Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **61**, 1-11; **http://www.math.u-szeged.hu/ejqtde/**

OSCILLATION CRITERIA FOR A CERTAIN SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENTS

AYDIN TIRYAKI

Abstract. In this paper, by using the generalized Riccati technique and the integral averaging technique, some new oscillation criteria for certain second order retarded differential equation of the form

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' + p(t)\,f(u(\tau(t))) = 0$$

are established. The results obtained essentially improve known results in the literature and can be applied to the well known half-linear and Emden-Fowler type equations.

1. Introduction

This paper is concerned with the problem of oscillatory behavior of the retarded functional differential equation

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' + p(t)\,f(u(\tau(t))) = 0,\ t \geq t_0 \geq 0. \tag{1.1}$$

We suppose throughout the paper that the following conditions hold.

(H_1) α is a positive number,

(H_2) $r \in C([t_0,\ \infty)),\ r(t) > 0,\ R(t) = \int_{t_0}^{t} r^{-1/\alpha}(s)\,ds \to \infty$ as $t \to \infty$,

(H_3) $p \in C([t_0,\ \infty)),\ p(t) > 0$,

(H_4) $\tau \in C^1([t_0,\ \infty)),\ \tau(t) \leq t,\ \tau'(t) > 0,\ \tau(t) \to \infty$ as $t \to \infty$,

(H_5) $f \in C((-\infty,\ \infty)),\ xf(x) > 0$ for $x \neq 0,\ f \in C^1(R_D)$, where $R_D = (-\infty,\ -D) \cup (D,\ \infty),\ D > 0$.

By a solution of (1.1), we mean a function $u \in C^1([T_u,\ \infty)),\ T_u \geq t_0$, which has the property $r(t)\left|u'(t)\right|^{\alpha-1}u'(t) \in C^1([T_u,\ \infty))$ and satisfies (1.1) on $[T_u,\ \infty)$. We consider only those solutions $u(t)$ of (1.1) which satisfy $\sup\{|u(t)| :\ t \geq T_u\} > 0$ for all $T_u \geq t_0$. We assume that (1.1) possesses such a solution. A nontrivial solution of (1.1) is said to be oscillatory if it has a sequence of zeros tending to infinity, otherwise it is called nonoscillatory. An equation is said to be oscillatory if all its solutions are oscillatory.

The oscillatory behavior of functional differential equations with deviating arguments has been the subject of intensive study in the last three decades, see for example the monographs Agarwal et al [2], Dosly and Rehak [9], Gyori and Ladas [13], and Ladde et al [20].

The study of oscillation of second order differential equations is of great interest. Many criteria have been found which involve the behavior of the integral of a

1991 *Mathematics Subject Classification.* 34C10, 34K11.
Key words and phrases. Oscillatory solution; Integral averaging technique; Second order retarded differential equation; Half-linear equation; Emden-Fowler equation.

combination of the coefficients. Recently Agarwal et al [4], Chern et al [8], Elbert [11], Kusano et al [16, 17, 18, 19], and Mirzov [23, 24] have observed some similar properties between the half-linear equation

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' + p(t)\left|u(\tau(t))\right|^{\alpha-1}u(\tau(t)) = 0 \tag{1.2}$$

and the corresponding linear equation

$$\left(r(t)\,u'(t)\right)' + p(t)\,u(\tau(t)) = 0. \tag{1.3}$$

Some of the above results are improved by Dzurina and Stavroulakis [10]. On the other hand Ladde et al [20] presented the following oscillatory criteria for Eq. (1.3)

$$\int^{\infty} R^{1-\epsilon}(\tau(t))\,p(t)\,dt = \infty,\ 0<\epsilon<1. \tag{1.4}$$

In this paper, we shall continue in this direction the study of oscillatory properties of (1.1). By using the generalized Riccati technique and the integral averaging technique, we shall establish some new oscillatory criteria. The first our purpose is to improve the above mentioned results. The second aim is to show that many others known criteria are included in the our obtained results. The third intention of paper, is to apply obtained results for investigation of oscillation of the generalized Emden-Fowler equation

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' + p(t)\left|u(\tau(t))\right|^{\beta-1}u(\tau(t)) = 0. \tag{1.5}$$

Note that, in this direction, although there is an extensive literature on the oscillatory behavior of Eq. (1.2) and (1.3), there is not much done for Eq. (1.5). We refer to the reader, see [1, 2, 3, 4, 26, 30] in delay case and [15, 29] in ordinary case.

