Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **54**, 1-18; **http://www.math.u-szeged.hu/ejqtde/**

On $\Psi-$stability of nonlinear Lyapunov matrix differential equations

Aurel Diamandescu
University of Craiova

Abstract

We prove necessary and sufficient conditions for $\Psi-$(uniform) stability of the trivial solution of a nonlinear Lyapunov matrix differential equation.

Key Words: $\Psi-$stability, $\Psi-$uniform stability, Lyapunov matrix differential equation.
2000 Mathematics Subject Classification: 34D20, 34D05.

1 Introduction

Recently, in [7] were studied the problems of $\Psi-$boundedness and $\Psi-$stability of the solutions of the corresponding Kronecker product system (3) associated with (1) in case F(t,X) = F(t) (i.e. a linear nonhomogeneous differential system of the form $x' = G(t)x + f(t)$). But the obtained results in [7] are particular cases of our general results stated in [3], [4]. Indeed, if in Theorems 2.1 and 2.2 ([3]), the fundamental matrix Y is replaced with the fundamental matrix $Z\otimes Y$ of the linear system (5), Theorems 1 and 2 ([7]) concerning $\Psi-$boundedness follow. Similarly, if in Theorems 3.1 and 3.3 ([4]), the fundamental matrix Y is replaced with the fundamental matrix $Z\otimes Y$ of the linear system (5), Theorems 3 and 4 ([7]) concerning $\Psi-$stability follow. In addition, in all results from [7] there are a few mistakes in connection with the matrix Ψ.

In our paper [5] it is proved a necessary and sufficient condition so that the linear nonhomogeneous Lyapunov matrix differential equation $Z' = A(t)Z + ZB(t) + F(t)$ have at least one $\Psi-$bounded solution on $\mathbb{R}_+$ for every Lebesgue $\Psi-$integrable matrix-valued function F on $\mathbb{R}_+$.

The purpose of present paper is to prove (necessary and) sufficient conditions for $\Psi-$(uniform) stability of trivial solution of nonlinear Lyapunov matrix differential equation

$$Z' = A(t)Z + ZB(t) + F(t,Z), \tag{1}$$

which can be seen as a perturbed equation of

$$Z' = A(t)Z + ZB(t). \tag{2}$$

We investigate conditions on the fundamental matrices of the systems $X' = A(t)X$ and $Y' = YB(t)$ and on the function F under which the trivial solutions of the equations (1) and (2) are $\Psi-$(uniformly) stable on $\mathbb{R}_+$. Here, Ψ is a matrix function whose introduction permits us obtaining a mixed asymptotic behavior for the components of solutions.

The main tool used in this paper is the technique of Kronecker product of matrices which has been successfully applied in various fields of matrix theory.

2 Preliminaries

In this section we present some basic definitions, notations, hypotheses and results which are useful later on.

Let $\mathbb{R}^n$ be the Euclidean n - space. For $x = (x_1, x_2, x_3, ..., x_n)^T \in \mathbb{R}^n$, let $\|x\| = \max\{|x_1|, |x_2|, |x_3|, ..., |x_n|\}$ be the norm of x (^{T} denotes transpose).

Let $\mathbb{M}_{m\times n}$ be the linear space of all m×n real valued matrices.

For a matrix $A = (a_{ij}) \in \mathbb{M}_{m\times n}$, we define the norm $|A|$ by $|A| = \sup\limits_{\|x\|\leq 1} \|Ax\|$.

It is well-known that $|A| = \max\limits_{1\leq i\leq n}\{\sum\limits_{j=1}^{n} |a_{ij}|\}$.

Definition 1. ([1]) Let $A = (a_{ij}) \in \mathbb{M}_{m\times n}$ and $B = (b_{ij}) \in \mathbb{M}_{p\times q}$. The Kronecker product of A and B, written $A\otimes B$, is defined to be the partitioned matrix

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}$$

Obviously, $A\otimes B \in \mathbb{M}_{mp\times nq}$.

Lemma 1. The Kronecker product has the following properties and rules, provided that the dimension of the matrices are such that the various expressions exist:

1). $A\otimes(B\otimes C) = (A\otimes B)\otimes C$;
2). $(A\otimes B)^T = A^T\otimes B^T$;
3). $(A\otimes B)\cdot(C\otimes D) = (A\cdot C)\otimes(B\cdot D)$;
4). $(A\otimes B)^{-1} = A^{-1}\otimes B^{-1}$;
5). $A\otimes(B + C) = A\otimes B + A\otimes C$;
6). $(A + B)\otimes C = A\otimes C + B\otimes C$;
7). $I_p\otimes A = \begin{pmatrix} A & O & \cdots & O \\ O & A & \cdots & O \\ \vdots & \vdots & \vdots & \vdots \\ O & O & \cdots & A \end{pmatrix}$;

8). $(A(t)\otimes B(t))' = A'(t)\otimes B(t) + A(t)\otimes B'(t)$; (here, $'$ denotes the derivative $\frac{d}{dt}$).
Proof. See in [1].

Definition 2. The application $\mathcal{V}ec : \mathbb{M}_{m\times n} \longrightarrow \mathbb{R}^{mn}$, defined by

$$\mathcal{V}ec(A) = (a_{11}, a_{21}, \cdots, a_{m1}, a_{12}, a_{22}, \cdots, a_{m2}, \cdots, a_{1n}, a_{2n}, \cdots, a_{mn})^T,$$

where $A = (a_{ij}) \in \mathbb{M}_{m\times n}$, is called the vectorization operator.

Lemma 2. ([5]) The vectorization operator $\mathcal{V}ec : \mathbb{M}_{n\times n} \longrightarrow \mathbb{R}^{n^2}$, is a linear and one-to-one operator. In addition, $\mathcal{V}ec$ and $\mathcal{V}ec^{-1}$ are continuous operators.

