

$$\exp\left\{\frac{1-\varepsilon}{3}\xi r_q^n\right\} < \left|\frac{A_0(z_q)}{A_1(z_q)} e^{Q(z_q) - P(z_q)}\right| < \exp\left\{\frac{4(1+\varepsilon)}{3}\xi r_q^n\right\}, \tag{17}$$

for sufficiently large q. Substituting (8)-(11) to (15), for sufficiently large q, we can get

$$\begin{aligned} & \left|\frac{\nu_f(r_q)}{z_q}(1 + o(1)) + \frac{A_0(z_q)}{A_1(z_q)} e^{Q(z_q) - P(z_q)}\right| \\ \leq & \left|\frac{e^{-P(z_q)}}{A_1(z_q)}\right| \left(2 r_q^{k\sigma(f)} + \Sigma_{j=2}^{k} |z_q|^{j(\sigma(f) - 1 + \varepsilon)} r_q^{t_j + \varepsilon} + \exp\{-r_q^{n - 2\varepsilon}\}\right) \\ \leq & \left|\frac{e^{-P(z_q)}}{A_1(z_q)}\right| r_q^{k\sigma(f) + t + \varepsilon}. \end{aligned} \tag{18}$$

By (14) and $0 < 4\varepsilon < \min\{1, n-\sigma, n-\sigma(H), n-t\}$, we have

$$\left|\frac{e^{-P(z_q)}}{A_1(z_q)}\right| r_q^{k\sigma+t+\varepsilon} \leq \frac{r_q^{k\sigma+t+\varepsilon}}{\exp\{\frac{(1-\varepsilon)}{3}\delta(P,\theta_0)r_q^n\}} \leq \exp\{-\frac{(1-2\varepsilon)}{3}\delta(P,\theta_0)r_q^n\}. \tag{19}$$

From (17)-(19) and (9), we can obtain

$$\begin{aligned} \exp\left\{\tfrac{(1-\varepsilon)}{3}\xi r_q^n\right\} &\leq \left|\tfrac{\nu_f(r_q)}{z_q}(1+o(1)) + \tfrac{A_0(z_q)}{A_1(z_q)}e^{Q(z_q)-P(z_q)} - \tfrac{\nu_f(r_q)}{z_q}(1+o(1))\right| \\ &\leq \exp\{-\tfrac{(1-2\varepsilon)}{3}\delta(P,\theta_0)r_q^n\} + 2r_q^{\sigma(f)} \leq 3r_q^{\sigma(f)}. \end{aligned} \tag{20}$$

Thus, we can get a contradiction.

Subcase 1.2. $\xi := \delta(Q-P,\theta_0) < 0$. Then from Lemma 2.2, for sufficiently large q, we have

$$\exp\left\{\frac{4(1+\varepsilon)}{3}\xi r_q^n\right\} \leq \left|\frac{A_0(z_q)}{A_1(z_q)}e^{Q(z_q)-P(z_q)}\right| \leq \exp\left\{\frac{(1-\varepsilon)}{3}\xi r_q^n\right\}. \tag{21}$$

From (21) and similar to (20), we can get

$$\frac{\nu_f(r_q)}{r_q}(1+o(1)) \leq \exp\left\{\frac{(1-\varepsilon)}{3}\xi r_q^n\right\} + \exp\left\{-\frac{(1-2\varepsilon)}{3}\delta(P,\theta_0)r_q^n\right\},$$

when q is large enough. Thus, we can get that $\nu_f(r_q) \to 0$ as $q \to \infty$, which is impossible.

Subcase 1.3. $\xi := \delta(Q-P,\theta_0) = 0$. From (12), we may construct another sequence of points $z_q^* = r_q e^{i\theta_q^*}$ with $\lim_{q\to\infty}\theta_q^* = \theta_0^*$ such that $\xi_1 := \delta(Q-P,\theta_0^*) > 0$. Without loss of generality, we may suppose that

$$\delta(Q-P,\theta) > 0, \quad \theta \in (\theta_0 + 2k\pi, \theta_0 + (2k+1)\pi),$$

$$\delta(Q-P,\theta) < 0, \quad \theta \in (\theta_0 + (2k-1)\pi, \theta_0 + 2k\pi),$$

which $k \in \mathbb{Z}$. When q is large enough, we have $|\theta_0 - \theta_q| \leq l_0$. Choose θ_q^* such that $\frac{l_0}{2} \leq \theta_q^* - \theta_q \leq l_0$, *i.e.*, $\theta_q + \frac{l_0}{2} \leq \theta_q^* \leq \theta_q + l_0$, then