2. Main Results

In this section we prove our main result.

Theorem 1. *Let there exist a constant $k>0$ such that*

$$f'(x)/\left|f(x)\right|^{1-\frac{1}{\alpha}} \geq k \text{ for all } x\in R_D. \tag{2.1}$$

If

$$\lim_{t\to\infty}\int_{t_0}^{t}\left(\frac{1}{r(s)}\int_{s}^{\infty}p(z)\,dz\right)^{1/\alpha}ds = \infty \tag{2.2}$$

and there exists a differentiable function $\rho:[t_0,\ \infty)\to(0,\infty)$ such that

$$\rho'(s)\geq 0 \text{ and } \limsup_{t\to\infty}\int_{t_0}^{t}\left\{p(s)\,\rho^{\alpha}(s)-\mu\frac{r(\tau(s))\,{\rho'}^{\alpha+1}(s)}{{\tau'}^{\alpha}(s)\,\rho(s)}\right\}ds = \infty, \tag{2.3}$$

where $\mu := \left(\dfrac{\alpha}{\alpha+1}\right)^{\alpha+1}\left(\frac{\alpha}{k}\right)^{\alpha}$, then Eq. (1.1) is oscillatory.

Proof. Assume the theorem false. Let $u(t)$ be a nonoscillatory solution of (1.1). Without loss of generality we may assume that $u(t) > 0$. This implies that

$$\left(r(t)\left|u'(t)\right|^{\alpha-1}u'(t)\right)' = -p(t)\,f(u(\tau(t))) < 0.$$

Hence the function $r(t)\left|u'(t)\right|^{\alpha-1}u'(t)$ is decreasing and therefore there are two cases $u'(t) > 0$ and $u'(t) < 0$. The case $u'(t) < 0$, by the hypothesis (H_2), is impossible and we see that $u'(t) > 0$ on $[t, \infty)$ for some $t_1 \geq t_0$. Define

$$w(t) := r(t)\frac{(\rho(t)\,u'(t))^{\alpha}}{f(u(\tau(t)))},\ t \in [t_1, \infty). \tag{2.4}$$

Then $w(t) > 0$. Differentiating $w(t)$ and using Eq. (1.1), since $r(t)\,u'^{\alpha}(t)$ is decreasing, we have Riccati type inequality

$$w'(t) \leq \frac{\alpha\rho'(t)}{\rho(t)}w(t) - \rho^{\alpha}(t)\,p(t) - \frac{(w(t))^{1+1/\alpha}\,\tau'(t)\,f'(u(\tau(t)))}{(r(\tau(t)))^{1/\alpha}\,\rho(t)\,f(u(\tau(t)))^{1-1/\alpha}}. \tag{2.5}$$

Let us assume that $u(t)$ is bounded. Then there exist some positive constants c_1 and c_2 such that for all $t \geq t_0$

$$c_2 \leq u(t) \leq c_1 \text{ and } c_2 \leq u(\tau(t)) \leq c_1.$$

Integrating Eq. (1.1) from t to ∞, we obtain

$$r(t)\,u'^{\alpha}(t)\Big|_t^{\infty} = -\int_t^{\infty} p(s)\,f(u(\tau(s)))\,ds.$$

Since $r(t)\,u'^{\alpha}(t)$ is positive and decreasing, we have

$$r(t)\,u'^{\alpha}(t) \geq \int_t^{\infty} p(s)\,f(u(\tau(s)))\,ds.$$

Integrating this inequality again from t_0 to t, we have

$$u(t) \geq \int_{t_0}^{t}\left(\frac{1}{r(s)}\int_s^{\infty} p(z)\,f(u(z))\,dz\right)^{1/\alpha} ds.$$

Denote $f_0 = \min\limits_{u\in[c_1,\ c_2]} f(u)$. Then from this inequality

$$c_1 \geq u(t) \geq f_0^{1/\alpha}\int_{t_0}^{t}\left(\frac{1}{r(s)}\int_s^{\infty} p(z)\,dz\right)^{1/\alpha} ds. \tag{2.6}$$

