Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **60**, 1-8; **http://www.math.u-szeged.hu/ejqtde/**

Forced Oscillation of Second-order Nonlinear Dynamic Equations on Time Scales *

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Abstract: In this paper, by defining a class of functions, we establish some oscillation criteria for the second order nonlinear dynamic equations with forced term

$$x^{\Delta\Delta}(t) + a(t)f(x(q(t))) = e(t)$$

on a time scale $\mathbb{T}$. Our results unify the oscillation of the second order forced differential equation and the second order forced difference equation. An example is considered to illustrate the main results.

Keywords: Oscillation; Forced term; Dynamic equations; Time scales
Mathematics Subject Classification 2000: 34N05; 39A10; 39A21

1 Introduction

The theory of time scales, which has recently received a lot of attention, was introduced by Hilger in his PhD thesis in 1988 in order to unify continuous and discrete analysis (see Hilger [19]). The area of dynamic equations on time scales is a new component of applied analysis that describes processes that feature both continuous and discrete elements. Several authors have expounded on various aspects of this new theory, see the survey paper by Agarwal et al. [1] and references cited therein. A book on the subject of time scales, by Bohner and Peterson [7], summarizes and organizes much of the time scale calculus. We refer also to the last book by Bohner and Peterson [8] for advances in dynamic equations on time scales.

*Corresponding author: Zhenlai Han, *e-mail: hanzhenlai@163.com*. This research is supported by the Natural Science Foundation of China (60774004), China Postdoctoral Science Foundation Funded Project (20080441126), Shandong Postdoctoral Funded Project (200802018) and supported by Shandong Research Funds (Y2008A28), also supported by University of Jinan Research Funds for Doctors (B0621, XBS0843).

We are concerned here with second-order forced nonlinear dynamic equations of the form

$$x^{\Delta\Delta}(t) + a(t)f(x(q(t))) = e(t) \tag{1.1}$$

on a time scale $\mathbb{T}$, where $a(t)$, $q(t)$ and $e(t)$ are real-valued rd-continuous functions defined on $\mathbb{T}$, the function $q(t)$ also satisfies $q : \mathbb{T} \to \mathbb{T}$, $q(t) \to \infty$ as $t \to \infty$, and $f \in C(\mathbb{R}, \mathbb{R})$, $xf(x) > 0$ whenever $x \neq 0$.

Agarwal and Grace [4], Ou and Wong [27], Wong [28] studied the n-order forced functional differential equations of the form

$$x^{(n)}(t) + a(t)f(x(q(t))) = e(t).$$

Çakmak and Tiryaki [11] studied (1.1) in the case when $\mathbb{T} = \mathbb{R}$. Sun and Saker [26] considered the n-order forced delay difference equations of the form

$$\Delta^m x(n) + q(n)f(x(n-\tau)) = e(n).$$

In recent years, there has been much research activity concerning the oscillation of solutions of various second order dynamic equations on time scales, we refer the reader to the articles [2, 3, 5, 6, 9, 10, 12–17, 20, 21, 23–25] and references cited therein. Agarwal et al. [3] investigated the second order forced dynamic equations with mixed nonlinearities

$$(r(t)\phi_\alpha(x^\Delta))^\Delta + f(t, x^\sigma) = e(t), \quad t \in \mathbb{T},$$

with $f(t,x) = q(t)\phi_\alpha(x) + \sum_{i=1}^n q_i(t)\phi_{\beta_i}(x)$, $\phi_*(u) = |u|^{*-1}u$. Anderson [5] considered the oscillations of the forced dynamic equations

$$(rx^\Delta)^\Delta(t) + p(t)|x(\tau(t))|^{\alpha-1}x(\tau(t)) + q(t)|x(\theta(t))|^{\beta-1}x(\theta(t)) = f(t), \quad t \in \mathbb{T},$$

and Anderson [6] studied the second order dynamic equations

$$(rx^\Delta)^\Delta(t) + f(t, x^\sigma(t), x^\Delta(t)) = 0, \quad t \in \mathbb{T}.$$

