*Electronic Journal of Qualitative Theory of Differential Equations*
2009, No. **55**, 1-8; **http://www.math.u-szeged.hu/ejqtde/**

# POSITIVE SOLUTIONS OF A SECOND-ORDER THREE-POINT BOUNDARY VALUE PROBLEM VIA FUNCTIONAL COMPRESSION-EXPANSION

JOHN EHRKE

Abstract. This paper examines a three point, nonlocal boundary value problem for a second order ordinary differential equation. We make use of a generalization of the fixed point theorem of compression and expansion of functional type to obtain the existence of positive solutions.

## 1. Introduction

The literature concerning nonlocal boundary value problems is extensive. Moreover, applications of fixed point theorems to positive solutions of such has sparked much interest, see [9, 10, 11, 12]. Many of the results involving nonlocal boundary value problems are summarized in [9]. A more general summary of positive solutions results for a variety of differential equation types is available in [1]. Our interest lies in the nature of nonlocal problems as being the limiting problem of more classical focal problems. For example, we will consider the second order, three-point BVP

$$x''(t) + f(x(t)) = 0, \quad t \in (0,1) \tag{1.1}$$

with nonlocal conditions

$$x(0) = 0 \text{ and } x(\eta) = x(1) \tag{1.2}$$

where, $\eta \in (0,1)$ is fixed. With the BVP (1.1),(1.2) we associate the right focal BVP (1.1) with boundary conditions $x(0) = 0, x'(1) = 0$. We say that this is the limiting problem for (1.1),(1.2) in the sense that the "limit" occurs when we think of $\eta \to 1^-$. Such problems and even characterizations are not new. For example, in [9] it is mentioned that for such problems often times numerical and experimental computation of $x'(1)$ is more difficult than that of $\dfrac{x(\eta) - x(1)}{\eta - 1}$. It is for this reason that we are keen to investigate problems of this type. We will make use of some newer results concerning the Green's function of such problems in guaranteeing the existence of positive solutions for (1.1), (1.2). This article differentiates itself from these results in the application of a functional expansion-compression argument to obtain positive solutions, due [5]. Where much of the current literature uses norm based, Krasnosel'skii fixed point results to obtain positive solutions for the cases when $f$ is superlinear or sublinear, we will only assume that $f : \mathbb{R} \to [0, \infty)$ is continuous and will be using a functional approach highlighted in [2, 3, 5]. Remarks concerning a larger class of problems will be given at the end.

## 2. Preliminaries

Our results are based on the functional compression-expansion fixed point theorem, due [5]. We will list some preliminary material to be used later. We begin by defining a set of properties to which our functionals must adhere. Throughout this paper we assume $0 < \eta \leq q < 1$ are fixed.

**Definition 1.** *A nonempty closed convex set $P \subset E$, real Banach space, is called a cone if it satisfies each of following conditions:*

(i) $\forall x \in P$, $\lambda \geq 0$, $\lambda x \in P$,

(ii) $\forall x \in P$, $-x \in P$, *implies* $x = 0$.

**Definition 2.** *A map $\alpha$ is said to be a nonnegative, continuous, concave functional on a cone $P \subset E$ if $\alpha : P \to [0, \infty)$ is continuous and satisfies the concave relationship*

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y),$$

$\forall x, y \in P$ *and* $t \in [0, 1]$.

**Definition 3.** *A map $\gamma$ is said to be a nonnegative, continuous, convex functional on a cone $P \subset E$ if $\gamma : P \to [0, \infty)$ is continuous and satisfies the convex relationship*

$$\gamma(tx + (1-t)y) \leq t\gamma(x) + (1-t)\gamma(y),$$

$\forall x, y \in P$ *and* $t \in [0, 1]$.

**Definition 4.** *A map $\psi$ is said to be a sublinear functional if*

$$\psi(tx) \leq t\psi(x),$$

$\forall x \in P$, $t \in [0, 1]$.

**Property A1.** Let $P$ be a cone in a real Banach space $E$ and $\Omega$ be a bounded open subset of $E$ with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy property A1 if one of the following conditions hold:

(i) $\beta$ is convex, $\beta(0) = 0, \beta(x) \neq 0$ if $x \neq 0$ and $\inf_{x \in P \cap \partial\Omega} \beta(x) > 0$,

(ii) $\beta$ is sublinear, $\beta(0) = 0$, $\beta(x) \neq 0$ if $x \neq 0$, and $\inf_{x \in P \cap \partial\Omega} \beta(x) > 0$,

(iii) $\beta$ is concave and unbounded.