$$\theta_0 + \frac{l_0}{2} \leq \theta_0^* \leq \theta_0 + l_0. \tag{22}$$

For sufficiently large q, we can get (12) for z_q^*, and $\xi_1 := \delta(Q-P,\theta_0^*) > 0$. Hence we can get

$$\left|\frac{H(z_q^*)}{f(z_q^*)}\right| \leq \frac{\nu_f(r)^{\frac{1}{8}-\delta}M(r_q,H)}{e^{-5\pi}M(r_q,f)^{1-C}}, \tag{23}$$

and

$$\exp\left\{\frac{(1-\varepsilon)}{3}\xi_1 r_q^n\right\} \leq \left|\frac{A_0(z_q^*)}{A_1(z_q^*)}e^{Q(z_q^*)-P(z_q^*)}\right| \leq \exp\left\{\frac{4(1+\varepsilon)}{3}\xi_1 r_q^n\right\}. \tag{24}$$

By virtue of [22], we may assume that $M(r_q,f) \geq \exp\{r_q^{\sigma(f)-\varepsilon}\}$. From the above argument, we can know that $z_q^* = r_q e^{i\theta_q^*}$ satisfies (9). Then from (23) and for all large enough q, we have

$$\left|\frac{H(z_q^*)}{f(z_q^*)}\right| \leq r_q^{(\sigma(f)+1)(\frac{1}{8}-\delta)}\frac{\exp\{r_q^{\sigma(H)+\varepsilon}\}}{\exp\{r_q^{\sigma(f)-\frac{3}{2}\varepsilon}\}} \leq \exp\{-r_q^{n-2\varepsilon}\}. \tag{25}$$

Taking now l_0 small enough, we have $\delta(P,\theta_0^*)>0$ by the continuity of $\delta(P,\theta)$. Thus, we have

$$\exp\left\{\frac{1-\varepsilon}{3}\delta(P,\theta_0^*)r_q^n\right\}<|A_1(z_q^*)e^{P(z_q^*)}|<\exp\left\{\frac{4(1+\varepsilon)}{3}\delta(P,\theta_0^*)r_q^n\right\}. \tag{26}$$

Substituting (10) and (25) into (15) and by (26), we have

$$\left|\frac{A_0(z_q^*)}{A_1(z_q^*)}e^{Q(z_q^*)-P(z_q^*)}\right|\leq \exp\left\{-\frac{(1-3\varepsilon)}{3}\delta(P,\theta_0^*)r_q^n\right\}+2r_q^{\sigma(f)}\leq 3r_q^{\sigma(f)}. \tag{27}$$

Combining (27) with (24), for large enough q, we can get a contradiction easily.

Case 2. Suppose that $\delta(P,\theta_0)<0$. Then from the continuity of $\delta(P,\theta)$ and Lemma 2.2, we have

$$\exp\left\{\frac{4(1+\varepsilon)}{3}\delta(P,\theta_0)r_q^n\right\}\leq |A_1(z_q)e^{P(z_q)}|\leq \exp\left\{\frac{(1-\varepsilon)}{3}\delta(P,\theta_0)r_q^n\right\} \tag{28}$$

for all sufficiently large q. From (5), we can get

$$\begin{aligned}\left|\frac{f^{(k)}(z_q)}{f(z_q)}+\sum_{j=2}^{k-1}h_j(z_q)\frac{f^{(j)}(z_q)}{f(z_q)}\right. &+ \left.A_0(z_q)e^{Q(z_q)}\right| \\ &\leq \left|A_1(z_q)e^{P(z_q)}\right|\left|\frac{f'(z_q)}{f(z_q)}\right|+\frac{|H(z_q)|}{M(r_q,f)}\end{aligned} \tag{29}$$

as $q\to\infty$. Again, we divide the proof in Case 2 in three subcases in the following.