Bohner and Tisdell [10] examined oscillation and nonoscillation for

$$(rx^\Delta)^\Delta(t) + p(t)x^\sigma(t) = e(t), \quad t \in \mathbb{T}.$$

Huang and Feng [20] considered the following second-order forced nonlinear dynamic equations

$$x^{\Delta\Delta}(t) + p(t)f(x^\sigma(t)) = e(t), \quad t \in \mathbb{T},$$

and in [21] the authors studied the oscillation of the forced dynamic equations

$$x^{\Delta\Delta}(t) + p(t)f(x(t)) = e(t), \quad t \in \mathbb{T}.$$

Oscillatory criteria for the forced dynamic equations

$$(a(t)x^\Delta(t))^\Delta + p(t)f(x(\sigma(t))) = r(t), \quad t \in \mathbb{T},$$

where $\int_{t_0}^{\infty} |r(s)|\Delta s < \infty$ are analyzed in [24].

Philos-type functions [22] are used extensively in the theory of oscillations, for example, Erbe et al. [12], Han et al. [17], Saker et al. [25] established some Philos-type oscillation criteria for the second-order delay dynamic equations on time scales. In [12], the authors utilized the class of functions as follows: Assume H is defined for $t_0 \le s \le \sigma(t)$, t, $s \in [t_0, \infty)_{\mathbb{T}}$, $H(t,s) \ge 0$, $H(\sigma(t), t) = 0$, $H^{\Delta_s}(t,s) \le 0$ for $t \ge s \ge t_0$, and for each fixed t, $H^{\Delta_s}(t,s)$ is delta integrable with respect to s.

As we are interested in oscillatory behavior, we assume throughout this paper that the given time scale $\mathbb{T}$ is unbounded above. We assume $t_0 \in \mathbb{T}$ and it is convenient to assume $t_0 > 0$. We define the time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$.

2 Main Results

In this section, we give some new oscillation criteria for (1.1). Motivated by [22], let us introduce the class of functions $\mathfrak{H}$ which will be extensively used in the sequel.

Let $\mathbb{D}_0 \equiv \{(t,s) \in \mathbb{T}^2 : t > s \geq t_0\}$ and $\mathbb{D} \equiv \{(t,s) \in \mathbb{T}^2 : t \geq s \geq t_0\}$. We say that the function $H \in C_{rd}(\mathbb{D}, \mathbb{R})$ belongs to the class $\mathfrak{H}$, if

(i) $H(t,t) = 0$, $t \geq t_0$, $H(t,s) > 0$ on $\mathbb{D}_0$,

(ii) H has a nonpositive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ and a nonnegative continuous second-order Δ-partial derivative $H^{\Delta^2_{s^2}}(t,s)$ with respect to the second variable,

(iii) $H^{\Delta_s}(\sigma(t), \sigma(t)) = 0$,

(iv) $\lim\limits_{t\to\infty} \frac{H^{\Delta_s}(\sigma(t),t_0)}{H(\sigma(t),t_0)} = O(1)$.

Theorem 2.1 *Let $a(t) \geq 0$. If there exists a function $H \in \mathfrak{H}$, such that*

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t),t_0)} \int_{t_0}^{\sigma(t)} H\left(\sigma(t), \sigma(s)\right) e(s) \Delta s = +\infty \tag{2.1}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(\sigma(t),t_0)} \int_{t_0}^{\sigma(t)} H\left(\sigma(t), \sigma(s)\right) e(s) \Delta s = -\infty, \tag{2.2}$$

then (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality we assume $x(t) > 0$, $x(q(t)) > 0$ for $t \geq t_0$. Multiplying (1.1) by $H\left(\sigma(t), \sigma(s)\right)$, and integrating from t_0 to $\sigma(t)$, we have

$$\begin{aligned}
&\int_{t_0}^{\sigma(t)} H\left(\sigma(t), \sigma(s)\right) x^{\Delta\Delta}(s) \Delta s \\
&+ \int_{t_0}^{\sigma(t)} H\left(\sigma(t), \sigma(s)\right) a(s) f(x(q(s))) \Delta s = \int_{t_0}^{\sigma(t)} H\left(\sigma(t), \sigma(s)\right) e(s) \Delta s.
\end{aligned} \tag{2.3}$$