Lemma 3. A function $F : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$ is a continuous (differentiable) matrix function on $\mathbb{R}_+$ if and only if the function $f = \mathcal{V}ec(F) : \mathbb{R}_+ \longrightarrow \mathbb{R}^{n^2}$ is a continuous (differentiable) vector function on $\mathbb{R}_+$.
Proof. It is a simple exercise.

We recall that the vectorization operator $\mathcal{V}ec$ has the following properties as concerns the calculations (see in [7]):

Lemma 4. If $A, B, M \in \mathbb{M}_{n\times n}$, then
1). $\mathcal{V}ec(AMB) = (B^T\otimes A)\cdot\mathcal{V}ec(M)$;
2). $\mathcal{V}ec(MB) = (B^T\otimes I_n)\cdot\mathcal{V}ec(M)$;
3). $\mathcal{V}ec(AM) = (I_n\otimes A)\cdot\mathcal{V}ec(M)$;
4). $\mathcal{V}ec(AM) = (M^T\otimes A)\cdot\mathcal{V}ec(I_n)$.
Proof. It is a simple exercise.

In the systems (1) and (2) we assume that A and B are continuous $n\times n$ matrices on $\mathbb{R}_+ = [0,\infty)$ and $F : \mathbb{R}_+ \times \mathbb{M}_{n\times n} \longrightarrow \mathbb{M}_{n\times n}$ is a continuous $n\times n$ matrix such that $F(t,O_n) = O_n$ (null matrix of order $n\times n$).

By a solution of the equation (1) we mean a continuous differentiable $n\times n$ matrix function satisfying the equation (1) for all $t \geq 0$.

Let $\Psi_i : \mathbb{R}_+ \longrightarrow (0,\infty)$, $i = 1,2,...,n$, be continuous functions and

$$\Psi = \text{diag}\,[\Psi_1, \Psi_2, \cdots \Psi_n].$$

Definition 3. ([4]). The trivial solution of the vector differential equation $x' = f(t,x)$ (where $x \in \mathbb{R}^n$ and f is a continuous n vector function) is said to be $\Psi-$stable on $\mathbb{R}_+$ if for every $\varepsilon > 0$ and every $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(\varepsilon,t_0) > 0$ such that any solution $x(t)$ of the equation which satisfies the inequality $\| \Psi(t_0)x(t_0)\| < \delta$, exists and satisfies the inequality $\| \Psi(t)x(t)\| < \varepsilon$ for all $t \geq t_0$.

The trivial solution of the vector differential equation $x' = f(t,x)$ is said to be $\Psi-$uniformly stable on $\mathbb{R}_+$ if it is $\Psi-$stable on $\mathbb{R}_+$ and the above δ is independent of t_0.

Extend this definition for a matrix differential equation.

Definition 4. The trivial solution of the matrix differential equation $X' = F(t,X)$ (where $X \in \mathbb{M}_{n\times n}$ and F is a continuous $n\times n$ matrix function) is said to be $\Psi-$stable on $\mathbb{R}_+$ if for every $\varepsilon > 0$ and every $t_0 \in \mathbb{R}_+$, there exists $\delta = \delta(\varepsilon,t_0) > 0$ such that any solution $X(t)$ of the equation which satisfies the inequality $|\,\Psi(t_0)X(t_0)| < \delta$, exists and satisfies the inequality $|\,\Psi(t)X(t)| < \varepsilon$ for all $t \geq t_0$.

The trivial solution of the matrix differential equation $X' = F(t,X)$ is said to be $\Psi-$uniformly stable on $\mathbb{R}_+$ if it is $\Psi-$stable on $\mathbb{R}_+$ and the above δ is independent of t_0.

The following lemmas play a vital role in the proofs of main results.

The first result is done in [7]. Because the proof is incomplete, we present it with a complete proof.

Lemma 5. The matrix function $Z(t)$ is a solution of (1) on the interval $J \subset \mathbb{R}_+$ if and only if the vector valued function $z(t) = \mathcal{V}\mathrm{ec}Z(t)$ is a solution of the differential system

$$z' = \left(I_n \otimes A(t) + B^T(t) \otimes I_n\right) z + f(t,z) \tag{3}$$

where $f(t,z) = \mathcal{V}\mathrm{ec}F(t,Z)$, on the same interval J.

Proof. It is similar with the proof of Lemma 7, [5].

Definition 5. The above system (3) is called "corresponding Kronecker product system associated with (1)".

Lemma 6. For every matrix function $M : \mathbb{R}_+ \longrightarrow \mathbb{M}_{n\times n}$, we have

$$\frac{1}{n}\,|\,\Psi(t)M(t)\,| \;\leq\; \|\,(I_n \otimes \Psi(t))\,\mathcal{V}\mathrm{ec}M(t)\,\|_{\mathbb{R}^{n^2}} \;\leq\; |\,\Psi(t)M(t)\,|\,,\ t \,\geq\, 0. \tag{4}$$

Proof. From the proof of Lemma 2, it results that

$$\frac{1}{n}\,|\,A\,| \;\leq\; \|\,\mathcal{V}\mathrm{ec}A\,\|_{\mathbb{R}^{n^2}} \;\leq\; |\,A\,|\,,$$

for every $A \in \mathbb{M}_{n\times n}$.

Setting $A = \Psi(t)M(t)$ for $t \geq 0$ and using Lemma 4, the inequality (4) follows immediately.

Lemma 7. The trivial solution of the equation (1) is $\Psi-$(uniformly) stable on $\mathbb{R}_+$ if and only if the trivial solution of the corresponding Kronecker product system (3) is $I_n \otimes \Psi-$(uniformly) stable on $\mathbb{R}_+$.