Subcase 2.1. $\delta(Q,\theta_0)>0$. From the continuity of $\delta(Q,\theta)$ and Lemma 2.2, for large enough q, we have

$$\exp\left\{\frac{(1-\varepsilon)}{3}\delta(q,\theta_0)r_q^n\right\}\leq \left|A_0(z_q)e^{Q(z_q)}\right|\leq \exp\left\{\frac{4(1+\varepsilon)}{3}\delta(Q,\theta_0)r_q^n\right\}. \tag{30}$$

Substituting (8)-(11) and (28) into (29), we get

$$\left|\frac{f^{(k)}(z_q)}{f(z_q)}+\sum_{j=2}^{k-1}h_j(z_q)\frac{f^{(j)}(z_q)}{f(z_q)}+A_0(z_q)e^{Q(z_q)}\right|\leq \exp\left\{-r_q^{n-3\varepsilon}\right\}. \tag{31}$$

From (8)-(11),(30),(31) and enough large q, we have

$$\begin{aligned}\exp\left\{\frac{(1-\varepsilon)}{3}\delta(Q,\theta_0)r_q^n\right\} \leq & \left|\frac{f^{(k)}(z_q)}{f(z_q)}+\sum_{j=2}^{k-1}h_j(z_q)\frac{f^{(j)}(z_q)}{f(z_q)}+A_0(z_q)e^{Q(z_q)}\right. \\ & \left.-\left(\frac{f^{(k)}(z_q)}{f(z_q)}+\sum_{j=2}^{k-1}h_j(z_q)\frac{f^{(j)}(z_q)}{f(z_q)}\right)\right|\end{aligned}$$

i.e.,

$$\exp\left\{\frac{(1-\varepsilon)}{3}\delta(Q,\theta_0)r_q^n\right\}\leq \exp\left\{-r_q^{n-3\varepsilon}\right\}+r_q^{k\sigma(f)+t+\varepsilon}.$$

Thus, we can get a contradiction.

Subcase 2.2. $\delta(Q,\theta_0) < 0$. By the continuity of $\delta(Q,\theta)$ and Lemma 2.2, for all sufficiently large q, we have

$$\exp\left\{\frac{4(1+\varepsilon)}{3}\delta(Q,\theta_0)r_q^n\right\} \le |A_0(z_q)e^{Q(z_q)}| \le \exp\left\{\frac{(1-\varepsilon)}{3}\delta(Q,\theta_0)r_q^n\right\}. \tag{32}$$

From (28), (29) and (32), we can get

$$2\left(\frac{\nu_f(r_q)}{z_q}\right)^k \le \exp\{-r_q^{n-2\varepsilon}\} + \left|\left(\frac{\nu_f(r_q)}{z_q}\right)^{k-1}\right| r_q^{t+\varepsilon} + \exp\left\{\frac{(1-\varepsilon)}{3}\delta(Q,\theta_0)r_q^n\right\}.$$

Since $0 < 4\varepsilon < \min\{1, n-\sigma, n-\sigma(H), n-t\}$, we can get a contradiction as $q \to \infty$.

Subcase 2.3. $\delta(Q,\theta_0) = 0$. Using the same argument as in Subcase 1.3, we can construct another sequence of points $z_q^* = r_q e^{i\theta_q^*}$ satisfying $\frac{l_0}{2} \le |\theta_q^* - \theta_q| \le l_0$ such that $\delta(P,\theta_0^*) < 0 < \delta(Q,\theta_0^*)$ where $\theta_0^* = \lim_{q\to\infty}\theta_q^*$. Then, we have (28) for $\delta(P,\theta_0^*)$ and (30) for $\delta(Q,\theta_0^*)$. Using the same argument as in Subcase 1.3, we also have (25) for the sequence of points z_q^*. From (29) and sufficiently large q, we have

$$|A_0(z_q^*)e^{Q(z_q^*)}| \le |A_1(z_q^*)e^{P(z_q^*)}| r_q^{\sigma(f)+\varepsilon} + \exp\{-r_q^{n-2\varepsilon}\} + r_q^{k\sigma(f)+t+\varepsilon}.$$

Thus, we can also get a contradiction.

Case 3. Suppose that $\delta(P,\theta_0) = 0$. We discuss three subcases according to $\delta(Q,\theta_0)$ as follows.

Subcase 3.1. $\delta(Q,\theta_0) > 0$. By the same argument as in Subcase 1.3, we can also construct another sequence of points $z_q^* = r_q e^{i\theta_q^*}$ with $\theta_0^* = \lim_{q\to\infty}\theta_q^*$ and $\frac{l_0}{2} \le |\theta_q^* - \theta_q| \le l_0$ such that z_q^* satisfies (25) and $\delta(P,\theta_0^*) < 0 < \delta(Q,\theta_0^*)$. Using the same argument as in Subcase 2.3, we can get a contradiction easily as $n \to \infty$.