Using the integration by parts formula two times, we get

$$\begin{aligned}
\int_{t_0}^{\sigma(t)} H\left(\sigma(t), \sigma(s)\right) x^{\Delta\Delta}(s) \Delta s &= -H(\sigma(t),t_0) x^{\Delta}(t_0) - \int_{t_0}^{\sigma(t)} H^{\Delta_s}(\sigma(t), s) x^{\Delta}(s) \Delta s \\
&= -H(\sigma(t),t_0) x^{\Delta}(t_0) + H^{\Delta_s}(\sigma(t),t_0) x(t_0) + \int_{t_0}^{\sigma(t)} H^{\Delta^2_{s^2}}(\sigma(t), s) x(\sigma(s)) \Delta s.
\end{aligned} \tag{2.4}$$

Substituting (2.4) into (2.3) and dividing through by $H(\sigma(t), t_0)$, we arrive at

$$\begin{aligned}
\frac{1}{H(\sigma(t),t_0)} \int_{t_0}^{\sigma(t)} H\left(\sigma(t), \sigma(s)\right) e(s) \Delta s &= -x^{\Delta}(t_0) + \frac{H^{\Delta_s}(\sigma(t),t_0)}{H(\sigma(t),t_0)} x(t_0) \\
&+ \frac{1}{H(\sigma(t),t_0)} \int_{t_0}^{\sigma(t)} \left[H^{\Delta^2_{s^2}}(\sigma(t), s) x^{\sigma}(s) + H\left(\sigma(t), \sigma(s)\right) a(s) f(x(q(s))) \right] \Delta s.
\end{aligned} \tag{2.5}$$

Taking lim inf as $t \to \infty$, we derive a contradiction. The proof is complete.

Remark 2.1 *This theorem is true for (1.1), i.e., $q(t) = t$, and true for $q(t) \leq t$ and for $q(t) \geq t$.*

When $\mathbb{T} = \mathbb{R}$, let $H(t, s) = (t - s)^\beta$, $\beta > 1$. It is easy to see that $H \in \mathfrak{H}$. Therefore, we have the following result.

Corollary 2.1 *Let $a(t) \geq 0$ and $\mathbb{T} = \mathbb{R}$. If*

$$\limsup_{t\to\infty} \frac{1}{(t - t_0)^\beta} \int_{t_0}^{t} (t - s)^\beta e(s) ds = +\infty$$

and

$$\liminf_{t\to\infty} \frac{1}{(t - t_0)^\beta} \int_{t_0}^{t} (t - s)^\beta e(s) ds = -\infty,$$

then (1.1) is oscillatory.

When $\mathbb{T} = \mathbb{Z}$, let $H(t, s) = (t - s)^{(k)} = (t - s)(t - s + 1) \cdots (t - s + k - 1)$, $k \geq 2$. It is easy to see that $H \in \mathfrak{H}$. Therefore, we have the following result.

Corollary 2.2 *Let $a(t) \geq 0$ and $\mathbb{T} = \mathbb{Z}$. If*

$$\limsup_{t\to\infty} \frac{1}{(t + 1 - t_0)^{(k)}} \sum_{s=t_0}^{t} (t - s)^{(k)} e(s) = +\infty$$

and

$$\liminf_{t\to\infty} \frac{1}{(t + 1 - t_0)^{(k)}} \sum_{s=t_0}^{t} (t - s)^{(k)} e(s) = -\infty,$$

then (1.1) is oscillatory.

Theorem 2.2 *Assume $H \in \mathfrak{H}$,*

(1) *$a(t) < 0$ for $t \geq t_0$, $q(t) = \sigma(t)$,*

(2) *there exist two positive constants c and λ such that either*

$$|f(x)| \geq c|x|^\lambda, \quad \lambda > 1, \tag{2.6}$$

or

$$|f(x)| \leq c|x|^\lambda, \quad 0 < \lambda < 1, \tag{2.7}$$

(3)

$$\limsup_{t\to\infty} \frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{\sigma(t)} \left(H(\sigma(t), \sigma(s))e(s) - G(t, s)\right) \Delta s = +\infty, \tag{2.8}$$

(4)

$$\liminf_{t\to\infty} \frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{\sigma(t)} \left(H(\sigma(t), \sigma(s))e(s) - G(t, s)\right) \Delta s = -\infty, \tag{2.9}$$

where

$$G(t, s) = (\lambda - 1)\lambda^{\frac{\lambda}{1-\lambda}} (H^{\Delta^2_{s^2}}(\sigma(t), s))^{\frac{\lambda}{\lambda-1}} (-cH(\sigma(t), \sigma(s))\, a(s))^{\frac{1}{1-\lambda}}.$$

