Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **59**, 1-12; **http://www.math.u-szeged.hu/ejqtde/**

AN OPTIMAL CONDITION FOR THE UNIQUENESS OF A PERIODIC SOLUTION FOR LINEAR FUNCTIONAL DIFFERENTIAL SYSTEMS

S. MUKHIGULASHVILI, I. GRYTSAY

Abstract. Unimprovable effective efficient conditions are established for the unique solvability of the periodic problem

$$u_i'(t) = \sum_{j=2}^{i+1} \ell_{i,j}(u_j)(t) + q_i(t) \qquad \text{for} \quad 1 \le i \le n-1,$$
$$u_n'(t) = \sum_{j=1}^{n} \ell_{n,j}(u_j)(t) + q_n(t),$$
$$u_j(0) = u_j(\omega) \qquad \text{for} \quad 1 \le j \le n,$$

where $\omega > 0$, $\ell_{ij} : C([0,\omega]) \to L([0,\omega])$ are linear bounded operators, and $q_i \in L([0,\omega])$.

2000 Mathematics Subject Classification: 34K06, 34K13

Key words and phrases: Linear functional differential system, periodic boundary value problem, uniqueness.

1. Statement of Problem and Formulation of Main Results

Consider on $[0,\omega]$ the system

$$\begin{aligned} u_i'(t) &= \sum_{j=2}^{i+1} \ell_{i,j}(u_j)(t) + q_i(t) \qquad \text{for} \quad 1 \le i \le n-1, \\ u_n'(t) &= \sum_{j=1}^{n} \ell_{n,j}(u_j)(t) + q_n(t), \end{aligned} \tag{1.1}$$

with the periodic boundary conditions

$$u_j(0) = u_j(\omega) \qquad \text{for} \quad 1 \le j \le n, \tag{1.2}$$

where $n \ge 2$, $\omega > 0$, $\ell_{i,j} : C([0,\omega]) \to L([0,\omega])$ are linear bounded operators and $q_i \in L([0,\omega])$.

By a solution of the problem (1.1), (1.2) we understand a vector function $u = (u_i)_{i=1}^n$ with $u_i \in \widetilde{C}([0,\omega])$ $(i = \overline{1,n})$ which satisfies system (1.1) almost everywhere on $[0,\omega]$ and satisfies conditions (1.2).

Much work had been carried out on the investigation of the existence and uniqueness of the solution for a periodic boundary value problem for systems of ordinary differential equations and many interesting results have been obtained (see, for instance, [1–3, 7–9, 11, 12, 17] and the references therein). However, an analogous problem for functional differential equations, remains investigated in less detail even for linear equations. In the present paper, we study problem (1.1) (1.2) under the assumption that $\ell_{n,1}, \ell_{i,i+1}$ ($i = \overline{1, n-1}$) are monotone linear operators. We establish new unimprovable integral conditions sufficient for unique solvability of the problem (1.1),(1.2) which generalize the well-known results of A. Lasota and Z. Opial (see Remark 1.1) obtained for ordinary differential equations in [13], and on the other hand, extend results obtained for linear functional differential equations in [5, 14–16]. These results are new not only for the systems of functional differential equations (for reference see [2, 4, 6, 10]), but also for the system of ordinary differential equations of the form

$$\begin{aligned} u_i'(t) &= \sum_{j=2}^{i+1} p_{i,j}(t)u_j(t) + q_i(t) \qquad \text{for} \quad 1 \le i \le n-1, \\ u_n'(t) &= \sum_{j=1}^{n} p_{n,j}(t)u_j(t) + q_n(t), \end{aligned} \tag{1.3}$$

where $q_i, p_{i,j} \in L([0, \omega])$ (see, for instance, [2, 7–9] and the references therein). The method used for the investigation of the problem considered is based on that developed in our previous papers [14–16] for functional differential equations.

The following notation is used throughout the paper: $N(R)$ is the set of all the natural (real) numbers; R^n is the space of n-dimensional column vectors $x = (x_i)_{i=1}^n$ with elements $x_i \in R$ ($i = \overline{1, n}$); $R_+ = [0, +\infty[$; $C([0, \omega])$ is the Banach space of continuous functions $u : [0, \omega] \to R$ with the norm $||u||_C = \max\{|u(t)| : 0 \le t \le \omega\}$; $C([0, \omega]; R^n)$ is the space of continuous functions $u : [0, \omega] \to R^n$; $\widetilde{C}([0, \omega])$ is the set of absolutely continuous functions $u : [0, \omega] \to R$; $L([0, \omega])$ is the Banach space of Lebesgue integrable functions $p : [0, \omega] \to R$ with the norm $\|p\|_L = \int_0^\omega |p(s)|ds$; if $\ell : C([0, \omega]) \to L([0, \omega])$ is a linear operator, then $\|\ell\| = \sup_{0<||x||_C \le 1} \|\ell(x)\|_L$.

Definition 1.1. We will say that a linear operator $\ell : C([0, \omega]) \to L([0, \omega])$ is *nonnegative* (*nonpositive*), if for any nonnegative $x \in C([0, \omega])$ the inequality $\ell(x)(t) \ge 0$ $(\ell(x)(t) \le 0)$ for $0 \le t \le \omega$ is satisfied. We will say that an operator ℓ is *monotone* if it is either nonnegative or nonpositive.