Letting $t \to \infty$ the last inequality contradicts to (2.2). Therefore, we conclude $u(t) \to \infty$ as $t \to \infty$. Thus $u(\tau(t)) \in R_D$ for all t large enough. Now it is easy to see that condition $f'(u(\tau(t)))\,/\,(f(u(\tau(t))))^{1-1/\alpha} \geq k$ implies

$$w'(t) \leq \frac{\alpha\rho'(t)}{\rho(t)}w(t) - \rho^{\alpha}(t)\,p(t) - k\frac{(w(t))^{1+1/\alpha}\,\tau'(t)}{(r(\tau(t)))^{1/\alpha}\,\rho(t)},\ t \geq t_1. \tag{2.7}$$

By using the inequality

$$Ax - Bx^{1+1/\alpha} \leq \frac{\alpha^{\alpha}}{(\alpha+1)^{\alpha+1}}A^{\alpha+1}B^{-\alpha},\ B > 0,\ A \geq 0,\ x \geq 0 \tag{2.8}$$

we get

$$w'(t) \le -\left\{ p(t)\rho^{\alpha}(t) - \mu \frac{r(\tau(t))\rho'^{\alpha+1}(t)}{\rho(t)\tau'^{\alpha}(t)} \right\}, \; t \ge t_1. \tag{2.9}$$

Integrating this inequality from t_1 to t, we get

$$w(t) \le w(t_1) - \int_{t_1}^{t} \left\{ \rho^{\alpha}(s)p(s) - \mu \frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)} \right\} ds.$$

Letting $\limsup_{t\to\infty}$, we get in view of (2.3) that $w(t) \to -\infty$, which contradicts $w(t) > 0$ and the proof is complete. ∎

The conclusions of Theorem 1 leads to the following.

Corollary 1. *Let the condition (2.3) in Theorem 1 be replaced by*

$$\int_{t_0}^{\infty} \rho^{\alpha}(s)p(s)\,ds = \infty, \;\; \textit{and} \;\; \int_{t_0}^{\infty} \frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)} ds < \infty, \tag{2.10}$$

then the conclusion of Theorem 1 holds.

Next we state the following result.

Corollary 2. *Assume that (2.1) and (2.2) are satisfied. If there exists a differentiable positive function ρ such that*

$$\rho'(t) > 0 \; \textit{for all} \; t \ge t_0, \tag{2.11}$$

$$\int_{t_0}^{\infty} \frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)} ds = \infty, \tag{2.12}$$

and

$$\liminf_{t\to\infty} \frac{\rho^{\alpha+1}(t)p(t)(\tau'(t))^{\alpha}}{r(\tau(t))(\rho'(t))^{\alpha+1}} > \mu, \tag{2.13}$$

then Eq. (1.1) is oscillatory.

Proof. It is enough to show that (2.12) and (2.13) together implies (2.3). From (2.13), it follows that there exist $\epsilon > 0$ such that for all large t

$$\frac{\rho^{\alpha+1}(t)p(t)(\tau'(t))^{\alpha}}{r(\tau(t))(\rho'(t))^{\alpha+1}} > \mu + \epsilon.$$

This means that

$$\rho^{\alpha}(t)p(t) - \mu \frac{r(\tau(t))(\rho'(t))^{\alpha+1}}{\rho(t)(\tau'(t))^{\alpha}} > \epsilon \frac{r(\tau(t))(\rho'(t))^{\alpha+1}}{\rho(t)(\tau'(t))^{\alpha}}.$$

Now, it is obvious that from (2.11), this inequality implies (2.3) and the assertion of this corollary follows from Theorem 1. ∎

If we choose $\rho(t) = R(\tau(t))$ in Theorem 1, Corollary 1 and Corollary 2, we have the following oscillation criteria.

Theorem 2. *Assume that (2.1) and (2.2) hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left\{R^{\alpha}\left(\tau\left(s\right)\right)p\left(s\right)-\mu\frac{\tau'\left(s\right)}{R\left(\tau\left(s\right)\right)r^{1/\alpha}\left(\tau\left(s\right)\right)}\right\}ds=\infty, \tag{2.14}$$

then Eq. (1.1) is oscillatory.

Corollary 3. *Assume that (2.1) and (2.2) hold. If there exists a positive constant ϵ such that*

$$\int_{t_0}^{\infty}R^{\alpha\epsilon}\left(\tau\left(s\right)\right)p\left(s\right)ds=\infty,\ 0<\epsilon<1, \tag{2.15}$$

then Eq. (1.1) is oscillatory.