Then (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality we assume $x(t) > 0$ for $t \geq t_0$. Assume first that (2.6) holds. Multiplying (1.1) by $H\left(\sigma(t), \sigma(s)\right)$, and integrating from t_0 to $\sigma(t)$, we have

$$\int_{t_0}^{\sigma(t)} H\left(\sigma(t), \sigma(s)\right) x^{\Delta\Delta}(s)\Delta s$$

$$+\int_{t_0}^{\sigma(t)} H\left(\sigma(t), \sigma(s)\right) a(s) f(x(\sigma(s)))\Delta s = \int_{t_0}^{\sigma(t)} H\left(\sigma(t), \sigma(s)\right) e(s)\Delta s. \tag{2.10}$$

Substituting (2.4) and (2.6) into (2.10) and note that $a(t) < 0$, we obtain

$$\frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{\sigma(t)} H\left(\sigma(t), \sigma(s)\right) e(s)\Delta s \leq -x^{\Delta}(t_0) + \frac{H^{\Delta_s}(\sigma(t), t_0)}{H(\sigma(t), t_0)} x(t_0)$$

$$+\frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{\sigma(t)} \left[H^{\Delta^2_{s^2}}(\sigma(t), s) x(\sigma(s)) + cH\left(\sigma(t), \sigma(s)\right) a(s) x^{\lambda}(\sigma(s))\right] \Delta s. \tag{2.11}$$

Set $F(x) = ax - bx^{\lambda}$ for $x > 0$, $a \geq 0$, $b > 0$. If $\lambda > 1$, then $F(x)$ has the maximum $F_{max} = (\lambda - 1)\lambda^{\frac{\lambda}{1-\lambda}} a^{\frac{\lambda}{\lambda-1}} b^{\frac{1}{1-\lambda}}$, (see [18]). Thus from (2.11), we have

$$\frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{\sigma(t)} \left(H(\sigma(t), \sigma(s)) e(s) - G(t, s)\right) \Delta s \leq -x^{\Delta}(t_0) + \frac{H^{\Delta_s}(\sigma(t), t_0)}{H(\sigma(t), t_0)} x(t_0).$$

Taking lim sup as $t \to \infty$ into the above inequality and applying (2.8), we obtain a desired contradiction. If condition (2.7) holds, substituting (2.4) and (2.6) into (2.10) and note that $a(t) < 0$, we obtain

$$\frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{\sigma(t)} H\left(\sigma(t), \sigma(s)\right) e(s)\Delta s \geq -x^{\Delta}(t_0) + \frac{H^{\Delta_s}(\sigma(t), t_0)}{H(\sigma(t), t_0)} x(t_0)$$

$$+\frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{\sigma(t)} \left[H^{\Delta^2_{s^2}}(\sigma(t), s) x(\sigma(s)) + cH\left(\sigma(t), \sigma(s)\right) a(s) x^{\lambda}(\sigma(s))\right] \Delta s. \tag{2.12}$$

Set $F(x) = ax - bx^{\lambda}$ for $x > 0$, $a \geq 0$, $b > 0$. If $0 < \lambda < 1$, then $F(x)$ has the minimum $F_{min} = (\lambda - 1)\lambda^{\frac{\lambda}{1-\lambda}} a^{\frac{\lambda}{\lambda-1}} b^{\frac{1}{1-\lambda}}$, (see [18]). Thus from (2.12), we have

$$\frac{1}{H(\sigma(t), t_0)} \int_{t_0}^{\sigma(t)} \left(H(\sigma(t), \sigma(s)) e(s) - G(t, s)\right) \Delta s \geq -x^{\Delta}(t_0) + \frac{H^{\Delta_s}(\sigma(t), t_0)}{H(\sigma(t), t_0)} x(t_0).$$

Taking lim inf as $t \to \infty$ into the above inequality and applying (2.9), we obtain a desired contradiction. The proof is complete.