Proof. First, suppose that the trivial solution of the equation (1) is $\Psi-$stable on $\mathbb{R}_+$. According to Definition 4, for a given $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, we choose $\delta_0(\varepsilon,t_0) = \frac{1}{n}\delta(\varepsilon,t_0)$. Let $z(t)$ be a solution of (3) such that $\|\,(I_n \otimes \Psi(t_0))z(t_0)\|_{\mathbb{R}^{n^2}} < \delta_0(\varepsilon,t_0)$. From Lemma 5 and Lemma 6, $Z(t) = \mathcal{V}\mathrm{ec}^{-1}z(t)$ is a solution on $\mathbb{R}_+$ of the equation (1) such that $|\,\Psi(t_0)Z(t_0)| < \delta(\varepsilon,t_0)$. It follows that $|\,\Psi(t)Z(t)| < \varepsilon$ for all $t \geq t_0$.

From Lemma 6, we have that $\| (I_n \otimes \Psi(t))z(t)\|_{\mathbb{R}^{n^2}} < \varepsilon$ for all $t \geq t_0$. Thus, the trivial solution of the system (3) is $I_n \otimes \Psi-$stable on $\mathbb{R}_+$.

Suppose, conversely, that the trivial solution of the system (3) is $I_n \otimes \Psi-$stable on $\mathbb{R}_+$. According to Definition 3, for a given $\varepsilon > 0$ and $t_0 \in \mathbb{R}_+$, we choose $\delta_0(\varepsilon,t_0) = \delta(\frac{\varepsilon}{n},t_0)$. Let Z(t) be a solution of (1) such that $| \Psi(t_0)Z(t_0)| < \delta_0$. From Lemma 5 and Lemma 6, $z(t) = \mathcal{V}ecZ(t)$ is a solution on $\mathbb{R}_+$ of the system (3) such that $\| (I_n \otimes \Psi(t_0))z(t_0)\|_{\mathbb{R}^{n^2}} < \delta_0$. It follows that $\| (I_n \otimes \Psi(t))z(t)\|_{\mathbb{R}^{n^2}} < \frac{\varepsilon}{n}$ for all $t \geq t_0$. From Lemma 6, we have that $| \Psi(t)Z(t)| < \varepsilon$ for all $t \geq t_0$. Thus, the trivial solution of the equation (1) is $\Psi-$stable on $\mathbb{R}_+$.

Since the proof of the $\Psi-$uniform stability is similar to the above proof, we omit it here.

The proof is now complete.

The next result is Lemma 1 of [7]. Because the proof is incomplete, we present it with a complete proof.

Lemma 8. Let X(t) and Y(t) be a fundamental matrices for the equations

$$X' = A(t)X \tag{5}$$

and

$$Y' = YB(t) \tag{6}$$

respectively.

Then, the matrix $Z(t) = Y^T(t) \otimes X(t)$ is a fundamental matrix for the system

$$z' = \left(I_n \otimes A(t) + B^T(t) \otimes I_n \right) z \tag{7}$$

Proof. See Lemma 6, [5].

3 $\Psi-$ stability of the linear Lyapunov matrix differential equations

The purpose of this section is to study conditions for $\Psi-$(uniform) stability of trivial solution of linear Lyapunov matrix differential equation (2). These conditions can be expressed in terms of a fundamental matrices for the equations (5) and (6).

Theorem 1. Let X(t) and Y(t) be a fundamental matrices for the equations (5) and (6) respectively. Then,

a). The trivial solution of (2) is $\Psi-$stable on $\mathbb{R}_+$ if and only if there exists a positive constant K such that

$$| Y^T(t) \otimes (\Psi(t)X(t)) | \ \leq \ K, \text{ for all } t \ \in \ \mathbb{R}_+. \tag{8}$$

b). The trivial solution of (2) is $\Psi-$uniformly stable on $\mathbb{R}_+$ if and only if there exists a positive constant K such that

$$| \mathrm{Y^T(t)(Y^T(s))^{-1}} \otimes (\Psi(\mathrm{t})\mathrm{X(t)X^{-1}(s)}\Psi^{-1}(\mathrm{s})) | \ \leq \ \mathrm{K}, \text{ for all } \mathrm{t} \ \geq \ \mathrm{s} \ \geq \ 0. \qquad (9)$$

Proof. a). Suppose that the trivial solution of (2) is $\Psi-$stable on $\mathbb{R}_+$. From Lemma 7, it follows that the trivial solution of (7) is $\mathrm{I_n} \otimes \Psi-$stable on $\mathbb{R}_+$. From Theorem 3.1 ([4]), it follows that the fundamental matrix Z(t) of (7) satisfies the condition

$$| (\mathrm{I_n} \otimes \Psi(\mathrm{t}))\mathrm{Z(t)} | \ \leq \ \mathrm{K}, \text{ for all } \mathrm{t} \ \in \ \mathbb{R}_+,$$

where K is a positive constant.

From Lemma 8, we replace $\mathrm{Z(t) = Y^T(t) \otimes X(t)}$ as a fundamental matrix on $\mathbb{R}_+$ for the system (7). After computation, it follows that (8) holds.

Now, suppose that (8) holds for some $\mathrm{K} > 0$ and for all $\mathrm{t} \in \mathbb{R}_+$. From Lemma 8 and Theorem 3.1 ([4]), it follows that the trivial solution of (7) is $\mathrm{I_n} \otimes \Psi-$stable on $\mathbb{R}_+$. From Lemma 7, it follows that the trivial solution of (2) is $\Psi-$stable on $\mathbb{R}_+$.

b). Since the proof for the $\Psi-$uniform stability is similar to the above proof, we omit it here.

The proof is now complete.

Remark. 1. In the same manner as in classical stability, we can speak about $\Psi-$

(uniform) stability of the linear differential equation (2).

2. It is easy to see that if $| \Psi(\mathrm{t}) |$ and $| \Psi^{-1}(\mathrm{t}) |$ are bounded on $\mathbb{R}_+$, then the $\Psi-$(uniform) stability of the linear equation (2) is equivalent with the classical (uniform) stability of the linear equation (2).

Remark. Theorem 1 generalizes Theorem 3.1 ([4]).