Corollary 4. *Assume that (2.1) and (2.2) hold. If*

$$\liminf_{t\to\infty}\frac{R^{\alpha+1}\left(\tau\left(t\right)\right)r^{1/\alpha}\left(\tau\left(t\right)\right)p\left(t\right)}{\tau'\left(t\right)}>\mu, \tag{2.16}$$

then Eq. (1.1) is oscillatory.

If we take $\rho\left(t\right)=\tau\left(t\right)$, the conclusions of Theorem 1, Corollary 1 and Corollary 2 lead to the following oscillation criteria.

Theorem 3. *Assume that (2.1) and (2.2) hold. If*

$$\limsup_{t\to\infty}\int_{t_0}^{t}\left\{\tau^{\alpha}\left(s\right)p\left(s\right)-\mu\frac{r\left(\tau\left(s\right)\right)\tau'\left(s\right)}{\tau\left(s\right)}\right\}ds=\infty, \tag{2.17}$$

then Eq. (1.1) is oscillatory.

Corollary 5. *Assume that (2.1) and (2.2) hold. If there exists a positive constant ϵ such that*

$$\int_{t_0}^{\infty}\tau^{\alpha\epsilon}\left(s\right)p\left(s\right)ds=\infty,\ 0<\epsilon<1, \tag{2.18}$$

then Eq. (1.1) with $r\left(t\right)\equiv1$ is oscillatory.

Corollary 6. *Assume that (2.1) and (2.2) hold. If*

$$\liminf_{t\to\infty}\frac{\tau^{\alpha+1}\left(t\right)p\left(t\right)}{\tau'\left(t\right)}>\mu, \tag{2.19}$$

then Eq. (1.1) with $r\left(t\right)\equiv1$ is oscillatory.

From Theorem 2, we get the following oscillation criterion.

Corollary 7. *Assume that (2.1) and (2.2) hold. If*

$$\liminf_{t\to\infty}R^{\alpha}\left(\tau\left(t\right)\right)\int_{t}^{\infty}p\left(s\right)ds>\mu\frac{1}{\alpha}, \tag{2.20}$$

then Eq. (1.1) is oscillatory.

Proof. It is sufficient to prove that (2.20) implies (2.14). Suppose that (2.14) fails, that is for all $\epsilon > 0$ there exists a t_1 such that for all $t \geq t_1$

$$\int_t^\infty R^\alpha(\tau(s))\left(p(s) - \mu\frac{\tau'(s)}{R^{\alpha+1}(\tau(s))\, r^{1/\alpha}(\tau(s))}\right) ds < \epsilon.$$

Since $R^\alpha(\tau(t))$ is nondecreasing, this inequality implies

$$R^\alpha(\tau(t))\int_t^\infty \left(p(s) - \mu\frac{\tau'(s)}{R^{\alpha+1}(\tau(s))\, r^{1/\alpha}(\tau(s))}\right) ds < \epsilon.$$

Therefore

$$R^\alpha(\tau(t))\int_t^\infty p(s)\, ds < \epsilon + \frac{\mu}{\alpha}$$

for all $\epsilon > 0$ which contradicts to (2.20). ∎

Similarly, we have the following result from Theorem 3.

Corollary 8. *Assume that (2.1) and (2.2) hold. If*

$$\liminf_{t\to\infty} \tau^\alpha(t)\int_t^\infty p(s)\, ds > \frac{\mu}{\alpha}, \tag{2.21}$$

then Eq. (1.1) with $r(t) \equiv 1$ is oscillatory.

Remark 1. *When $\alpha = 1$ and $r(t) \equiv 1$, Theorem 1 and Corollary 2 with $\rho(t) = t$, Theorem 3, Corollary 6 reduce to Theorem 2.1, Corollary 2.1, Theorem 2.2, and Corollary 2.5 in* [7]*, respectively.*

3. Philos, Kamenev- type oscillation criteria for Equation (1.1)

In this section, by using the generalized Riccatti technique (2.4) and the integral averaging technique, similar to that Grace [12], Kamenev [14], Philos [25], Rogovchenko [27], Tiryaki [28], and Wong [31], we give new oscillation criteria for Eq. (1.1) thereby improving our main results.

For this purpose, we first define the sets

$$D_0 = \{(t,s) : t > s \geq t_0\} \text{ and } D = \{(t,s) : t \geq s \geq t_0\}.$$

We introduce a general class of parameter functions $H : D \to R$ which have continuous partial derivative on D with respect to the second variable and satisfy

H_1 : $H(t,\ t) = 0$ for $t \geq t_0$ and $H(t,s) > 0$ for all $(t,s) \in D_0$,

H_2 : $-\dfrac{\partial H(t,s)}{\partial s} \geq 0$ for all $(t,s) \in D$.