Theorem 2.3 *Assume $a(t) < 0$, $q(t) = \sigma(t)$, $H \in \mathfrak{H}$, conditions (2.1) and (2.2) hold. If $|f(x)| \geq |x|$ holds with*

$$H^{\Delta^2_{s^2}}(\sigma(t), s) + H(\sigma(t), \sigma(s)) a(s) \leq 0, \tag{2.13}$$

or $|f(x)| \leq |x|$ holds with

$$H^{\Delta^2_{s^2}}(\sigma(t), s) + H(\sigma(t), \sigma(s)) a(s) \geq 0, \tag{2.14}$$

then (1.1) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of (1.1). Without loss of generality we assume $x(t) > 0$ for $t \geq t_0$. Substituting (2.4) and $|f(x)| \geq |x|$ into (2.10) and note that $a(t) < 0$, we obtain

$$\frac{1}{H(\sigma(t),t_0)} \int_{t_0}^{\sigma(t)} H(\sigma(t),\sigma(s))\, e(s)\Delta s \leq -x^{\Delta}(t_0) + \frac{H^{\Delta_s}(\sigma(t),t_0)}{H(\sigma(t),t_0)} x(t_0)$$

$$+\frac{1}{H(\sigma(t),t_0)} \int_{t_0}^{\sigma(t)} \left[H^{\Delta_{s^2}^2}(\sigma(t),s)x(\sigma(s)) + H(\sigma(t),\sigma(s))\, a(s)x(\sigma(s))\right] \Delta s. \tag{2.15}$$

From (2.13), taking lim sup as $t \to \infty$ into (2.15) and applying (2.1), we obtain a desired contradiction. Substituting (2.4) and $|f(x)| \leq |x|$ into (2.10) and note that $a(t) < 0$, we obtain

$$\frac{1}{H(\sigma(t),t_0)} \int_{t_0}^{\sigma(t)} H(\sigma(t),\sigma(s))\, e(s)\Delta s \geq -x^{\Delta}(t_0) + \frac{H^{\Delta_s}(\sigma(t),t_0)}{H(\sigma(t),t_0)} x(t_0)$$

$$+\frac{1}{H(\sigma(t),t_0)} \int_{t_0}^{\sigma(t)} \left[H^{\Delta_{s^2}^2}(\sigma(t),s)x(\sigma(s)) + H(\sigma(t),\sigma(s))\, a(s)x(\sigma(s))\right] \Delta s. \tag{2.16}$$

Taking lim inf as $t \to \infty$ into (2.16) and applying (2.2), we obtain a desired contradiction. The proof is complete.

Remark 2.2 *The results in Saker [24] cannot be applied in (1.1) when* $\int_{t_0}^{\infty} |r(s)|\Delta s = \infty$.

3 Example

As an application, we consider the following example.

Example Consider the equation

$$x^{\Delta\Delta}(t) + a(t)f(x(q(t))) = t^{\alpha} \sin t, \tag{3.1}$$

where $a(t) \geq 0$, $\alpha > 0$. When $\mathbb{T} = \mathbb{R}$, by Corollary 2.1, (3.1) is oscillatory for $\alpha > 1$. When $\mathbb{T} = \mathbb{Z}$, by Corollary 2.2, we have that (3.1) is oscillatory for $\alpha > 1$.

Acknowledgments

The authors sincerely thank the reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original manuscript.

References

[1] R. P. Agarwal, M. Bohner, D. O'Regan, A. Peterson, Dynamic equations on time scales: a survey. J. Comput. Appl. Math., 141 (2002), 1–26.

[2] R. P. Agarwal, M. Bohner, S. H. Saker, Oscillation of second order delay dynamic equations, Can. Appl. Math. Q. 13 (2005), 1–18.

[3] R. P. Agarwal, A. Zafer, Oscillation criteria for second-order forced dynamic equations with mixed nonlinearities, Advances in Difference Equations, Volume 2009, Article ID 938706, 20 pages.

[4] R. P. Agarwal, S. R. Grace, Forced oscillation of nth order nonlinear differential equations, Appl. Math. Lett. 13 (2000), 53–57.

[5] D. R. Anderson, Oscillation of second-order forced functional dynamic equations with oscillatory potentials, J. Difference Equ. Appl. 13 (5) (2007), 407–421.