Definition 1.2. With system (1.1) we associate the matrix $A_1 = (a^{(1)}_{i,j})^n_{i,j=1}$ defined by the equalities

$$a^{(1)}_{1,1} = -1, \quad a^{(1)}_{n,1} = \frac{1}{4}||\ell_{n,1}||, \quad a^{(1)}_{i,1} = 0 \quad \text{for} \quad 2 \le i \le n-1,$$

$$a^{(1)}_{i+1,i+1} = ||\ell_{i+1,i+1}|| - 1, \ \ a^{(1)}_{i,i+1} = \frac{1}{4}||\ell_{i,i+1}|| \quad \text{for} \quad 1 \le i \le n-1, \tag{1.4}$$

$$a^{(1)}_{i,j} = 0 \ \text{ for } \ i+2 \le j \le n, \ \ a^{(1)}_{i,j} = ||\ell_{i,j}|| \ \text{ for } \ 3 \le j+1 \le i \le n.$$

and the matrices $A_k = (a^{(k)}_{i,j})^n_{i,j=1}$, $(k = \overline{2,n})$ given by the recurrence relations

$$A_2 = A_1, \tag{1.5}$$

$$a^{(k+1)}_{i,j} = a^{(k)}_{i,j} \qquad \text{for} \quad i \le k \quad \text{or} \quad j \notin \{k, k+1\}, \tag{1.6}$$

$$a^{(k+1)}_{i,j} = a^{(k)}_{i,j} + \frac{a^{(k)}_{k,j}}{|a^{(k)}_{k,k}|} a^{(k)}_{i,k} \qquad \text{for} \quad k+1 \le i \le n, \quad k \le j \le k+1. \tag{1.7}$$

Theorem 1.1. *Let* $\ell_{n,1}, \ell_{i,i+1} : C([0,\omega]) \to L([0,\omega])$ $(i = \overline{1,n-1})$ *be linear monotone operators,*

$$\int\limits_0^\omega \ell_{n,1}(1)(s)ds \ne 0, \quad \int\limits_0^\omega \ell_{i,i+1}(1)(s)ds \ne 0 \quad \textit{for} \quad 1 \le i \le n-1, \tag{1.8}$$

and

$$a^{(k)}_{k,k} < 0 \quad \textit{for} \quad 2 \le k \le n. \tag{1.9}$$

where the matrices A_k *are defined by relations* (1.4)-(1.7)*. Let, moreover,*

$$\int_0^\omega |\ell_{n,1}(1)(s)|ds \prod_{j=1}^{n-1} \int_0^\omega |\ell_{j,j+1}(1)(s)|ds < 4^n \prod_{j=2}^{n} |a^{(j)}_{j,j}|. \tag{1.10}$$

Then problem (1.1), (1.2) *has a unique solution.*

Definition 1.3. For the system (1.3) we define the matrix $A_1 = (a^{(1)}_{i,j})^n_{i,j=1}$ by the equalities (1.4)-(1.7) with

$$\ell_{i,j}(x)(t) = p_{i,j}(t)x(t) \quad \text{for} \quad i,j \in \overline{1,n}, \quad x \in C([0,\omega]). \tag{1.11}$$

Corollary 1.1. *Let*

$$0 \le \sigma_n p_{n,1}(t) \not\equiv 0, \ \ 0 \le \sigma_i p_{i,i+1}(t) \not\equiv 0 \qquad \textit{for} \quad 1 \le i \le n-1 \tag{1.12}$$

where $\sigma_i \in \{-1, 1\}$ $(i = \overline{1,n})$*, the matrices* A_k *are defined by the relations* (1.5)-(1.7), (1.11) *and*

$$a^{(k)}_{k,k} < 0 \qquad \textit{for} \quad 2 \le k \le n. \tag{1.13}$$

Let, moreover,

$$\int_0^\omega |p_{n,1}(s)|ds \prod_{j=1}^{n-1} \int_0^\omega |p_{j,j+1}(s)|ds < 4^n \prod_{j=2}^{n} |a_{j,j}^{(j)}|. \tag{1.14}$$

Then problem (1.3), (1.2) *has a unique solution.*

Now, assume that

$$\begin{gathered} \ell_{1,j} \equiv 0 \ \text{ for } \ j \neq 2, \quad \ell_{i,j} \equiv 0 \ \text{ for } \ j \notin \{i, i+1\}, \ \ i = \overline{2, n-1}, \\ \ell_{n,j} = 0 \ \text{ for } \ j = \overline{2, n-1}. \end{gathered} \tag{1.15}$$