Indeed, in particular case $\mathrm{B(t) = O_n}$, we have $\mathrm{Y = I_n}$ and then $\mathrm{Z(t) = I_n \otimes X(t)}$. On the other hand, it is easy to see that the solutions of (2) are $\mathrm{Z(t) = X(t)C}$, C being a n$\times$n constant matrix. Now, the conditions (8) and (9) become the conditions from Theorem 3.1 ([4]) concerning $\Psi-$(uniform) stability of the linear system $\mathrm{x' = A(t)x}$, because $| \mathrm{I_n \otimes M} | = |\mathrm{M}|$

Corollary. If the equation (5) is $\Psi-$(uniformly) stable on $\mathbb{R}_+$ and the equation (6) is (uniformly) stable on $\mathbb{R}_+$, then the equation (2) is $\Psi-$(uniformly) stable on $\mathbb{R}_+$.

Proof. It results from the above Theorem 1 and from the inequality $| \mathrm{A \otimes B} | \leq |\mathrm{A}||\mathrm{B}|$, for all $\mathrm{A, B} \in \mathbb{M}_{\mathrm{n\times n}}$.

Sufficient conditions for the $\Psi-$(uniform) stability of trivial solution of the equation (2) are given by the next theorems.

Theorem 2. Let X(t) and Y(t) be a fundamental matrices for the equations (5) and (6) respectively. Suppose that there exist a continuous function $\varphi : \mathbb{R}_+$

$\longrightarrow (0,\infty)$ and the constants $p \geq 1$ and $M > 0$ which fulfil one of the following conditions:

(i). $\displaystyle\int_0^t \varphi(s) \mid (Y^T(t)(Y^T(s))^{-1}) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)) \mid^p ds \ \leq \ M$, for all $t \geq 0$,

(ii). $\displaystyle\int_0^t \varphi(s) \mid ((Y^T(s))^{-1}Y^T(t)) \otimes (X^{-1}(s)\Psi^{-1}(s)\Psi(t)X(t)) \mid^p ds \ \leq \ M$,for all $t \geq 0$.

Then, the trivial solution of the equation (2) is $\Psi-$stable on $\mathbb{R}_+$.

Proof. From the hypotheses, Lemma 8 and Theorem 3.3 ([4]), it follows that the system (7) is $I_n \otimes \Psi-$stable on $\mathbb{R}_+$. From this and Lemma 7, it follows that the equation (2) is $\Psi-$stable on $\mathbb{R}_+$.

Remark. 1. The function φ can serve to weaken the required hypotheses on the fundamental metrices X and Y.

2. In the conditions of Theorem 2, the equation (2) can not be $\Psi-$uniformly stable on $\mathbb{R}_+$. This is shown in the next example, adapted from J. L. Massera and J. J. Schäffer [6].

Example. Let $a(t)$ be a real, continuously differentiable function, equal to 1 except in the intervals $J_n = [n - 2^{-4n}, n + 2^{-4n}]$, $n = 1, 2, \ldots$; in J_n, $a(t)$ lies between 1 and 4^n and $a(n) = 4^n$.

Consider the equation (2) with

$$A(t) = \begin{pmatrix} -(3 + \frac{a'(t)}{a(t)}) & 0 \\ 0 & -5 \end{pmatrix}, \ B(t) = \begin{pmatrix} 4 & 3 \\ -10 & -7 \end{pmatrix}.$$

The matrix B has the eigenvalues $\lambda_1 = -1$, $\lambda_2 = -2$ and the Jordan canonical form $L = \mathrm{diag}\,[-1, -2]$. We have $B^T = ULU^{-1}$, where $U = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$.

The fundamental matrices for the equations (5) and (6) are

$$X(t) = \begin{pmatrix} \frac{e^{-3t}}{a(t)} & 0 \\ 0 & e^{-5t} \end{pmatrix}$$

and

$$Y^T(t) = Ue^{Lt}U^{-1} = \begin{pmatrix} 6e^{-t} - 5e^{-2t} & 10e^{-2t} - 10e^{-t} \\ 3e^{-t} - 3e^{-2t} & 6e^{-2t} - 5e^{-t} \end{pmatrix}$$

respectively.

Consider

$$\Psi(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}.$$

For $t \geq s \geq 0$, we have

$$\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s) = \begin{pmatrix} \frac{a(s)}{a(t)}e^{-2(t-s)} & 0 \\ 0 & e^{-6(t-s)} \end{pmatrix}$$

and

$$Y^T(t)(Y^T(s))^{-1} = \begin{pmatrix} 6e^{-(t-s)} - 5e^{-2(t-s)} & 10e^{-2(t-s)} - 10e^{-(t-s)} \\ 3e^{-(t-s)} - 3e^{-2(t-s)} & 6e^{-2(t-s)} - 5e^{-(t-s)} \end{pmatrix}.$$

It follows that

$$| (Y^T(t)(Y^T(s))^{-1}) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)) | \le$$
$$\le 31 \max \left\{ \frac{a(s)}{a(t)} e^{-3(t-s)}, \; e^{-7(t-s)} \right\}, \; t \ge s \ge 0,$$

and then

$$\int_0^t | (Y^T(t)(Y^T(s))^{-1}) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)) | \, ds \le 31, \text{ for all } t \ge 0.$$

Thus, the condition (i) of Theorem 2 is satisfied with $\varphi = 1$, $p = 1$, $M = 31$. It follows that the equation (2) is $\Psi-$stable on $\mathbb{R}_+$.
On the other hand, for $s = n$, $t = n + 2^{-4n}$ we have

$$| (Y^T(t)(Y^T(s))^{-1}) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)) | \ge$$
$$\ge \left| \left(6e^{-2^{-4n}} - 5e^{-2^{-4n+1}} \right) 4^n e^{-2^{-4n+1}} \right| \longrightarrow \infty.$$

From Theorem 1, it follows that the equation (2) is not $\Psi-$uniformly stable on $\mathbb{R}_+$.