Suppose that $h : D_0 \to R$ is a continuous function such that

$$\alpha\frac{\rho'(s)}{\rho(s)}H(t,s) - \frac{\partial H(t,s)}{\partial s} = h(t,s)\,(H(t,s))^{\alpha/\alpha+1} \text{ for all } (t,s) \in D_0.$$

Note that, by choosing specific functions H, it is possible to derive several oscillation criteria for a wide range of differential equations, see [5, 6, 21, 22, 27]. More general types of such functions have been constructed in [28, 31].

Theorem 4. *Assume that (2.1) and (2.2) hold. If there exist a positive differentiable function $\rho \in C^1([t_0, \infty), R^+)$ and H such that (H_1) and (H_2) hold, and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \left\{ p(s)\rho^{\alpha}(s) - \mu \frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\tau'^{\alpha}(s)\rho(s)} \right\} ds = \infty, \tag{3.1}$$

then Eq. (1.1) is oscillatory.

Proof. Using the function $w(t)$ defined in (2.4) and proceeding similarly as in the proof of Theorem 1, we have inequality (2.9).

$$w'(t) + \left\{ p(t)\rho^{\alpha}(t) - \mu \frac{r(\tau(t))\rho'^{\alpha+1}(t)}{\rho(t)\tau'^{\alpha}(t)} \right\} \le 0, \ t \ge t_1.$$

Multiplying this inequality by $H(t,s)$, $t > s$, and next integrating from t_1 to t after simple computation we have

$$\begin{aligned} \int_{t_1}^{t} H(t,s) \left\{ p(s)\rho^{\alpha}(s) - \mu \frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)} \right\} ds &\le H(t,t_1) w(t_1) \\ &\le H(t,t_0) w(t_1). \end{aligned}$$

Therefore

$$\begin{aligned} &\int_{t_0}^{t} H(t,s) \left\{ p(s)\rho^{\alpha}(s) - \mu \frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)} \right\} ds \\ = \ &\int_{t_0}^{t_1} H(t,s) \left\{ p(s)\rho^{\alpha}(s) - \mu \frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)} \right\} ds \\ &+ \int_{t_1}^{t} H(t,s) \left\{ p(s)\rho^{\alpha}(s) - \mu \frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)} \right\} ds \\ \le \ &H(t,t_0) \int_{t_0}^{t_1} p(s)\rho^{\alpha}(s)\, ds + H(t,t_0) w(t_1) \ \text{ for all } t \ge t_0. \end{aligned} \tag{3.2}$$

This gives

$$\begin{aligned} &\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s) \left\{ p(s)\rho^{\alpha}(s) - \mu \frac{r(\tau(s))\rho'^{\alpha+1}(s)}{\rho(s)\tau'^{\alpha}(s)} \right\} ds \\ \le \ &\int_{t_0}^{t_1} p(s)\rho^{\alpha}(s)\, ds + w(t_1) \end{aligned}$$

which contradicts (3.1). This completes the proof of the theorem. ∎

Theorem 5. *Assume that (2.1) and (2.2) hold. If there exist a positive differentiable function $\rho \in C^1([t_0, \infty), R^+)$ and H such that (H_1), (H_2) and (H_3) hold, and*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left\{ H(t,s)p(s)\rho^{\alpha}(s) - \mu_1 \frac{r(\tau(s))\rho^{\alpha}(s)}{\tau'^{\alpha}(s)} (h(t,s))^{\alpha+1} \right\} ds = \infty, \tag{3.3}$$

where $\mu_1 := \dfrac{1}{(\alpha+1)^{\alpha+1}}\left(\dfrac{\alpha}{k}\right)^{\alpha}$, *then Eq. (1.1) is oscillatory.*

Proof. Using the function $w(t)$ defined in (2.4) and proceeding similarly as in the proof of Theorem 1, we obtain inequality (2.7).

$$w'(t) \le \alpha\frac{\rho'(t)}{\rho(t)}w(t) - \rho^{\alpha}(t)\,p(t) - k\frac{(w(t))^{1+1/\alpha}\,\tau'(t)}{r(\tau(t))^{1/\alpha}\,\rho(t)},\ t \ge t_1. \tag{2.7}$$