Then system (1.1) is of the following type

$$\begin{aligned} u_1'(t) &= \ell_{1,2}(u_2)(t) + q_1(t), \\ u_i'(t) &= \ell_{i,i}(u_i)(t) + \ell_{i,i+1}(u_{i+1})(t) + q_i(t) \quad \text{for } \ 2 \leq i \leq n-1, \\ u_n'(t) &= \ell_{n,1}(u_1)(t) + \ell_{n,n}(u_n)(t) + q_n(t), \end{aligned} \tag{1.16}$$

and from Theorem 1.1 we obtain

Corollary 1.2. *Let* $\ell_{n,1}, \ell_{i,i+1}$ $(i = \overline{1, n-1})$ *be linear monotone operators, the conditions* (1.8) *hold and*

$$\int_0^\omega |\ell_{k,k}(1)(s)|ds < 1 \quad \textit{for} \quad 2 \leq k \leq n. \tag{1.17}$$

Let, moreover,

$$\begin{gathered} \int_0^\omega |\ell_{n,1}(1)(s)|ds \prod_{j=1}^{n-1} \int_0^\omega |\ell_{j,j+1}(1)(s)|ds < \\ < 4^n \prod_{j=2}^{n} \Big(1 - \int_0^\omega |\ell_{j,j}(1)(s)|ds\Big). \end{gathered} \tag{1.18}$$

Then problem (1.16), (1.2) *has a unique solution.*

For the cyclic feedback system

$$\begin{aligned} u_i'(t) &= \ell_i(u_{i+1})(t) + q_i(t) \qquad \text{for} \quad 1 \leq i \leq n-1, \\ u_n'(t) &= \ell_n(u_1)(t) + q_n(t), \end{aligned} \tag{1.19}$$

Corollary 1.2 yields

Corollary 1.3. *Let* $\ell_i : C([0,\omega]) \to L([0,\omega])$ $(i = \overline{1, n})$ *be linear monotone operators,*

$$||\ell_i|| \neq 0 \quad \textit{for} \quad i = \overline{1, n}, \tag{1.20}$$

and

$$\prod_{i=1}^{n} ||\ell_i|| < 4^n. \tag{1.21}$$

Then problem (1.19), (1.2) *has a unique solution.*

Remark 1.1. The problem

$$u''(t) = p(t)u(t) + q(t), \quad u(0) = u(\omega), \; u'(0) = u'(\omega), \tag{1.22}$$

is equivalent to the problem (1.19), (1.2) with $n = 2$, $\ell_1(x)(t) = x(t)$, $\ell_2(x)(t) = p(t)x(t)$, $q_1 \equiv 0$ and $q_2 \equiv q$.

Then if $p, q \in L([0,\omega])$, $p(t) \leq 0$ and $\int_0^\omega p(s)ds \neq 0$ from the corollary 1.3 it follows that problem (1.19), (1.2) and therefore problem (1.22), has a unique solution if the condition $\int_0^\omega |p(s)|ds < \frac{16}{\omega}$ is fulfilled. This follows from the well-known result of A. Lasota and Z. Opial (see [13]).

Example 1.1. The example below shows that condition (1.21) in Corollary 1.3 is optimal and cannot be replaced by the condition

$$\prod_{i=1}^{n} ||\ell_i|| \leq 4^n. \tag{1.21$_1$}$$

Define the function $u_0 \in \widetilde{C}([0,1])$ on $[0,1/2]$, and extend it to $[1/2,1]$ by the equalities

$$u_0(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1/8 \\ \sin \pi(1-4t) & \text{for } 1/8 < t \leq 3/8 \\ -1 & \text{for } 3/8 < t \leq 1/2 \end{cases},$$

$$u_0(t) = u_0(1-t) \quad \text{for} \quad 1/2 < t \leq 1.$$

Now let measurable functions $\tau_i : [0,1] \to [0,1]$ and the linear nonnegative operators $\ell_i : C([0,1]) \to L([0,1])(i = \overline{1,n})$ be given by the equalities

$$\tau_i(t) = \begin{cases} 1/8i & \text{for } 0 \leq u_0'(t) \\ 1/2 - 1/8i & \text{for } 0 > u_0'(t) \end{cases}, \; \ell_i(x)(t) = |u_0'(t)|x(\tau_i(t)).$$

Then it is clear that $u_0(0) = u_0(1)$, $\ell_i \neq \ell_j$ if $i \neq j$, and $||\ell_i|| = \int_0^1 |\ell_i(1)(s)|ds = 16\pi \int_{1/8}^{1/4} \cos \pi(1-4s)ds = 4$ for $i = \overline{1,n}$. Thus, all the assumptions of Corollary 1.3 are satisfied except (1.21), instead of which condition (1.21$_1$) is fulfilled with $\omega = 1$. On the other hand, from the relations $u_0'(t) = |u_0'(t)|u_0(\tau_i(t)) = \ell_i(u_0)(t)$ $(i = \overline{1,n})$, it follows that the vector function $(u_i(t))_{i=1}^n$ if $u_i(t) \equiv u_0(t)$ $(i = \overline{1,n})$ is a nontrivial solution of problem (1.1), (1.2) with $\omega = 1$, $q(t) \equiv 0$, which contradicts the conclusion of Corollary 1.3.