[6] D. R. Anderson, Interval criteria for oscillation of nonlinear second-order dynamic equations on time scales, Nonlinear Analysis 69 (2008) 4614–4623.

[7] M. Bohner, A. Peterson, Dynamic equations on time scales, an introduction with applications, Birkhäuser, Boston, 2001.

[8] M. Bohner, A. Peterson, Advances in dynamic equations on time scales, Birkhäuser, Boston, 2003.

[9] M. Bohner, S. H. Saker, Oscillation of second order nonlinear dynamic equations on time scales, Rocky Mountain J. Math. 34 (2004), 1239–1254.

[10] M. Bohner, C. Tisdell, Oscillation and nonoscillation of forced second order dynamic equations, Pacific J. Math. 230 (2007), 59–71.

[11] D. Çakmak, A. Tiryaki, Oscillation criteria for certain forced second-order nonlinear differential equations with delayed argument, Comput. Math. Appl. 49 (2005), 1647–1653.

[12] L. Erbe, A. Peterson, S. H. Saker, Oscillation criteria for second-order nonlinear delay dynamic equations, J. Math. Anal. Appl. 333 (2007), 505–522.

[13] L. Erbe, T. S. Hassan, A. Peterson, S. H. Saker, Interval oscillation criteria for forced second-order nonlinear delay dynamic equations with oscillatory potential, Dynamics of Continuous, Discrete, and Impulsive Systems, to appear.

[14] S. R. Grace, R. P. Agarwal, B. Kaymakcalan, Wichuta Sae-Jie, Oscillation criteria for scond order nonlinear dynamic equations, Canadian Applied Mathematics Quarterly, 16 (1) (2008), 59–76.

[15] S. R. Grace, R. P. Agarwal, M. Bohner, D. O'Regan, Oscillation of second-order strongly superlinear and strongly sublinear dynamic equations, Communications in Nonlinear Science and Numerical Simulation, 14 (2009), 3463–3471.

[16] Zhenlai Han, Shurong Sun, Bao Shi, Oscillation criteria for a class of second order Emden-Fowler delay dynamic equations on time scales, J. Math. Anal. Appl. 334 (2007), 847–858.

[17] Zhenlai Han, Tongxing Li, Shurong Sun, Chenghui Zhang, Oscillation for second-order nonlinear delay dynamic equations on time scales, Advances in Difference Equations, Volume 2009, Article ID 756171, 13 pages.

[18] G. H. Hardy, J. E. Littlewood, G. Polya, Inequalities, second ed., Cambridge University Press, Cambridge, 1988.

[19] S. Hilger, Analysis on measure chains—a unified approach to continuous and discrete calculus, Results Math. 18 (1990), 18–56.

[20] Mugen Huang, Weizhen Feng, Oscillation for forced second-order nonlinear dynamic equations on time scales, Electronic Journal of Differential Equations, 145 (2006), 1–8.

[21] Mugen Huang, Weizhen Feng, Forced oscillation of second order nonlinear dynamic equations on time scales, Electronic Journal of Qualitative Theory of Differential Equations, 36 (2008), 1–13.

[22] Ch. G. Philos, Oscillation theorems for linear differential equation of second order, Archive for Mathematical Logic, 53 (1989), 483–492.

[23] Y. Sahiner, Oscillation of second-order delay differential equations on time scales, Nonlinear Anal, TMA 63 (2005), 1073–1080.

[24] S. H. Saker, Oscillation of second-order forced nonlinear dynamic equations on time scales, Electronic Journal of Qualitative Theory of Differential Equations, 23 (2005), 1–17.

[25] S. H. Saker, R. P. Agarwal, D. O'Regan, Oscillation results for second-order nonlinear neutral delay dynamic equations on time scales, Applicable Analysis, 86 (1) (2007), 1–17.

[26] Y. G. Sun, S. H. Saker, Forced oscillation of higher order nonlinear difference equations, Appl. Math. Comput., 187 (2007), 868–872.

[27] C. H. Ou, J. S. W. Wong, Forced oscillation of nth-order functional differential equations, J. Math. Anal. Appl., 262 (2001), 722–732.

[28] J. S. W. Wong, Oscillation criteria for a forced second-order linear differential equation, J. Math. Anal. Appl., 231 (1999), 235–240.

(Received August 26, 2009)