Multiplying this inequality by H and next integrating from t_1 to t we have

$$\begin{aligned}&\int_{t_1}^{t} H(t,s)\,\rho^{\alpha}(s)\,p(s)\,ds \le H(t,t_1)\,w(t_1)\\ &-\int_{t_1}^{t}\left[\left(\frac{\partial H(t,s)}{\partial s} - \alpha\frac{\rho'(s)}{\rho(s)}H(t,s)\right)w(s)\right.\\ &\left.+H(t,s)\,\frac{\tau'(s)\,k}{r(\tau(s))^{1/\alpha}\,\rho(s)}\,(w(s))^{1+1/\alpha}\right]ds.\end{aligned}$$

By using the inequality (2.8), we get

$$\int_{t_1}^{t} H(t,s)\,\rho^{\alpha}(s)\,p(s)\,ds \le H(t,t_1)\,w(t_1) - \mu_1\int_{t_1}^{t}\rho^{\alpha}(s)\,\frac{r(\tau(s))}{(\tau'(s))^{\alpha}}\,(h(t,s))^{\alpha+1}\,ds$$

Next proceeding similarly as in the inequality (3.2) we obtain

$$\begin{aligned}&\int_{t_0}^{t}\left\{H(t,s)\,\rho^{\alpha}(s)\,p(s) - \mu_1\rho^{\alpha}(s)\,\frac{r(\tau(s))}{(\tau'(s))^{\alpha}}\,(h(t,s))^{\alpha+1}\right\}ds\\ \le\ & H(t,t_0)\left(\int_{t_0}^{t_1}\rho^{\alpha}(s)\,p(s)\,ds + w(t_1)\right)\end{aligned}$$

which contradicts (3.3). Thus, the proof is complete. ∎

Corollary 9. *The conclusion of Theorem 5 remains valid , if assumption (3.3) is replaced by*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t} H(t,s)\,p(s)\,\rho^{\alpha}(s)\,ds = \infty \tag{3.4}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{r(\tau(s))\,\rho^{\alpha}(s)}{(\tau'(s))^{\alpha}}\,(h(t,s))^{\alpha+1}\,ds < \infty. \tag{3.5}$$

Note that, by choosing specific functions ρ and H, it is possible to derive several oscillation criteria for Eq. (1.1) and its special cases Half-linear Eq. (1.2) and generalized Emden-Fowler Eq. (1.5) with $\beta \ge \alpha$.

In particular, by choosing $\alpha = 1$, $r(t) \equiv 1$, $H(t,s) = (t-s)^n$, $\rho(t) = t$ in Theorem 4 and by choosing $\alpha = 1$, $r(t) \equiv 1$, $\rho(t) = t$ and also $\rho(t) = \tau(t)$ in Theorem 5 we obtain the corresponding results given in [7], respectively.

Remark 2. *If we take* $f(x) = |x|^{\alpha-1} x$*, then* $\dfrac{f'(x)}{|f(x)|^{1-1/\alpha}} = k = \alpha$ *is satisfied for all* $x \neq 0 \in R$*. Therefore the condition (2.2) is not required. That is, all above oscillation criteria are valid for the half-linear Eq. (1.2) without using the condition (2.2). Hence, Theorem 2, Corollary 3, Theorem 3, and Corollary 5, improve Theorem 1, Corollary 1, Theorem 3, and Corollary 2 in* [10]*, respectively. Also Theorem 2 improves Theorem 1 in* [8] *and Theorem 2.3 in* [4]*. Finally, Theorem 1 gives Corollary 1 in* [3] *by changing* $\rho(t)$ *with* $\rho^{1/\alpha}(t)$ *and Corollary 8 reduces Theorem 3 in* [17]*.*

Remark 3. *Let* $\beta \geq \alpha$*. If we take* $f(x) = |x|^{\beta-1} x$*, then* $\dfrac{f'(x)}{|f(x)|^{1-1/\alpha}} = \beta |x|^{\beta/\alpha-1}$ *and we can find always a positive constant* k *large enough such that* $f'(x)/|f(x)|^{1-1/\alpha} \geq k$ *for all* $x \in R_D$*. Hence, all above oscillation criteria are valid for the generalized Emden-Fowler Eq. (1.5). We note that these conclusions, for Eq. (1.5) with* $\beta \geq \alpha$*, complement Theorem 3.1 in* [4] *and they do not appear to follow from the known oscillation criteria in the literature.*