2. Auxiliary Propositions

Lemma 2.1. *Let the matrices* A_k $(k=\overline{1,n})$ *be defined by equalities* (1.4)-(1.7). *Then the following relations hold:*

$$a_{i,j}^{(m)} \geq 0 \qquad \text{for} \quad i \neq j, \ \ m=\overline{1,n}, \tag{2.1_m}$$

$$a_{n,1}^{(1)} = a_{n,1}^{(n)} \tag{2.2_0}$$

$$a_{i,j}^{(\lambda)} \leq a_{i,j}^{(m)} \qquad \text{for} \quad i \geq m \geq 2, \ \ j \geq m, \ \ \lambda \leq m. \tag{2.2_m}$$

Proof. It immediately follows from the definition of A_1, A_2 that inequalities (2.1_1) and (2.2_2) are true. Now, we assume that (2.1_m) holds for $m=3,..,m_0$ $(m_0<n)$ and prove (2.1_{m_0+1}). If $i \leq m_0$ or $j \notin \{m_0, \ m_0+1\}$, relation (1.6) implies inequality (2.1_{m_0+1}), and if $i \geq m_0+1$, $j \in \{m_0, \ m_0+1\}$, then (2.1_{m_0+1}) follows from (1.7).

Now we prove inequality (2.2_m). First assume that $j \geq m+1$. Then from (1.6) it is clear that

$$a_{i,j}^{(\lambda)} = a_{i,j}^{(\lambda+1)} = \ldots = a_{i,j}^{(m)} \qquad \text{for} \quad j \geq m+1, \ i \geq m, \ \lambda \leq m. \tag{2.3}$$

Now, let $j=m$. Then from (1.6) we get $a_{i,m}^{(\lambda)} = a_{i,m}^{(\lambda+1)} = \ldots = a_{i,m}^{(m-1)}$ for $i \geq m$, $\lambda \leq m$. By the last equalities and (2.1_m), from (1.7) it follows

$$a_{i,m}^{(m)} = a_{i,m}^{(m-1)} + \frac{a_{m-1,m}^{(m-1)}}{|a_{m-1,m-1}^{(m-1)}|} a_{i,m-1}^{(m-1)} \geq a_{i,m}^{(m-1)} = a_{i,m}^{(\lambda)} \quad \text{for} \quad i \geq m, \ \lambda \leq m,$$

From this inequality and (2.3) we conclude that (2.2_m) is fulfilled for all $j \geq m$ and $i \geq m$. Equality (2.2_0) follows immediately from (1.5) and (1.6). □

Also we need the following simple lemma proved in the paper [17].

Lemma 2.2. *Let* $\sigma \in \{-1,1\}$ *and* $\sigma\ell : C([0,\omega]) \to L([0,\omega])$ *be a nonnegative linear operator. Then*

$$-m|\ell(1)(t)| \leq \sigma\ell(x)(t) \leq M|\ell(1)(t)| \quad \text{for} \quad 0 \leq t \leq \omega, \quad x \in C([0,\omega]),$$

where $m = -\min\limits_{0 \leq t \leq \omega}\{x(t)\}$, $M = \max\limits_{0 \leq t \leq \omega}\{x(t)\}$.

Now, consider on $[0,\omega]$ the homogeneous problem

$$v_i'(t) = \sum_{j=2}^{i+1} \ell_{i,j}(v_j)(t) \qquad \text{for} \quad 1 \leq i \leq n, \tag{2.4_i}$$

$$v_j(0) = v_j(\omega) \qquad \text{for} \quad 1 \leq j \leq n, \tag{2.5}$$

where the operator $\ell_{n,n+1}$ and function v_{n+1} are defined by the equalities $\ell_{n,n+1} \equiv \ell_{n,1}$ and $v_{n+1} \equiv v_1$. Also define the functional Δ_i :

$C([0,\omega];R^n) \to R_+$ by the equality $\Delta_i(v) = \max\limits_{0\le t\le\omega}\{v_i(t)\} - \min\limits_{0\le t\le\omega}\{v_i(t)\}$ $(i=\overline{1,n})$ for any vector function $v=(v_i)_{i=1}^n$ and put $\Delta_{n+1}\equiv\Delta_1$.

Lemma 2.3. *Let* $\ell_{i,i+1} : C([0,\omega]) \to L([0,\omega])$ $(i=\overline{1,n})$ *be linear monotone operators,*

$$\int_0^\omega \ell_{i,i+1}(1)(s)ds \neq 0 \qquad \text{for} \quad 1\le i\le n, \tag{2.6}$$

the matrices A_k *be defined by the equalities* (1.4)-(1.7) *and*

$$a_{k,k}^{(k)} < 0 \qquad \text{for} \quad 2\le k\le n. \tag{2.7}$$

Let, moreover $v=(v_i)_{i=1}^n$ *be a nontrivial solution of the problem* $((2.4_i))_{i=1}^n$, (2.5) *for which there exists a* $k_1\in\{2,...,n\}$ *such that* $v_{k_1}\not\equiv 0$. *Then if*

$$k_0 = \min\{k\in\{2,...,n\} : v_k \not\equiv 0\}, \tag{2.8}$$

the inequalities

$$0 < ||v_k||_C \le \Delta_k(v) \qquad \text{for} \quad k=1, \ \ k_0\le k\le n, \tag{2.9$_k$}$$

$$0 \le a_{k,k}^{(k)}\Delta_k(v) + a_{k,k+1}^{(k)}\Delta_{k+1}(v) \qquad \text{for} \quad k_0\le k\le n, \tag{2.10$_k$}$$

hold, where $a_{n,n+1}^{(1)} = a_{n,1}^{(1)}$.