**Property A2.** Let $P$ be a cone in a real Banach space $E$ and $\Omega$ be a bounded open subset of $E$ with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy property A2 if one of the following conditions hold:

(i) $\beta$ is convex, $\beta(0) = 0$ and $\beta(x) \neq 0$, if $x \neq 0$,

(ii) $\beta$ is sublinear, $\beta(0) = 0$ and $\beta(x) \neq 0$ if $x \neq 0$,

(iii) $\beta(x + y) \geq \beta(x) + \beta(y)$ for all, $x, y \in P$, $\beta(0) = 0, \beta(x) \neq 0$ if $x \neq 0$.

**Theorem 1.** *Let $\Omega_1$ and $\Omega_2$ be two bounded open sets in a Banach space $E$ such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subseteq \Omega_2$ and $P$ is a cone in $E$. Suppose $A : P \cap (\overline{\Omega_2} - \Omega_1) \to P$ is completely continuous, $\alpha$ and $\gamma$ are non-negative continuous functionals on $P$, and one of the two conditions:*

(C1) *$\alpha$ satisfies Property A1 with $\alpha(Ax) \geq \alpha(x)$, for all $x \in P \cap \partial\Omega_1$, and $\gamma$ satisfies Property A2 with $\gamma(Ax) \leq \gamma(x)$, for all $x \in P \cap \partial\Omega_2$*

(C2) *$\alpha$ satisfies Property A2 with $\alpha(Ax) \leq \alpha(x)$, for all $x \in P \cap \partial\Omega_1$, and $\gamma$ satisfies Property A1 with $\gamma(Ax) \geq \gamma(x)$, for all $x \in P \cap \partial\Omega_2$,*

*is satisfied. Then $A$ has at least one fixed point in $P \cap (\overline{\Omega_2} - \Omega_1)$.*

We will obtain positive solutions for (1.1), (1.2) by looking for a fixed point of a completely continuous operator which satisfies certain properties. A solution to (1.1), (1.2) is given by

$$x(t) = \int_0^1 G^*(t, s) f(x(s))\, ds \tag{2.1}$$

where $G^*(t, s)$ is the Green's function for the operator $Lx := -x'' = 0$ satisfying boundary conditions (1.2). Recent results in the literature, though are characterizing Green's functions for nonlocal problems as being part local and part nonlocal. For example, throughout this paper we will think of $G^*(t, s)$ as being given by

$$G^*(t, s) = G(t, s) + \frac{1}{1 - \eta} G(\eta, s) \tag{2.2}$$

where $G(t, s)$ is the Green's function for the problem with local conditions $x(0) = x(1) = 0$, given by

$$G(t, s) := \begin{cases} t(1 - s) & t \leq s \\ s(1 - t) & s \leq t \end{cases} \tag{2.3}$$

Some useful properties of the Green's function in this case are:

(G1) $\displaystyle\int_0^1 G(t,s)\,ds = \frac{t(1-t)}{2}$ for $0 \le t, s \le 1$

(G2) $G^*(t,s) \ge 0$ for $0 \le t, s \le 1$

(G3) $G^*(t,s) \le G^*(s,s)$ for $0 \le t, s \le 1$

(G4) $G^*(t,s) \le \dfrac{2-\eta}{1-\eta}$ for $0 \le t, s \le 1$

The construction of Green's functions for multi-point problems, using two-point BVP Green's functions is considered in various places in the literature. The benefits of this approach are two-fold. We can easily make use of the positivity of the Green's function in this case and results using these Green's function methods are more easily generalizable to higher order (or larger multiple-point) results.

## 3. Main Result

We will make use of the generalization of the fixed point theorem of compression and expansion of functional type to obtain positive solutions, but first we define the parameters under which we will satisfy the theorem.