Subcase 3.2. $\delta(Q,\theta_0) < 0$. By Lemma 2.2, we first define

$$\delta'(P,\theta) := -n\alpha\sin(n\theta) - n\beta\cos(n\theta)$$

where $a_n = \alpha + i\beta$. Since $a_n \ne 0$, we have $\delta'(P,\theta_0) \ne 0$. Take $z_q' = r_q e^{i\theta_q'}$ satisfying $0 < |\theta_q' - \theta_0| \le l_0$, we have (25) for z_q' and $\delta(P,\theta_q') \ne 0$. By the continuity of $\delta(Q,\theta)$, we may assume that $\delta(Q,\theta_q') < 0 < \delta(P,\theta_q')$ for a suitable l_0, $0 < \theta_q' - \theta_0 \le l_0$. For a suitable l_0, we have $\delta'(P,\theta_0) > 0$ and

$$\frac{1}{3}\delta'(P,\theta_0) \le \delta'(P,\theta) \le \frac{4}{3}\delta'(P,\theta_0), \quad \theta \in (\theta_0, \theta_0 + l_0). \tag{33}$$

Since we have $|f(z_q)| = M(r_q,f)$ and $\theta_q \to \infty$ as $q \to \infty$ for the sequence of points z_q, we have $|f(r_q e^{i\theta_0})| \ge M(r_q,f)\nu_f(r_q)^{-\frac{1}{8}+\delta}$ for sufficiently large q. From (7), we have

$$\begin{aligned}\left|\frac{f'(z_q')}{f(z_q')}\right| \le\ & \left|\frac{e^{-P(z_q')}}{A_1(z_q')}\right| \left(\left|\frac{f^{(k)}(z_q)}{f(z_q)}\right| + \sum_{j=2}^{k-1}|h_j(z_q)|\left|\frac{f^{(j)}(z_q)}{f(z_q)}\right|\right. \\ & \left. + \left|\frac{H(z_q')}{M(r_q,f)}\right| + |A_0(z_q')e^{Q(z_q')}|\right).\end{aligned} \tag{34}$$

By Lemma 2.2, we have

$$\exp\left\{-(1+\varepsilon)\delta(P,\theta_q')r_q^n\right\} \le \left|\frac{e^{-P(z_q')}}{A_1(z_q')}\right| \le \exp\left\{-(1-\varepsilon)\delta(P,\theta_q')r_q^n\right\} \tag{35}$$

and

$$\exp\{(1+\varepsilon)\delta(Q,\theta_q')r_q^n\} \le |A_0(z_q')e^{Q(z_q')}| \le \exp\{(1-\varepsilon)\delta(Q,\theta_q')r_q^n\} \tag{36}$$

for sufficiently large q. From (9),(25),(35),(36) and (34), we can get

$$\left|\frac{f'(z_q')}{f(z_q')}\right| \le \exp\{-(1-2\varepsilon)\delta(P,\theta_q')r_q^n\}.$$

Since θ_q' is arbitrary in (θ_0,θ_0+l_0), for sufficiently large r_q, we can obtain

$$\left|\frac{f'(r_qe^{i\theta})}{f(r_qe^{i\theta})}\right| \le \exp\{-(1-2\varepsilon)\delta(P,\theta)r_q^n\}, \quad \theta\in(\theta_0,\theta_0+l_0). \tag{37}$$

Therefore, for $\theta\in(\theta_0,\theta_0+l_0)$, we have

$$\gamma(r_q,\theta) = r_q\int_{\theta_0}^{\theta}\left|\frac{f'(r_qe^{i\theta})}{f(r_qe^{i\theta})}\right|d\theta \le r_q\int_{\theta_0}^{\theta}e^{\kappa_1(\theta)r_q^n}d\theta = \int_{\theta_0}^{\theta}\frac{-1}{\kappa_2(\theta)r_q^{n-1}}e^{\kappa_1(\theta)r_q^n}d(\kappa_1(\theta)r_q^n),$$

where $\kappa_1(\theta) = -(1-2\varepsilon)\delta(P,\theta)$, $\kappa_2(\theta) = (1-2\varepsilon)\delta'(P,\theta)$.