Example 1. *Consider the generalized Emden-Fowler delay differential equation*

$$\left(|x'(t)|^{\alpha-1} x'(t)\right)' + p(t)\left|x\left(\frac{t}{2}\right)\right|^{\beta-1} x\left(\frac{t}{2}\right) = 0, \tag{3.6}$$

where α, β *are positive constants such that* $\beta \geq \alpha$ *and* $p \in C([1, \infty), R^+)$. $f(u) = |u|^{\beta-1} u$ *and there exists a* $k > 0$ *such that* $f'(u) \geq k$ *for all* $u \in R_D$, k *large enough. Hence (2.1) holds.*

Here we choose $\rho(t) = t^{\lambda/\alpha}$ *and* $H(t,s) = (t-s)^{\lambda}$*, for* $t \geq s \geq 1$ *such that* $\alpha + 1 < \lambda < \alpha^2$ *and* $\rho(t)p(t) \geq \dfrac{c}{t}$ *where* c *is a positive constant.*

$$\begin{aligned}
\lim_{t\to\infty} \int_1^t \left(\int_s^\infty p(z)\,dz\right)^{1/\alpha} ds &\geq \lim_{t\to\infty} \int_1^t \left(\int_s^\infty c z^{-1-\lambda/\alpha} dz\right)^{1/\alpha} ds \\
&= \lim_{t\to\infty} \frac{\alpha^2}{\alpha^2 - \lambda} \left(\frac{c\alpha}{\lambda}\right)^{1/\alpha} \left(t^{1-\frac{\lambda}{\alpha^2}} - 1\right) = \infty.
\end{aligned}$$

Using the inequality

$$(t-s)^{\lambda} \geq t^{\lambda} - \lambda s t^{\lambda-1}, \text{ for } t \geq s \geq 1$$

given in [21]*, we have*

$$\limsup_{t\to\infty} \frac{1}{t^{\lambda}} \int_1^t (t-s)^{\lambda} \rho(s) p(s)\,ds \geq \limsup_{t\to\infty} \frac{c}{t^{\lambda}} \int_1^t \frac{t^{\lambda} - \lambda s t^{\lambda-1}}{s} ds = \infty.$$

On the other hand, observing that $H(t,s) = \lambda (t-s)^{\frac{\lambda-\alpha-1}{\alpha+1}} t s^{-1}$*, we obtain*

$$\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t^{\lambda}} \int_1^t \frac{s^{\lambda}}{2^{-\alpha}} \left[\lambda (t-s)^{\frac{\lambda-\alpha-1}{\alpha+1}} t s^{-1}\right]^{\alpha+1} ds \\
\leq\ &\limsup_{t\to\infty} \frac{2^{\alpha} t^{\lambda+1}}{\lambda - \alpha} \left(1 - \frac{t_0}{t}\right)^{\lambda-\alpha-1} \left(t^{\lambda-\alpha} - t_0^{\lambda-\alpha}\right) < \infty.
\end{aligned}$$

Consequently, all condition of Corollary 9 are satisfied and hence Eq. (3.6) is oscillatory.

Note that criteria reported in the references do not apply to Eq. (3.6).

References

[1] Agarwal, R.P., Dontha, S., and Grace, S.R., Linearization of second order nonlinear oscillation theorems, Pan. Amer. Math. Journal, 15 (2005), 1-15.

[2] Agarwal, R.P., Grace, S.R., and O'Regan, D., Oscillation Theory for Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations, Kluwer Academic Publishers, London, (2002).

[3] Agarwal, R.P., Grace, S.R., and O'Regan, D., Oscillation Criteria for certain n th order differential equations with deviating arguments, J. Math. Anal. Appl., 262 (2001), 601-622.

[4] Agarwal, R.P., Shieh, S.L., and Yeh, C.C., Oscillation criteria for second-order retarded differential equations, Math. Comput. Modell, 26 (1997), 1-11.

[5] Ayanlar, B., and Tiryaki, A., Oscillation theorems for nonlinear second order differential equation with damping, Acta Math. Hungar, 89 (2000), 1-13.

[6] Ayanlar, B., and Tiryaki, A. Oscillation theorems for nonlinear second-order differential equations, Comput. Math. Appl., 44 (2002), no. 3-4, 529–538.