Let $E = C[0,1]$ endowed with the maximum norm be our Banach space. We define our cone $P$ as follows:

$$P := \left\{ x \in E \;\middle|\; \begin{array}{l} x \text{ is non-decreasing on } [0,q] \\ x \text{ is non-increasing on } [q,1] \\ x \text{ is non-negative on } [0,1] \\ x(0) = 0. \end{array} \right\} \tag{3.1}$$

The functionals which will satisfy the requirements of the theorem are given by

$$\alpha(x) = \min_{t \in [\eta,1]} x(t) = x(\eta) = x(1) \tag{3.2}$$

and

$$\gamma(x) = \max_{t \in [0,1]} x(t) = \|x\| = x(q) \tag{3.3}$$

which satisfy properties $A1(iii)$ and $A2(i)$, respectively. We will consider the sets

$$\Omega_1 = \{x : \gamma(x) < r\}, \quad \Omega_2 = \{x : \alpha(x) < R\} \tag{3.4}$$

as well. We are now ready to impose growth conditions on $f(x)$ to apply the theorem.

**Theorem 2.** *Suppose there exists positive numbers $r$ and $R$ such that $0 < r < \eta R$, and suppose that $f$ satisfies the following growth conditions:*

(i) $f(x) \geq \dfrac{2R}{\eta(2-\eta)}$ *for all* $x \in [0, R]$.

(ii) $f(x) \leq \dfrac{2r}{q(2-q)}$ *for all* $x \in [r, R]$.

*Then the second order BVP,* (1.1), (1.2) *has at least one positive solution in* $P \cap (\overline{\Omega_2} - \Omega_1)$.

*Proof.* We will verify each of the hypotheses of the functional compression-expansion theorem to obtain the result.

**Claim 1.** *$A : P \to P$ is a completely continuous operator.*

Standard arguments show that $A$ is completely continuous. Let $x \in P$. Then by the properties of $G^*(t,s)$, $Ax(t) \geq 0$, $(Ax)''(t) = -f(x(t)) \leq 0$. Moreover, there exists a $q \in [0,1]$ such that by the properties of

$$\frac{\partial G^*(t,s)}{\partial t} = \begin{cases} 1-s, & t \leq s \\ -s, & s \leq t \end{cases}$$

$Ax$ is non-decreasing on $[0,q]$ and non-increasing on $[q,1]$. Hence, $Ax \in P$ giving $A : P \to P$.

**Claim 2.** *If $x \in P \cap \partial\Omega_1$, then $\gamma(Ax) \leq \gamma(x)$.*

We employ properties of the Green's function as well as our growth conditions on $f$ to resolve this claim:

$$\begin{aligned}
\gamma(Ax) &= Ax(q) \\
&= \int_0^1 G^*(q,s)f(x(s))\ ds \\
&= \int_0^1 G(q,s)f(x(s))\ ds + \frac{1}{1-\eta}\int_0^1 G(\eta,s)f(x(s))\ ds \\
&\leq \left(\frac{2r}{q(2-q)}\right)\left[\int_0^1 G(q,s)\ ds + \frac{1}{1-\eta}\int_0^1 G(\eta,s)\ ds\right] \\
&= \left(\frac{2r}{q(2-q)}\right)\left(\frac{q(1-q)}{2} + \frac{1}{1-\eta}\cdot\frac{\eta(1-\eta)}{2}\right) \\
&\leq \left(\frac{2r}{q(2-q)}\right)\left(\frac{q(1-q)}{2} + \frac{q}{2}\right) \\
&= \left(\frac{2r}{q(2-q)}\right)\left(\frac{q(2-q)}{2}\right) \\
&= r = \gamma(x)
\end{aligned}$$

**Claim 3.** *If $x \in P \cap \partial\Omega_2$, then $\alpha(Ax) \geq \alpha(x)$.*

As above we use the Green's function properties and growth conditions on $f$ to resolve this claim:

$$\begin{aligned}
\alpha(Ax) &= Ax(\eta) \\
&= \int_0^1 G^*(n,s)f(x(s))\ ds \\
&= \int_0^1 G(\eta,s)f(x(s))\ ds + \frac{1}{1-\eta}\int_0^1 G(\eta,s)f(x(s))\ ds \\
&= \left(\frac{2-\eta}{1-\eta}\right)\int_0^1 G(\eta,s)f(x(s))\ ds \\
&\geq \left(\frac{2-\eta}{1-\eta}\right)\cdot\left(\frac{2R}{\eta(2-\eta)}\right)\int_0^1 G(\eta,s)\ ds \\
&= R = \alpha(x)
\end{aligned}$$

Having satisfied the hypotheses for the functional compression-expansion theorem for appropriately conditioned functionals, $\alpha$ and $\gamma$, we can conclude that the operator $A$ has a fixed point in $\overline{\Omega_2} - \Omega_1$.