Since $\delta(P,\theta)>0$ for all $\theta\in(\theta_0,\theta_0+l_0)$, we can get

$$0\le\gamma(r_q,\theta)\le\frac{2}{(1-2\varepsilon)\delta'(P,\theta_0)r_q^{n-1}}(e^{\kappa_1(\theta_0)r_q^n}-e^{\kappa_1(\theta)r_q^n}).$$

For sufficiently large q, we can get

$$0\le\gamma(r_q,\theta)\le\frac{2}{\kappa_2(\theta_0)}. \tag{38}$$

By the proof of Lemma 2.4 in REF.[22], we have

$$\nu_f(r_q)^{-\frac{1}{8}+\delta'}M(r_q,f) = \exp\{-2\pi-2/\kappa_2(\theta_0)\}\nu_f(r_q)^{-\frac{1}{8}+\delta}M(r_q,f) \le |f(r_qe^{i\theta})| \tag{39}$$

for $\theta\in(\theta_0,\theta_0+l_0)$, where $0<\delta'<\delta<\frac{1}{8}$. Therefore, we can take the sequence of points $z_q^* = r_qe^{i\theta_q^*}$ satisfying $\theta_q^* = \frac{l_0}{2}+\theta_0$ and (25) for z_q^*. Furthermore, from (39), we have (8) for z_q^* when q is sufficiently large. Thus, from (8) and (37), we can deduce that $\nu_f(r_q)\to\infty$ as $q\to\infty$, which is impossible.

When $\delta(Q,\theta_q')<0<\delta(P,\theta_q')$ for $-l_0<\theta_q'-\theta_0<0$. Then, we deduce that $\gamma(r_q,\theta)\le0$ for all $\theta\in(\theta_0-l_0,\theta_0)$. Similarly, we can get

$$\nu_f(r_q)^{-\frac{1}{8}+\delta'}M(r_q,f) = \exp\{-2\pi\}\nu_f(r_q)^{-\frac{1}{8}+\delta}M(r_q,f) \le |f(r_qe^{i\theta})| \tag{40}$$

for $\theta\in(\theta_0-l_0,\theta_0)$, where $0<\delta'<\delta<\frac{1}{8}$. Thus, we can also get a contradiction.

Subcase 3.3. $\delta(Q,\theta_0)=0$. We have $a_n=cb_n$ and $c\in\mathbb{R}\setminus\{0,1\}$. Then we have

$$P(z)=cb_nz^n+\cdots+a_1z+a_0,\quad Q(z)-P(z)=(1-c)b_nz^n+R_{n-1}(z),$$

where $R_{n-1}(z)$ is a polynomial of degree at most $n-1$.

If $c<0$, we may take l_0 small enough such that $\delta(Q,\theta)<0<\delta(P,\theta)$, provided that either $\theta\in(\theta_0,\theta_0+l_0)$ or $\theta\in(\theta_0-l_0,\theta_0)$. Using the same argument as in Subcase 3.2, we can get (37) and (39). Therefore, by a standard Wiman-Valiron theory, we can deduce that $\nu_f(r_q)\to\infty$ as $q\to\infty$. Thus, we can get a contradiction.

If $0<c<1$, for small enough l_0, we also obtain $\delta(Q-P,\theta)>0$ and $\delta(P,\theta)>0$, provided that either $\theta\in(\theta_0,\theta_0+l_0)$ or $\theta\in(\theta_0-l_0,\theta_0)$. Using the same argument as in Subcase 1.3, we can get a contradiction easily.

If $c>1$, from the above argument, we can obtain $\delta(Q-P,\theta)<0<\delta(P,\theta)$ provided that either $\theta\in(\theta_0,\theta_0+l_0)$ or $\theta\in(\theta_0-l_0,\theta_0)$. Furthermore, we can take the sequence of points $z'_q=r_qe^{i\theta'_q}$ satisfying (25), provided that either $\theta'_q\in(\theta_0,\theta_0+l_0)$ or $\theta'_q\in(\theta_0-l_0,\theta_0)$. Therefore, from (7), we have

$$\begin{aligned}\left|\frac{f'(z'_q)}{f(z'_q)}\right| \quad &\le \quad \left|\frac{A_0(z'_q)}{A_1(z'_q)}e^{(1-c)b_n(z'_q)^n+R_{n-1}(z'_q)}\right|+\left|\frac{e^{-P(z'_q)}}{A_1(z'_q)}\right|+\left|\frac{H(z'_q)}{f(z'_q)}\right|\\ &\quad +\left(\left|\frac{f^{(k)}(z_q)}{f(z_q)}\right|+\sum_{j=2}^{k-1}|h_j(z_q)|\left|\frac{f^{(j)}(z_q)}{f(z_q)}\right|\right).\end{aligned}$$

Similarly as in Subcase 3.2, we get (37) and (39). By the Wiman-Valiron theory, we can also get a contradiction.