Remark. Theorem 2 generalizes Theorem 3.3 ([4]).

Theorem 3. Suppose that the linear equation (2) is $\Psi-$uniformly stable on $\mathbb{R}_+$ and A_1, B_1 are continuous $n \times n$ matrices on $\mathbb{R}_+$ such that

$$\int_0^\infty | \; I_n \otimes (\Psi(t)A_1(t)\Psi^{-1}(t)) + B_1^T(t) \otimes I_n \,| \, dt < +\infty.$$

Then, the linear Lyapunov matrix differential equation

$$Z' = (A(t) + A_1(t))Z + Z(B(t) + B_1(t)) \tag{10}$$

is also $\Psi-$uniformly stable on $\mathbb{R}_+$.

Proof. The corresponding Kronecker product system associated with (10) can be seen in the form

$$z' = \left(I_n \otimes A(t) + B^T(t) \otimes I_n \right) z + \left(I_n \otimes A_1(t) + B_1^T(t) \otimes I_n \right) z, \tag{11}$$

i.e. as a perturbed system of the corresponding Kronecker product system associated with (2),

$$z' = \left(I_n \otimes A(t) + B^T(t) \otimes I_n \right) z. \tag{12}$$

From the hypothesis and Lemma 7, this system is $I_n \otimes \Psi-$uniformly stable on $\mathbb{R}_+$.

On the other hand, from hypothesis, the matrix $I_n \otimes A_1(t) + B_1^T(t) \otimes I_n$ satisfies the conditions of Theorem 3.4 ([4]).

Now, from this Theorem it follows that the system (11) is $I_n \otimes \Psi-$uniformly stable on $\mathbb{R}_+$.

From Lemma 7 again, it follows that the system (10) is $\Psi-$uniformly stable on $\mathbb{R}_+$.

The proof is now complete.

Remark. If the linear equation (2) is only $\Psi-$stable on $\mathbb{R}_+$, then the linear equation (10) can not be $\Psi-$stable on $\mathbb{R}_+$. This is shown by the next Example, adapted from an example due to O. Perron [8] and Example 3.5 ([4]).

Example. Consider the equation (10) with

$$A(t) = \begin{pmatrix} -1 & 0 \\ 0 & \sin\ln(t+1) + \cos\ln(t+1) - \frac{3}{2} \end{pmatrix}, \quad A_1(t) = \begin{pmatrix} 0 & 0 \\ e^{-\frac{1}{2}(t+1)} & 0 \end{pmatrix},$$

$B(t) = \frac{1}{2}I_2$ and $B_1(t) = O_2$.

Consider $\Psi(t) = \begin{pmatrix} e^{\frac{1}{2}(t+1)} & 0 \\ 0 & 1 \end{pmatrix}$.

The equation (2) becomes

$$Z' = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \sin\ln(t+1) + \cos\ln(t+1) - 1 \end{pmatrix} Z.$$

From the Example 3.5, ([4]), it follows that the equation (2) is $\Psi-$stable on $\mathbb{R}_+$, but it is not $\Psi-$uniformly stable on $\mathbb{R}_+$. In addition, the condition

$$\int_0^\infty |\ I_2 \otimes (\Psi(t)A_1(t)\Psi^{-1}(t)) + B_1^T(t) \otimes I_n \,|\, dt \ < \ +\infty$$

holds.

Finally, the perturbed equation (10) is not $\Psi-$stable on $\mathbb{R}_+$ (see the Example 3.5, ([4])).

Remark. Theorem 3 is no longer true if we require that

$$\lim_{t\longrightarrow\infty} |\ I_n \otimes (\Psi(t)A_1(t)\Psi^{-1}(t)) + B_1^T(t) \otimes I_n\ | = 0,$$

instead of the condition

$$\int_0^\infty |\ I_n \otimes (\Psi(t)A_1(t)\Psi^{-1}(t)) + B_1^T(t) \otimes I_n \,|\, dt \ < \ +\infty.$$

This is shown by the next Example, adapted from Example 3.6 ([4]).

Example. Consider the equation (10) with

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 & -\frac{2}{t+1} \end{pmatrix},\ A_1(t) = \begin{pmatrix} -\frac{1}{t+1} & 0 \\ 0 & \frac{1}{t+1} \end{pmatrix},$$

$B(t) = -\frac{1}{t+1}I_2$ and $B_1(t) = \frac{2}{t+1}I_2$.

Consider $\Psi(t) = \begin{pmatrix} t+1 & 0 \\ 0 & t+1 \end{pmatrix}$.

For the corresponding equation (2), we have that

$$X(t) = \begin{pmatrix} \frac{\sin(t+1)}{t+1} & \frac{\cos(t+1)}{t+1} \\ \frac{(t+1)\cos(t+1) - \sin(t+1)}{(t+1)^2} & -\frac{(t+1)\sin(t+1) + \cos(t+1)}{(t+1)^2} \end{pmatrix}$$

and $Y(t) = \frac{1}{t+1}I_2$ are the fundamental matrices for the systems (5) and (6) respectively.

From the Theorem 1, it follows that the equation (2) is $\Psi-$uniformly stable on $\mathbb{R}_+$.

In addition, we have that

$$\int_0^\infty \mid\ I_2 \otimes (\Psi(t)A_1(t)\Psi^{-1}(t)) + B_1^T(t) \otimes I_2\ \mid dt = +\infty$$

and

$$\lim_{t\longrightarrow\infty} \mid\ I_2 \otimes (\Psi(t)A_1(t)\Psi^{-1}(t)) + B_1^T(t) \otimes I_2\ \mid\ = 0.$$

On the other hand, the solutions of the equation (10) are

$$Z(t) = \begin{pmatrix} \sin t & \cos t \\ \cos t & -\sin t \end{pmatrix} \cdot C,$$

where C is a 2×2 constant matrix.

It is easy to see that the equation (10) is not $\Psi-$stable on $\mathbb{R}_+$.