$\square$

## 4. Remarks

Similar results have been proved using fixed point theorems based on the Krasnosel'skii result. Echoing the sentiment expressed in [2, 3, 5] these arguments greatly simplify the results of [9, 12] as well as others in establishing existence results for nonlocal boundary value problems. Let us consider for a moment the right focal "limit" problem (1.1) with boundary conditions $x(0) = x'(1) = 0$. In [3], this problem is considered where the growth conditions for the nonlinearity look like

(i) $f(x) \geq 2R, \quad x \in [0, R]$

(ii) $f(x) \leq \dfrac{r}{\eta(1-\eta)}, \quad x \in [r, R].$

Notice that as $\eta \to 1^-$ the growth conditions, $(i)$ and $(ii)$, coincide. Likewise when we examine the BVP considered in [5] with boundary conditions $x(0) = x(1) = 0$ we note that this problem is related to our result for when $\eta \to 0^+$ and $q \to 1/2$. In this case, it is interesting to note the growth conditions for the nonlinearity look like

(i) $f(x) \geq 16R, \quad x \in [R, 2R]$

(ii) $f(x) \leq 8r, \quad x \in [0, r].$

As above it is interesting to note that in our result above if you let $q \to 1/2$ then growth condition $(ii)$ can be replaced by $f(x) \leq \frac{8}{3}r$ which is very similar to what is obtained in [5]. The motivation for the work in this regard is a unifying theorem that simultaneously gives existence results for all three problem types. As seen in this paper these ideas are not without some foundation. The additional freedom in choosing functionals provided by the result of [5] makes this possible and deserves more attention in the future.

## References


[1] Agarwal, R.P., O'Regan, D., Wong, P.; Positive Solutions of Differential, Difference, and Integral Equations, Kluwer Acad. Publ., Boston, 1999.

[2] Anderson, D., Avery R.; Fixed point theorem of cone expansion and compression of functional type, *Journal of Difference Equations and Applications*, **8** (2002), 1073-1083.

[3] Anderson, D., Avery R., Krueger, R.; An extension of the fixed point theorem of cone expansion and compression of functional type, *Comm. on Appl. Nonl. Anal.*, **12** (2003), 83-94.

[4] Anderson, D., Avery R.; A Fourth-Order, Four-Point Right Focal Boundary Value Problem, *Rocky Mountain Journal of Mathematics*, **36** (2006), 367-380.
[5] Avery, R., Henderson, J., O'Regan, D.; Functional Compression-Expansion Fixed Point Theorem, *Electronic Journal of Differential Equations*, **22** (2008), 1-12.
[6] Avery, R., Henderson, J., O'Regan, D.; Four functionals fixed point theorem, *Mathematical and Computer Modelling*, **48** (2008), 1081-1089.
[7] Guo, D., Lakshmikantham, V.; Nonlinear problems in abstract cones, Academic Press, San Diego, 1988.
[8] Leggett, R., Williams, L.; Multiple Positive Fixed Points of Nonlinear Operators on Ordered Banach Spaces, *Indiana University Mathematics Journal*, **28** (1979), 673-688.
[9] Ma, R.; A Survey On Nonlocal Boundary Value Problems, *Applied Mathematics E-Notes*, **7** (2007), 257-279.
[10] Ma, R.; Existence of positive solutions for superlinear semipositone $m$-point boundary value problems, *Proc. Edinburgh Math. Soc.*, **46** (2003), 279-292.
[11] Ma, R., Thompson, B.; Positive solutions of nonlinear three point boundary value problem, *J. Math. Anal. Appl.*, **279** (2003), 1216-1227.
[12] Sun, J., Wei, J.; Positive Solutions to Systems of Nonlinear Second-Order Three-Point Boundary Value Problems, *Applied Mathematics E-Notes*, **9** (2009), 55-62.

(Received July 14, 2009)

Abilene Christian University, Abilene, TX, USA.
*E-mail address*: john.ehrke@acu.edu