Proof. Define the numbers $M_k, m_k \in R$, $t_k', t_k'' \in [0,\omega]$ by the relations

$$M_k = v_k(t_k') = \max_{0\le t\le\omega}\{v_k(t)\}, \quad -m_k = v_k(t_k'') = \min_{0\le t\le\omega}\{v_k(t)\}, \tag{2.11$_k$}$$

and introduce the sets $I_k^{(1)} = [t_k',\ t_k'']$, $I_k^{(2)} = I \setminus I_k^{(1)}$ for $t_k' < t_k''$. It is clear from (2.8) that

$$v_{k_0} \not\equiv 0. \tag{2.12}$$

On the other hand, from (2.4_{k_0-1}) by (2.8) we obtain

$$\int_0^\omega \ell_{k_0-1,k_0}(v_{k_0})(s)ds = 0. \tag{2.13}$$

Equality (2.13), in view of (2.6) and Lemma 2.2 guarantees the existence of a $t_0\in[0,\ \omega]$ such that $v_{k_0}(t_0)=0$. Then from (2.12) we get (2.9_{k_0}).

Let the numbers $M_{k_0}, m_{k_0} \in R$, $t_{k_0}', t_{k_0}'' \in [0,\omega]$ be defined by the relations (2.11_{k_0}) and $t_{k_0}' < t_{k_0}''$ (the case $t_{k_0}'' < t_{k_0}'$ can be considered analogously). The integration of (2.4_{k_0}) on $I_{k_0}^{(r)}$, by virtue of (2.5) and (2.8) results in

$$\Delta_{k_0}(v) = (-1)^r\Big[\int\limits_{I_{k_0}^{(r)}} \ell_{k_0,k_0}(v_{k_0})(s)ds + \int\limits_{I_{k_0}^{(r)}} \ell_{k_0,k_0+1}(v_{k_0+1})(s)ds\Big] \tag{2.14}$$

for $r = 1, 2$. From the last equality, by virtue of (1.4), (2.7), (2.9_{k_0}) and (2.2_{k_0}) with $\lambda = 1,\ i = j = k_0$ we get

$$0 < -a^{(k_0)}_{k_0,k_0}\Delta_{k_0}(v) \leq (-1)^r \int_{I^{(r)}_{k_0}} \ell_{k_0,k_0+1}(v_{k_0+1})(s)ds \qquad (2.15_r)$$

for $r = 1, 2$. Assume that v_{k_0+1} is a constant sign function. Then in view of the fact that the operator ℓ_{k_0,k_0+1} is monotone we get the contradiction with (2.15_1) or (2.15_2), i.e., v_{k_0+1} changes its sign. Then

$$M_{k_0+1} > 0, \quad m_{k_0+1} > 0, \qquad (2.16)$$

and the inequality (2.9_{k_0+1}) holds ((2.9_1) if $k_0 = n$). If ℓ_{k_0,k_0+1} is a nonnegative operator, from (2.15_r) ($r = 1, 2$) in view of (2.16) by Lemma 2.2 we get $0 < -a^{(k_0)}_{k_0,k_0}\Delta_{k_0}(v) \leq m_{k_0+1}\int_{I^{(1)}_{k_0}} |\ell_{k_0,k_0+1}(1)(s)|ds$, $\quad 0 < -a^{(k_0)}_{k_0,k_0}\Delta_{k_0}(v) \leq M_{k_0+1}\int_{I^{(2)}_{k_0}} |\ell_{k_0,k_0+1}(1)(s)|ds$. By multiplying these estimates and applying the numerical inequality $4AB \leq (A + B)^2$, in view of the notations (1.4) we obtain $0 \leq a^{(k_0)}_{k_0,k_0}\Delta_{k_0}(v) + \frac{1}{4}(M_{k_0+1} + m_{k_0+1})\Big(\int_{I^{(1)}_{k_0}} |\ell_{k_0,k_0+1}(1)(s)|ds + \int_{I^{(2)}_{k_0}} |\ell_{k_0,k_0+1}(1)(s)|ds\Big) = a^{(k_0)}_{k_0,k_0}\Delta_{k_0}(v) + a^{(1)}_{k_0,k_0+1}\Delta_{k_0+1}(v)$, ($0 \leq a^{(n)}_{n,n}\Delta_n(v) + a^{(1)}_{n,1}\Delta_1(v)$ if $k_0 = n$), from which by (2.2_0) if $k_0 = n$ and (2.2_{k_0}) with $\lambda = 1,\ i = k_0,\ j = k_0 + 1$ if $k_0 < n$, follows (2.10_{k_0}). Analogously, from (2.15_r) we get (2.10_{k_0}) in the case when the operator ℓ_{k_0,k_0+1} is nonpositive.