Remark. Theorem 3 generalizes Theorem 3.4 ([4]).

Theorem 4. Suppose that:

1). There exist a continuous function $\varphi : \mathbb{R}_+ \longrightarrow (0,\infty)$ and a positive constant M such that the fundamental matrices X and Y for the equations (5) and (6) respectively satisfy the condition

$$\int_0^t \varphi(s) \mid (Y^T(t)(Y^T(s))^{-1}) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)) \mid ds\ \leq\ M, \text{ for all } t \geq 0;$$

2). $A_1(t)$ and $B_1(t)$ are continuous $n\times n$ matrices on $\mathbb{R}_+$ such that

$$\sup_{t\geq0} \frac{1}{\varphi(t)} \mid\ I_n \otimes (\Psi(t)A_1(t)\Psi^{-1}(t)) + B_1^T(t) \otimes I_n\ \mid\ <\ \frac{1}{M}.$$

Then, the linear Lyapunov matrix differential equation (10) is $\Psi-$stable on $\mathbb{R}_+$.

Proof. From the hypothesis 1) and Theorem 2, it follows that the equation (2) is $\Psi-$stable on $\mathbb{R}_+$. Thereafter, from Lemma 7, it follows that the equation (12) is $I_n \otimes \Psi-$stable on $\mathbb{R}_+$. From Theorem 3.7 ([4]), it follows that the system (11) is $I_n \otimes \Psi-$stable on $\mathbb{R}_+$. From Lemma 7 again, the equation (10) is $\Psi-$stable on $\mathbb{R}_+$.

The proof is now complete.

Remark. Theorem 4 is no longer true if we require that the equation (2) be $\Psi-$(uniformly) stable on $\mathbb{R}_+$, instead of the condition 1) of Theorem. This is shown by the next simple example, adapted from Example 3.8 ([4]).

Example. Consider the equation (2) with $A(t) = B(t) = O_2$. Then, a fundamental matrices for the equations (5) and (6) are $X(t) = I_2$, $Y(t) = I_2$ respectively.

Consider $\Psi(t) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{t+1} \end{pmatrix}$.

Therefore,

$$(Y^T(t)(Y^T(s))^{-1}) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{s+1}{t+1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{s+1}{t+1} \end{pmatrix}$$

is bounded for $0 \leq s \leq t < +\infty$.

From Theorem 1, it follows that the equation (2) is $\Psi-$uniformly stable on $\mathbb{R}_+$.

Now, we consider the equation (10) with

$$A_1(t) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{a}{\sqrt{t+1}} \end{pmatrix} \text{ and } B_1(t) = O_2,$$

a being a positive constant.

It is easy to see (by reduction to absurdity) there is no function φ which satisfies the conditions of Theorem 4.

The solutions of the equation (10) are

$$Z(t) = \begin{pmatrix} 1 & 0 \\ 0 & e^{2a\sqrt{t+1}} \end{pmatrix} \cdot C,$$

where C is a 2×2 constant matrix.

It is easy to see that the trivial solution of (10) is not $\Psi-$stable on $\mathbb{R}_+$.

Finally, we have

$$\sup_{t\geq 0} \mid I_2 \otimes (\Psi(t)A_1(t)\Psi^{-1}(t)) + B_1^T(t) \otimes I_2 \mid = a$$

and

$$\lim_{t\longrightarrow\infty} \mid I_2 \otimes (\Psi(t)A_1(t)\Psi^{-1}(t)) + B_1^T(t) \otimes I_2 \mid = 0.$$

Remark. Theorem 4 generalizes Theorem 3.7 ([4]).

4 $\Psi-$ stability of the nonlinear Lyapunov matrix differential equations

The purpose of this section is to study the $\Psi-$(uniform) stability of trivial solution of the equation (1), where A, B and F are matrix functions. It will be assumed that A and B are continuous for $t \in \mathbb{R}_+$ and that F is continuous for $t \in \mathbb{R}_+$ and $Z \in \mathbb{M}_{n\times n}$. This will ensure the local existence of a solution passing through any given point (t_0,Z_0) of the domain of definition of F, but it does not guarantee that the solution is unique or that it can be continued for all large values of t.

Thus, we state the following hypothesis:

(H) For all $t_0 \in \mathbb{R}_+$ and $Z_0 \in \mathbb{M}_{n\times n}$, then there exists a unique solution Z(t) of the equation (1) such that $Z(t_0) = Z_0$.

Theorem 5. Suppose that:
a). the hypothesis (H) is satisfied;
b). the trivial solution of (2) is $\Psi-$uniformly stable on $\mathbb{R}_+$;
c). the matrix function F satisfies the inequality

$$\mid \Psi(t)F(t,Z) \mid \ \leq \ \gamma(t) \mid \Psi(t)Z \mid,$$

for all $t \in \mathbb{R}_+$ and for all $Z \in \mathbb{M}_{n\times n}$, where γ is a continuous nonnegative function on $\mathbb{R}_+$ such that $L = \int_0^\infty \gamma(t)dt < +\infty$.

Then, the trivial solution of the equation (1) is $\Psi-$uniformly stable on $\mathbb{R}_+$.

Proof. From Lemma 5, it follows that if Z(t) is the unique solution of (1) with $Z(t_0) = Z_0$, then, $z(t) = \mathcal{V}ecZ(t)$ is the unique solution of the system (3) with $z(t_0) = z_0 = \mathcal{V}ecZ_0$. Therefore, z(t) is also a solution of the inhomogeneous linear system

$$z' = \Big(I_n \otimes A(t) + B^T(t) \otimes I_n\Big) z + f(t,z(t)),\ t \ \in \ [t_0,t_1), \tag{13}$$

$[t_0,t_1)$ being the existence interval of solution z(t).