[7] Baculikova, B., Oscillation criteria for second order nonlinear differential equations, Arch. Math(Brno)., 42 (2006), 141-149.

[8] Chern, J.L., Lian, Ch.W., and Yeh, C.C., Oscillation criteria for second-order half-linear differential equations with functional arguments, Publ. Math. Debrecen, 48 (1996), 209-216.

[9] Dosly, O., and Rehak, P., Half-Linear Differential Equations, Mathematics Studies. 202, North-Holland, 2005.

[10] Dzurina, J., and Stavroulakis, I.P., Oscillation criteria for second-order delay differential equations, Apply. Math. Comput, 140 (2003), 445-453.

[11] Elbert, A., Oscillation and nonoscillation theorems for some nonlinear ordinary differential equations, in: Ordinary and Partial Differential Equations, Lecture Notes in Mathematics, 964 (1982), 187-212.

[12] Grace, S.R., Oscillation theorems for nonlinear differential equations of second order, J. Math. Anal. Appl., 171 (1992), 220-241.

[13] Gyori, I., and Ladas, G., Oscillation Theory of Delay Differential Equations With Applications, Clarendon Press, Oxford. 1991.

[14] Kamenev, I.V., An integral test for conjugacy for second order linear differential equations, Math. Zemetki, 23 (1978), 249-251(in Russian).

[15] Kitano, M., and Kusano, T., On a class of second order quasilinear ordinary differential equations, Hiroshima Math. J., 25 (1995), 321-355.

[16] Kusano, T., and Naito, Y. Oscillation and nonoscillation criteria for second order quasilinear differential equations, Acta Math. Hungar, 76 (1997), 81-99.

[17] Kusano, T., and Wang, J., Oscillation properties of half-linear functional differential equations of the second order, Hiroshima Math. J., 25 (1995), 371-385.

[18] Kusano, T., and Yoshida, N., Nonoscillation theorems for a class of quasilinear differential equations of second order, J. Math. Anal. Appl., 189 (1995), 115–127.

[19] Kusano, T., Naito Y., and Ogata, A., Strong oscillation and nonoscillation of quasilinear differential equations of second order, Diff. Equat. Dyn. Syst., 2 (1994), 1–10.

[20] Ladde, G.S., Lakshmikantham, V., and Zhang, B.G., Oscillation theory of differential equations with deviating arguments, Marcel Dekker, Inc., Newyork., (1987).

[21] Li, H.J., Oscillation criteria for second order linear differential equations, J. Math. Anal. Appl., 194 (1995), 217-234.

[22] Li, H.J., Oscillation criteria for half-linear second order differential equations, Hiroshima Math. J., 25 (1995), 571-583.

[23] Mirzov, D.D., The oscillation of the solutions of a certain system of differential equations, Math. Zametki, 23 (1978), 401-404.

[24] Mirzov, D.D., The oscillatoriness of the solutions of a system of nonlinear differential equations, Differentsial nye Uravneniya, 9 (1973), 581-583.
[25] Philos, Ch.G., Oscillation theorems for linear differential equation of second order, Arch. Math., 53 (1989), 482-492.
[26] Philos, Ch.G., and Sficas, Y.G., Oscillatory and asymptotic behavior of second and third order retarded differential equations, With a loose Russian summary. Czech. Math. J., 32(107) (1982), 169–182.
[27] Rogovchenko, Yu.V., Oscillation criteria for certain nonlinear differential equations, J. Math. Anal. Appl., 229 (1999), 399-416.
[28] Tiryaki, A., Çakmak, D., and Ayanlar, B., On the oscillation of certain second-order nonlinear differential equation, J. Math. Anal. Appl., 281 (2003), 565-574.
[29] Zhiting, X., and Yong, X., Kamenev-type oscillation criteria for second-order quasilinear differential equations, Electronic Journal of Diff. Equat., 27 (2005), 1-9.
[30] Wang, J., Oscillation and nonoscillation theorems for a class of second order quasilinear functional-differential equations, Hiroshima Math. J., 27 (1997), 449–466.
[31] Wong, J.S.W., On Kamenev-type oscillation theorems for second-order differential equations with damping, J. Math. Anal. Appl., 258 (2001), 244-257.

(Received October 17, 2009)

Department of Mathematics and Computer Sciences, Izmir University, Faculty of Arts and Sciences, 35350 Uckuyular, Izmir, Turkey
E-mail address: aydin.tiryaki@izmir.edu.tr