Consequently, we have proved the proposition:

i. Let $2 \leq k_0 \leq n$, then the inequalities (2.9_{k_0}), (2.9_{k_0+1}) ((2.9_1) if $k_0 = n$) and (2.10_{k_0}) hold.

Now, we shall prove the following proposition:

ii. Let $k_1 \in \{k_0, ..., n-1\}$ be such that the inequalities (2.9_k),(2.10_k) for ($k = \overline{k_0, k_1}$), and (2.9_{k_1+1}) hold. Then the inequalities (2.9_{k_1+2}) if $k_1 \leq n - 2$, (2.9_1) if $k_1 = n - 1$ and (2.10_{k_1+1}) hold too.

Define the numbers $M_{k_1+1}, m_{k_1+1} \in R$, $t'_{k_1+1}, t''_{k_1+1} \in [0, \omega]$ by the relations (2.11_{k_1+1}) and let $t'_{k_1+1} < t''_{k_1+1}$ (the case $t''_{k_1+1} < t'_{k_1+1}$ can be proved analogously). The integration of (2.4_{k_1+1}) on $I^{(r)}_{k_1+1}$, by virtue of (2.5) and (2.8) results in

$$\Delta_{k_1+1}(v) = (-1)^r \sum_{j=k_0}^{k_1+2} \int_{I^{(r)}_{k_1+1}} \ell_{k_1+1,j}(v_j)(s)ds \qquad (2.17)$$

for $r = 1, 2$. From this equality, by the conditions (1.4),(2.7),(2.9_k) with $k = k_0, ..., k_1 + 1$, and (2.2_{k_0}) with $\lambda = 1,\ i = k_1 + 1,\ j = k_0, ..., k_1 + 1$

we get

$$0 \le \sum_{j=k_0}^{k_1+1} a_{k_1+1,j}^{(k_0)} \Delta_j(v) + (-1)^r \int_{I_{k_1+1}^{(r)}} \ell_{k_1+1,k_1+2}(v_{k_1+2})(s)ds \qquad (2.18)$$

for $r = 1, 2$. By multiplying (2.10_k) with $a_{k_1+1,k}^{(k)}/|a_{k,k}^{(k)}|$ for $k \in \{k_0, ..., k_1\}$ in view of the inequalities (2.7) we obtain

$$0 \le -a_{k_1+1,k}^{(k)} \Delta_k(v) + \frac{a_{k,k+1}^{(k)}}{|a_{k,k}^{(k)}|} a_{k_1+1,k}^{(k)} \Delta_{k+1}(v). \qquad (2.19_k)$$

Now, summing (2.18) and (2.19_{k_0}) by virtue of (1.7) with $k = k_0,\ i = k_1 + 1,\ j = k_0 + 1$, we get

$$0 \le a_{k_1+1,k_0+1}^{(k_0+1)} \Delta_{k_0+1}(v) + \sum_{j=k_0+2}^{k_1+1} a_{k_1+1,j}^{(k_0)} \Delta_j(v) +$$

$$+(-1)^r \int_{I_{k_1+1}^{(r)}} \ell_{k_1+1,k_1+2}(v_{k_1+2})(s)ds,$$

from which by (2.2_{k_0+1}) with $i = k_1 + 1,\ j \ge k_0 + 2,\ \lambda = k_0$, we obtain

$$0 \le \sum_{j=k_0+1}^{k_1+1} a_{k_1+1,j}^{(k_0+1)} \Delta_j(v) + (-1)^r \int_{I_{k_1+1}^{(r)}} \ell_{k_1+1,k_1+2}(v_{k_1+2})(s)ds \qquad (2.20)$$

for $r = 1, 2$. Analogously, by summing (2.20) and the inequalities (2.19_k) for all $k = k_0 + 1, ..., k_1$ we get

$$0 < -a_{k_1+1,k_1+1}^{(k_1+1)} \Delta_{k_1+1}(v) \le (-1)^r \int_{I_{k_1+1}^{(r)}} \ell_{k_1+1,k_1+2}(v_{k_1+2})(s)ds \qquad (2.21)$$

for $r = 1, 2$. In the same way as the inequality (2.9_{k_0+1}) and (2.10_{k_0}) follow from (2.15_r), the inequalities (2.9_{k_1+2}) ((2.9_1) if $k_0 = n - 1$) and (2.10_{k_1+1}) follow from (2.21).

From the propositions i. and ii. by the the method of mathematical induction we obtain that the inequalities (2.9_1), (2.9_k) and (2.10_k) ($k = \overline{k_0, n}$) hold. □

3. Proofs

Proof of Theorem 1.1. It is known from the general theory of boundary value problems for functional differential equations that if $\ell_{i,j}$ ($i, j = \overline{1, n}$) are strongly bounded linear operators, then problem (1.1), (1.2) has the Fredholm property (see [6]). Thus, problem (1.1), (1.2) is uniquely solvable iff the homogeneous problem $(2.4_i)_{i=1}^n$, (2.5) has only the trivial solution.