If U(t) is a fundamental matrix for the system (7), i.e. the homogeneous system associated with (13), by the variation of constant formula ([2], Chapter II, section 2(8)),

$$z(t) = U(t)U^{-1}(t_0)z_0 + \int_{t_0}^t U(t)U^{-1}(s)f(s,z(s))ds,\ t \ \in \ [t_0,t_1).$$

From Lemma 8, we replace $U(t) = Y^T(t)\otimes X(t)$, X(t) and Y(t) being fundamental matrices for the equations (5) and (6) respectively. After computation, it follows that

$$z(t) = ((Y^T(t)(Y^T(t_0))^{-1}) \otimes (X(t)X^{-1}(t_0)))z_0 +$$

$$+\int_{t_0}^{t}((Y^T(t)(Y^T(s))^{-1})\otimes(X(t)X^{-1}(s)))f(s,z(s))ds,\ t\ \in\ [t_0,t_1). \qquad (14)$$

From hypothesis b) and Theorem 1, it follows that there exists a positive constant K such that

$$|\ Y^T(t)(Y^T(s))^{-1}\otimes(\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s))\ |\ \leq\ K,$$

for all $t \geq s \geq 0$.

From (14), it follows that

$$(I_n\otimes\Psi(t))z(t)=((Y^T(t)(Y^T(t_0))^{-1})\otimes(\Psi(t)X(t)X^{-1}(t_0)\Psi^{-1}(t_0)))(I_n\otimes\Psi(t_0))z_0+$$

$$+\int_{t_0}^{t}((Y^T(t)(Y^T(s))^{-1})\otimes(\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)))(I_n\otimes\Psi(s))f(s,z(s))ds,\ t\in[t_0,t_1)$$

and then,

$$\|\ (I_n\otimes\Psi(t))z(t)\ \|_{\mathbb{R}^{n^2}}\leq K\|\ (I_n\otimes\Psi(t_0))z_0\ \|_{\mathbb{R}^{n^2}}+$$

$$+K\int_{t_0}^{t}\|\ (I_n\otimes\Psi(s))f(s,z(s))\ \|_{\mathbb{R}^{n^2}}ds,\ t\in[t_0,t_1).$$

From hypothesis c) and Lemma 6, it follows that

$$\|\ (I_n\otimes\Psi(t))f(t,z)\ \|_{\mathbb{R}^{n^2}}\ =\ \|\ (I_n\otimes\Psi(t))\mathcal{V}ecF(t,Z)\ \|_{\mathbb{R}^{n^2}}\ \leq$$

$$\leq\ |\ \Psi(t)F(t,Z)\ |\ \leq\ \gamma(t)\ |\ \Psi(t)Z\ |\ \leq$$

$$\leq\ n\gamma(t)\ \|\ (I_n\otimes\Psi(t))\mathcal{V}ecZ\ \|_{\mathbb{R}^{n^2}}\ =n\gamma(t)\ \|\ (I_n\otimes\Psi(t))z\ \|_{\mathbb{R}^{n^2}},$$

for $t\in\mathbb{R}_+$ and $z\in\mathbb{R}^{n^2}$.

Then,

$$\|\ (I_n\otimes\Psi(t))z(t)\ \|_{\mathbb{R}^{n^2}}\ \leq\ K\ \|\ (I_n\otimes\Psi(t_0))z_0\ \|_{\mathbb{R}^{n^2}}\ +$$

$$+\ nK\int_{t_0}^{t}\gamma(s)\ \|\ (I_n\otimes\Psi(s))z(s)\ \|_{\mathbb{R}^{n^2}}\ ds,\ \ t\in[t_0,t_1)$$

and therefore by Gronwall's inequality,

$$\|\ (I_n\otimes\Psi(t))z(t)\ \|_{\mathbb{R}^{n^2}}\ \leq\ K\ \|\ (I_n\otimes\Psi(t_0))z_0\ \|_{\mathbb{R}^{n^2}}\cdot e^{nK\int_{t_0}^{t}\gamma(s)ds}\ \leq$$

$$\leq\ Ke^{nKL}\ \|\ (I_n\otimes\Psi(t_0))z_0\ \|_{\mathbb{R}^{n^2}}\ ,\ \text{for all } t\in[t_0,t_1).$$

This inequality shows that $t_1 = +\infty$ and hence, the solution z is defined on $[t_0, +\infty)$.

Therefore, it follows that the trivial solution of the system (13) is $I_n\otimes\Psi-$uniformly stable on $\mathbb{R}_+$.

From Lemma 7, it follows that the trivial solution of (1) is $\Psi-$uniformly stable on $\mathbb{R}_+$.

The proof is now complete.

Example. Consider the equation (1) with

$$\mathrm{A(t)} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},\ \mathrm{B(t)} = \begin{pmatrix} -6 & -3 \\ 10 & 5 \end{pmatrix} \text{ and } \mathrm{F(t,Z)} = \begin{pmatrix} \mathrm{e^{-t}z_1} & 0 \\ 0 & \frac{\sin z_4}{1+\mathrm{t}^2} \end{pmatrix}.$$

The matrices A, $\mathrm{B^T}$ have the eigenvalues $\lambda_1 = 1 + \mathrm{i}$, $\lambda_2 = 1 - \mathrm{i}$ and $\mu_1 = -1$, $\mu_2 = 0$ respectively. The fundamental matrices for the systems (5) and (6) are

$$\mathrm{X(t)} = \begin{pmatrix} \mathrm{e^t}\cos \mathrm{t} & \mathrm{e^t}\sin \mathrm{t} \\ -\mathrm{e^t}\sin \mathrm{t} & \mathrm{e^t}\cos \mathrm{t} \end{pmatrix} \text{ and } \mathrm{Y(t)} = \begin{pmatrix} 2\mathrm{e^{-t}} & \mathrm{e^{-t}} \\ 5 & 3 \end{pmatrix}$$

respectively.