Thus,from the above argument, we can prove that every solution f of equation (4) satisfies $\sigma(f)=\infty$.

The exponent of convergence of the zero points

Rewrite (4) as

$$\frac{1}{f}=\frac{1}{H}\left(\frac{f^{(k)}}{f}+\sum_{j=2}^{k-1}h_j\frac{f^{(j)}}{f}+A_1e^{P(z)}\frac{f'}{f}+A_0e^{Q(z)}\right). \tag{41}$$

If f has z_0 as its zeros with multiplicity of $s(>k)$, then z_0 is the zeros of H with order $s-k$. Therefore, we have

$$N\left(r,\frac{1}{f}\right)\le k\overline{N}\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{H}\right). \tag{42}$$

On the other hand, from (41), we have

$$m\left(r,\frac{1}{f}\right)\le m\left(r,\frac{1}{H}\right)+\sum_{j=2}^{k-1}m(r,h_j)+m(r,A_1e^P)+m(r,A_0e^Q)+S(r,f). \tag{43}$$

Since $\sigma(f)=\infty$, $\sigma:=\max\{\sigma(A_j), j=0,1\}<n$ and $\sigma(H)<n$, and from (42) and (43), we have

$$T(r,f)=T\left(r,\frac{1}{f}\right)+O(1)\le (k+4)k\overline{N}\left(r,\frac{1}{f}\right)+S(r,f).$$

Thus, by Lemma 2.9, we can get $\sigma(f)=\lambda(f)=\overline{\lambda}(f)=\infty$ and $\sigma_2(f)=\lambda_2(f)=\overline{\lambda}_2(f)\le n$.

Next, we will prove that $\sigma(f)=\overline{\lambda}(f-\varphi)=\infty$ and $\sigma_2(f)=\overline{\lambda}_2(f-\varphi)\le n$.

First, setting $\omega_0=f-\varphi$. Since $\sigma(\varphi)<\infty$, then we have $\sigma(\omega_0)=\sigma(f)$. From (4), we have

$$\omega_0^{(k)}+\sum_{j=2}^{k-1}h_j\omega_0''+A_1e^P\omega_0'+A_0e^Q\omega_0=H-(A_0e^Q\varphi+A_1e^P\varphi'+\sum_{j=2}^{k-1}h_j\varphi^{(j)}+\varphi^{(k)}).$$

Since $\sigma(H)<n$, $\sigma<n$, $\sigma(\varphi)<\infty$ and $a_n\ne b_n$, we have $H-(A_0e^Q\varphi+A_1e^P\varphi'+\sum_{j=2}^{k-1}h_j\varphi^{(j)}+\varphi^{(k)})\not\equiv 0$ whether $H\not\equiv 0$ or $H\equiv 0$. Thus,

$$\frac{1}{\omega_0}=\frac{1}{H-(A_0e^Q\varphi+A_1e^P\varphi'+\sum_{j=2}^{k-1}h_j\varphi^{(j)}+\varphi^{(k)})}\left(\frac{\omega_0^{(k)}}{\omega_0}+\sum_{j=2}^{k-1}h_j\frac{\omega_0^{(j)}}{\omega_0}+A_1e^P\frac{\omega_0'}{\omega_0}+A_0e^Q\right). \tag{44}$$

If ω_0 has z_1 as its zero with multiplicity of $l(>k)$, then z_1 is the zeros of $H-(A_0e^Q\varphi+A_1e^P\varphi'+\sum_{j=2}^{k-1}h_j\varphi^{(j)}+\varphi^{(k)})$ with multiplicity $l-k$. Then, we have

$$N\left(r,\frac{1}{\omega_0}\right)\le k\overline{N}\left(r,\frac{1}{\omega_0}\right)+N\left(r,\frac{1}{H-(A_0e^Q\varphi+A_1e^P\varphi'+\sum_{j=2}^{k-1}h_j\varphi^{(j)}+\varphi^{(k)})}\right).$$

On the other hand, from (44), we have

$$\begin{aligned} m\left(r,\frac{1}{\omega_0}\right) &\le m\left(r,\frac{1}{H-(A_0e^Q\varphi+A_1e^P\varphi'+\sum_{j=2}^{k-1}h_j\varphi^{(j)}+\varphi^{(k)})}\right)+m(r,A_1e^P)\\ &\quad +m(r,A_0e^Q)+\sum_{j=2}^{k-1}m(r,h_j)+S(r,f).\end{aligned}$$

Using the above argument, we obtain

$$T(r,\omega_0)=T(r,f)+S(r,f)\le K_1\overline{N}\left(r,\frac{1}{\omega_0}\right)+S(r,f)=K_1\overline{N}\left(r,\frac{1}{f-\varphi}\right)+S(r,f),$$

where K_1 is a constant.