Assume that, on the contrary, the problem $(2.4_i)_{i=1}^n$,(2.5) has a nontrivial solution $v = (v_i)_{i=1}^n$. Let

$$v_1 \not\equiv 0, \quad v_i \equiv 0 \quad \text{for} \quad 2 \leq i \leq n. \tag{3.1}$$

Thus from (2.4_1) and (2.4_n) it follows that $v_1'(t) \equiv 0$ and $\ell_{n,1}(v_1)(t) \equiv 0$, i.e., in view of the fact that the operator $\ell_{n,1}$ satisfies (1.8) we obtain that $v_1 \equiv 0$, which contradicts (3.1). Consequently there exists $k_0 \in \{2, ..., n\}$ such that $v_{k_0} \not\equiv 0$. Then all the conditions of Lemma 2.3 are satisfied, from which it follows that $0 < ||v_1||_C \leq \Delta_1(v)$, i.e., $v_1 \not\equiv Const$ and in view of the condition (2.5) the function v_1' changes its sign. Thus from (2.4_1) by the monotonicity of the operator $\ell_{1,2}$, we get that v_2 changes its sign too. Consequently if M_2, m_2 are the numbers defined by the equalities (2.11_2) then

$$M_2 > 0, \;\; m_2 > 0, \tag{3.2}$$

and if k_0 is the number defined by the equality (2.8), then $k_0 = 2$. Thus from Lemma 2.3 it follows that the inequalities (2.9_1), (2.9_k) and (2.10_k) $(k = \overline{2,n})$ hold.

Now, assume that the numbers M_1, m_1, and $t_1', t_1'' \in [0, \omega[$ are defined by the equalities (2.11_1) and $t_1' < t_1''$ (the case $t_1'' < t_1'$ can be proved analogously). By integration of (2.4_1) on the set $I_1^{(r)}$ we obtain

$$\Delta_1(v) = (-1)^r \int_{I_1^{(r)}} \ell_{1,2}(v_2)(s)ds \tag{3.3}$$

for $r = 1, 2$. First assume that the operator $\ell_{1,2}$ is nonnegative (the case of nonpositive $\ell_{1,2}$ can be proved analogously), then from (3.3) by (2.9_1), (3.2) and the Lemma 2.2 we obtain

$$0 < \Delta_1(v) \leq m_2 \int_{I_1^{(1)}} |\ell_{1,2}(1)(s)|ds, \;\; 0 < \Delta_1(v) \leq M_2 \int_{I_1^{(2)}} |\ell_{1,2}(1)(s)|ds.$$

By multiplying these estimates and applying the numerical equality $4AB \leq (A+B)^2$ and the equalities (1.4) we get $0 \leq a_{1,1}^{(1)}\Delta_1(v) + \frac{1}{4}(m_2 + M_2)\Big(\int_{I_1^{(1)}} |\ell_{1,2}(1)(s)|ds + \int_{I_1^{(2)}} |\ell_{1,2}(1)(s)|ds\Big) = a_{1,1}^{(1)}\Delta_1(v) + a_{1,2}^{(1)}\Delta_2(v)$, i.e., all the inequalities (2.10_k) $(k = \overline{1,n})$ are satisfied.

On the other hand from (1.4)–(1.6) and Lemma 2.1 it is clear that

$$a_{1,1}^{(1)} = -1, \;\; a_{n,1}^{(n)} = a_{n,1}^{(1)}, \;\; a_{k,k+1}^{(k)} = a_{k,k+1}^{(1)} = \frac{1}{4}||\ell_{k,k+1}|| \tag{3.4}$$

for $1 \leq k \leq n-1$. By multiplying all the estimates (2.10_k) $(k = \overline{1,n})$ and applying (3.4) we get the contradiction with condition (1.10). Thus our assumption fails, and hence $v_i \equiv 0$ $(i = \overline{1,n})$. □

Proof of Corollary 1.1. From (1.11) and (1.12) it is clear that $\ell_{n,1}$ and $\ell_{i,i+1}$ are monotone operators and (1.8) holds. Also, from (1.13) and (1.14), the conditions (1.9) and (1.10) follow. Consequently all the conditions of Theorem 1.1 are fulfilled for system (1.3). □

Proof of Corollary 1.2. From (1.4), (1.6), and (1.15) it is clear that

$$a_{k,k}^{(k-1)} = a_{k,k}^{(k-2)} = \dots = a_{k,k}^{(1)} = ||\ell_{k,k}|| - 1 \quad \text{for} \quad 2 \le k \le n, \tag{3.5}$$

and

$$\begin{gathered} a_{k,k-i}^{(k-i-1)} = a_{k,k-i}^{(k-i-2)} = \dots = a_{k,k-i}^{(1)} = 0 \quad \text{for} \quad 3 \le k-i \le n, \\ a_{2,1}^{(1)} = 0. \end{gathered} \tag{3.6}$$

From (1.7), (1.15) and the first equality of (3.6) we get

$$\begin{aligned} a_{k,k-1}^{(k-1)} &= a_{k,k-1}^{(k-2)} + \frac{a_{k-2,k-1}^{(k-2)}}{|a_{k-2,k-2}^{(k-2)}|} a_{k,k-2}^{(k-2)} = \frac{a_{k-2,k-1}^{(k-2)}}{|a_{k-2,k-2}^{(k-2)}|} a_{k,k-2}^{(k-2)} = \\ &= \frac{a_{k-2,k-1}^{(k-2)}}{|a_{k-2,k-2}^{(k-2)}|} \frac{a_{k-3,k-2}^{(k-3)}}{|a_{k-3,k-3}^{(k-3)}|} a_{k,k-3}^{(k-3)} = \dots = a_{k,2}^{(2)} \prod_{j=2}^{k-2} \frac{a_{j,j+1}^{(j)}}{|a_{j,j}^{(j)}|} = 0 \end{aligned} \tag{3.7}$$

for $k \ge 3$. From (3.7) and the second equality of (3.6) it is clear that

$$a_{k,k-1}^{(k-1)} = 0 \qquad \text{for} \qquad 2 \le k \le n \tag{3.8}$$

Then from (1.7) by (3.5) and (3.8) we obtain

$$a_{k,k}^{(k)} = a_{k,k}^{(k-1)} + a_{k-1,k}^{(k-1)} a_{k,k-1}^{(k-1)} / |a_{k-1,k-1}^{(k-1)}| = ||\ell_{k,k}|| - 1.$$

Thus from the conditions (1.17) and (1.18) it follows that (1.9) and (1.10) hold. Consequently all the conditions of Theorem 1.1 are fulfilled for the system (1.16). □

Acknowledgement

The research was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503, and the Grant of Georgian National Scientific Foundation # GNSF/ST06/3 – 002.

References

[1] F. Bates , Y. Ward, *Periodic solutions of higher order systems,* Pacific J. Math., **84** (1979), 275–282.

[2] T. A. Burton, *Stability and periodic solutions of ordinary and functional-differential equations.* Mathematics in Science and Engineering, **178.** Academic Press, Inc., Orlando, FL, 1985.

[3] A. Capietto, D. Qian, F. Zanolin, *Periodic solutions for differential systems of cyclic feedback type,* Differential Equations Dyn. Systems, **7**(1)(1999), 99–120.

[4] Sh. Gelashvili and I. Kiguradze, *On multi-point boundary value problems for systems of functional differential and difference equations.* Mem. Differential Equations Math. Phys. **5** (1995), 1–113.

[5] R. Hakl, *Periodic boundary-value problem for third-order linear functional differential equations.* Ukrainian Mathematical Journal **60** (2008), No. 3, 481–494.

[6] I. Kiguradze, B. Půža, *On boundary value problems for systems of linear functional differential equations,* Czechoslovak Math. J., **47** (1976), 341–373.

[7] I. Kiguradze, *On bounded and periodic solutions of linear higher order differential equations.* (Russian) Mat. Zametki **37** (1985), No. 1, 48–62; English transl.: Math. Notes **37** (1985), 28–36.

[8] I. Kiguradze, *Periodic solutions of systems of nonautonomous ordinary differential equations.* (Russian) Mat. Zametki **39** (1986), No. 4, 562–575; English transl.: Math. Notes **39** (1986), 308–315.

[9] I. Kiguradze, B. Půža, *On some boundary value problems for systems of ordinary differential equations,* Differentsial'nye Uravnenija, **12** (1996), 2139–2148 (in Russian).

[10] I. Kiguradze and B. Půža, *Boundary value problems for systems of linear functional differential equations.* Masaryk University, Brno, 2003.

[11] I. Kiguradze, S. Mukhigulashvili, *On periodic solutions of two-dimensional nonautonomous differential systems,* Nonlinear Analysis, **60** A(2005), 241–256.

[12] M. A. Krasnosel'skii, *Translation along trajectories of differential equations.* American Mathematical Society Translations **19** (1968).

[13] A. Lasota, Z. Opial, *Sur les solutions periodiques des equations differentielles ordinaires,* Ann. Polon. Math., **16** (1964), 69–94.

[14] A. Lomtatidze, S. Mukhigulashvili, *On periodic solutions of second order functional differential equations,* Mem. Differential Equations Math. Phys. **5** (1995), 125–126.

[15] S. Mukhigulashvili, *On a periodic solutions of second order functional differential equations,* Italian J. of Pure and Appl. Math., (2006), No 20, 29–50.

[16] S. Mukhigulashvili, *On a periodic boundary value problem for cyclic feedback type linear functional differential systems,* Arch. Math.,**87** (2006), 255–260.

[17] A. Ronto and M. Ronto, *On certain symmetry properties of periodic solutions,* Nonlinear Oscillations, **6** (2003), No.1, 82–107.

Authors' addresses:

Sulkhan Mukhigulashvili

1. Mathematical Institute, Academy of Sciences of the Czech Republic, Žižkova 22, 616 62 Brno, Czech Republic.

2. I. Chavchavadze State University, Faculty of physics and mathematics, I. Chavchavadze Str. No.32, 0179 Tbilisi, Georgia.

E-mail: `mukhig@ipm.cz`

Iryna Grytsay

1. Taras Shevchenko National University of Kyiv, Faculty of Cybernetics, Department of Mathematical Analysis, Vladimirskaya Street, 64, 01033, Kyiv, Ukraine

E-mail: `grytsay@mail.univ.kiev.ua`

(Received July 30, 2009)