Thus, by Lemma 2.9, we can get $\sigma(f)=\sigma(\omega_0)=\overline{\lambda}(f-\varphi)=\infty$ and $\sigma_2(f)=\overline{\lambda}_2(f-\varphi)\le n$.

Hence, we can get $\sigma(f)=\lambda(f)=\overline{\lambda}(f)=\overline{\lambda}(f-\varphi)=\infty$ and $\sigma_2(f)=\lambda_2(f)=\overline{\lambda}_2(f)=\overline{\lambda}_2(f-\varphi)\le n$.

Thus, we can complete the proof of Theorem 1.1. □

4 The proof of Theorem 1.2

Proof: Let f be a nontrivial solution of (5) with finite order. By [19], similar to Theorem 1.1, we can get $\sigma(f) \geq n$. Now rewrite (5) as

$$(45)\quad \frac{f^{(k)}}{f} + \sum_{j=2}^{k-1} h_j \frac{f^{(j)}}{f} + \Big(A_1(z)e^{P(z)} + D_1(z)\Big)\frac{f'}{f} + \Big(A_0(z)e^{Q(z)} + D_0(z)\Big) = \frac{H(z)}{f}.$$

Since $D = \max\{\sigma(D_j), j = 0,1\} < n$, then for any $\varepsilon(0 < 4\varepsilon < \min\{1, n-\sigma, n-\sigma(H), n-t, n-D\})$, we have

$$(46)\qquad |D_j(z)| \leq \exp\{r^{D+\varepsilon}\}, \quad j = 0,1.$$

Similarly as in the proof of Theorem 1.1, we can take a sequence of points $z_q = r_q e^{i\theta_q}, r_q \to \infty$, such that $\lim_{q\to\infty}\theta_q = \theta_0$ and

$$|f(z_q)| = M(r_q, f), \quad r_q \in (E_3 \cap E_4) \setminus (E_1 \cup E_2),$$

and the sequence of points satisfies (8)-(12).

Suppose that $a_n/b_n = c < 0$, we will discuss three cases according to the signs of $\delta(P,\theta_0)$ and $\delta(Q,\theta_0)$ as follows.

Case 1. Suppose that $\delta(P,\theta_0) < 0 < \delta(Q,\theta_0)$. By Lemma 2.2 and the continuity of $\delta(P,\theta), \delta(Q,\theta)$, we have

$$(47)\qquad \exp\left\{\frac{4(1+\varepsilon)}{3}\delta(P,\theta_0)r_q^n\right\} \leq |A_1(z_q)e^{P(z_q)}| \leq \exp\left\{\frac{(1-\varepsilon)}{3}\delta(P,\theta_0)r_q^n\right\}$$

and

$$(48)\qquad \exp\left\{\frac{(1-\varepsilon)}{3}\delta(Q,\theta_0)r_q^n\right\} \leq |A_0(z_q)e^{Q(z_q)}| \leq \exp\left\{\frac{4(1+\varepsilon)}{3}\delta(Q,\theta_0)r_q^n\right\}$$

for all sufficiently large q. From (45), we have

$$(49)\qquad \begin{aligned} |A_0(z_q)e^{Q(z_q)} + D_0(z_q)| \leq & \left|\tfrac{f^{(k)}(z_q)}{f(z_q)}\right| + \textstyle\sum_{j=2}^{k-1}|h_j(z_q)|\left|\tfrac{f^{(j)}(z_q)}{f(z_q)}\right| + \tfrac{|H(z_q)|}{M(r_q,f)} \\ & + \left|(|A_1(z_q)e^{P(z_q)} + D_1(z_q)|)\tfrac{f'(z_q)}{f(z_q)}\right|. \end{aligned}$$

From (46),(47) and (48), we have

$$(50)\qquad |A_1(z_q)e^{P(z_q)} + D_1(z_q)| \leq \exp\{r^{D+2\varepsilon}\}$$

and

$$(51)\qquad |A_0(z_q)e^{Q(z_q)} + D_0(z_q)| \geq \exp\left\{\frac{(1-2\varepsilon)}{3}\delta(Q,\theta_0)r_q^n\right\}$$

for large enough q.

Substituting (10),(11),(47) and (48) into (49), we can obtain

$$\begin{aligned} \exp\left\{\frac{(1-\varepsilon)}{3}\delta(Q,\theta_0)r_q^n\right\} &\leq r_q^{k\sigma(f)+\varepsilon} + \exp\{r_q^{D+2\varepsilon}\} + (k-2)r_q^{(k-1)\sigma(f)+t+\varepsilon} + \exp\{-r_q^{n-2\varepsilon}\} \\ &\leq \exp\{r_q^{D+3\varepsilon}\}. \end{aligned}$$

Since $D < n$, we can obtain a contradiction.

Case 2. Suppose that $\delta(Q, \theta_0) < 0 < \delta(P, \theta_0)$. By Lemma 2.2 and the continuity of $\delta(P, \theta), \delta(Q, \theta)$, we have

$$\exp\left\{\frac{(1-\varepsilon)}{3}\delta(P,\theta_0)r_q^n\right\} \le |A_1(z_q)e^{P(z_q)}| \le \exp\left\{\frac{4(1+\varepsilon)}{3}\delta(P,\theta_0)r_q^n\right\} \tag{52}$$

and

$$\exp\left\{\frac{4(1+\varepsilon)}{3}\delta(Q,\theta_0)r_q^n\right\} \le |A_0(z_q)e^{Q(z_q)}| \le \exp\left\{\frac{(1-\varepsilon)}{3}\delta(Q,\theta_0)r_q^n\right\} \tag{53}$$

for all sufficiently large q. From (45), we have

$$\begin{aligned}\left|(A_1(z_q)e^{P(z_q)} + D_1(z_q))\tfrac{f'(z_q)}{f(z_q)}\right| \le \quad & \left|\tfrac{f^{(k)}(z_q)}{f(z_q)}\right| + \textstyle\sum_{j=2}^{k-1}|h_j(z_q)|\left|\tfrac{f^{(j)}(z_q)}{f(z_q)}\right| + \tfrac{|H(z_q)|}{M(r_q,f)} \\ & +|A_0(z_q)e^{Q(z_q)} + D_0(z_q)|.\end{aligned} \tag{54}$$

From (46), (52) and (53), we have

$$|A_0(z_q)e^{Q(z_q)} + D_0(z_q)| \le \exp\{r^{D+2\varepsilon}\} \tag{55}$$

and

$$|A_1(z_q)e^{P(z_q)} + D_1(z_q)| \ge \exp\left\{\frac{(1-2\varepsilon)}{3}\delta(P,\theta_0)r_q^n\right\} \tag{56}$$

for large enough q.

Substituting (10),(11),(55) and (56) into (54), we obtain

$$\nu_f(r_q) \le 2r_q \exp\left\{-\frac{(1-2\varepsilon)}{3}\delta(P,\theta_0)r_q^n\right\}\left(2Kr_q^{k\sigma(f)+t+\varepsilon} + \exp\{r_q^{D+2\varepsilon}\}\right) \tag{57}$$

for sufficiently large q, where K is a constant. From (9), (57) and $D < n$, we can deduce that $\nu_f(r_q) \to 0$ as $q \to \infty$, which is a contradiction.

Case 3. Suppose that $\delta(Q, \theta_0) = 0 = \delta(P, \theta_0)$. Similarly as in Subcase 1.3 of the proof of Theorem 1.1, from (12), we can construct a sequence of points $z_q^* = r_q e^{i\theta_q^*}$ with $\lim_{q\to\infty}\theta_q^* = \theta_0$ such that $\delta(P, \theta_0^*) < 0$ and (25) holds for z_q^*.

Without loss of generality, we can assume that

$$\delta(P,\theta) > 0, \quad \theta \in (\theta_0 + 2m\pi, \theta_0 + (2m+1)\pi)$$

and

$$\delta(P,\theta) < 0, \quad \theta \in (\theta_0 + (2m-1)\pi, \theta_0 + 2m\pi)$$

for all $m \in \mathbb{Z}$.

For sufficiently large q, we can have $|\theta_0 - \theta_q| \le l_0$. Taking θ_0^* such that $\frac{l_0}{2} \le \theta_q - \theta_q^* \le l_0$, then $\theta_0 - l_0 \le \theta_0^* \le \theta_0 - \frac{l_0}{2}$ and $\delta(P, \theta_0^*) < 0$. Since $\delta(Q, \theta_0^*) > 0$, by using the same argument as in Case 2, we can get a contradiction easily.

□

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