Consider

$$\Psi(\mathrm{t}) = \begin{pmatrix} \mathrm{e^{-t}} & 0 \\ 0 & \mathrm{e^{-t}} \end{pmatrix}.$$

We have

$$\Psi(\mathrm{t})\mathrm{X(t)X^{-1}(s)}\Psi^{-1}(\mathrm{s}) = \begin{pmatrix} \cos(\mathrm{t}-\mathrm{s}) & \sin(\mathrm{t}-\mathrm{s}) \\ -\sin(\mathrm{t}-\mathrm{s}) & \cos(\mathrm{t}-\mathrm{s}) \end{pmatrix}$$

and

$$\mathrm{Y^T(t)(Y^T(s))^{-1}} = \begin{pmatrix} 6\mathrm{e}^{-(\mathrm{t}-\mathrm{s})} - 5 & 10 - 10\mathrm{e}^{-(\mathrm{t}-\mathrm{s})} \\ 3\mathrm{e}^{-(\mathrm{t}-\mathrm{s})} - 3 & 6 - 5\mathrm{e}^{-(\mathrm{t}-\mathrm{s})} \end{pmatrix}.$$

From Corollary of Theorem 1, it follows that the equation (2) is $\Psi-$uniformly stable on $\mathbb{R}_+$.

Further, the function F satisfies a Lipschitz condition and

$$|\ \Psi(\mathrm{t})\mathrm{F(t,Z)}\ | \ = \max\left\{\mathrm{e^{-2t}}\ |\ \mathrm{z_1}\ |, \frac{\mathrm{e^{-t}}\ |\ \sin \mathrm{z_4}\ |}{1+\mathrm{t}^2}\right\} \ \le\ \frac{1}{1+\mathrm{t}^2}\ |\ \Psi(\mathrm{t})\mathrm{Z}\ |$$

for all $\mathrm{t} \in \mathbb{R}_+$ and $\mathrm{Z} \in \mathbb{M}_{2\times 2}$.

From these, it is easy to see that the function F satisfies all the hypotheses of Theorem 5.

Thus, the trivial solution of the equation (1) is $\Psi-$uniformly stable on $\mathbb{R}_+$.

Theorem 6. Suppose that:
a). the hypothesis (H) is satisfied;
b). the trivial solution of (5) is $\Psi-$uniformly stable on $\mathbb{R}_+$;
c). the matrix function F satisfies the inequality

$$|\ \Psi(\mathrm{t})\mathrm{F(t,Z)}\ | \ \le\ \gamma(\mathrm{t})\ |\ \Psi(\mathrm{t})\mathrm{Z}\ |,$$

for all $t \in \mathbb{R}_+$ and for all $Z \in \mathbb{M}_{n\times n}$, where γ is a continuous nonnegative function on $\mathbb{R}_+$ such that $\int_0^\infty \gamma(t)dt < +\infty$;

d). the matrix function B satisfies the condition $\int_0^\infty |B(t)|dt < +\infty$.

Then, the trivial solution of the equation (1) is $\Psi-$uniformly stable on $\mathbb{R}_+$.

Proof. It is similar to the proof of the above Theorem 5.

Remark. Theorems 5 and 6 generalize Theorem 6 ([2], Chapter III, Section 3).

Theorem 7. Suppose that:

a). the hypothesis (H) is satisfied;

b). there exist a continuous function $\varphi : \mathbb{R}_+ \longrightarrow (0,\infty)$ and a positive constant M such that the fundamental matrices X and Y for the equations (5) and (6) respectively satisfy the condition

$$\int_0^t \varphi(s) \mid (Y^T(t)(Y^T(s))^{-1}) \otimes (\Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)) \mid ds \ \leq \ M, \text{ for all } t \geq 0;$$

c). the matrix function F satisfies the inequality

$$\mid \ \Psi(t)F(t,Z) \ \mid \ \leq \ \gamma(t) \mid \ \Psi(t)Z \ \mid,$$

for all $t \in \mathbb{R}_+$ and for all $Z \in \mathbb{M}_{n\times n}$, where $\gamma : \mathbb{R}_+ \longrightarrow [0,+\infty)$ is a continuous function such that

$$q = \sup_{t\geq 0} \frac{\gamma(t)}{\varphi(t)} \ < \ \frac{1}{nM}.$$

Then, the trivial solution of the system (1) is $\Psi-$stable on $\mathbb{R}_+$.

Proof. From Lemma 5, it follows that if Z(t) is the unique solution of (1) with $Z(t_0) = Z_0$, then, $z(t) = \mathcal{V}ecZ(t)$ is the unique solution of the system (3) with $z(t_0) = z_0 = \mathcal{V}ecZ_0$. Therefore, z(t) is also a solution of the inhomogeneous linear equation

$$z' = \Big(I_n \otimes A(t) + B^T(t) \otimes I_n\Big)\, z + f(t,z(t)),\ t \ \in \ [t_0,t_1),$$

$[t_0,t_1)$ being the existence interval of solution z(t).

If U(t) is a fundamental matrix for the system (7), by the variation of constant formula ([2], Chapter II, section 2(8)),

$$z(t) = U(t)U^{-1}(t_0)z_0 + \int_{t_0}^t U(t)U^{-1}(s)f(s,z(s))ds,\ t \ \in \ [t_0,t_1).$$

From Lemma 8, we replace $U(t) = Y^T(t)\otimes X(t)$, as a fundamental matrix on $\mathbb{R}_+$ for the system (7). After computation, it follows that

$$z(t) = (Y^T(t) \otimes X(t)) \cdot ((Y^T(t_0))^{-1} \otimes X^{-1}(t_0)) \cdot z_0 +$$

$$+ \int_{t_0}^t (Y^T(t) \otimes X(t)) \cdot ((Y^T(s))^{-1} \otimes X^{-1}(s))f(s,z(s))ds,\ t \ \in \ [t_0,t_1). \qquad (15)$$

From hypothesis b) and Theorems 2 and 1, it follows that there exists a positive constant K such that

$$| \ \mathrm{Y^T(t) \otimes (\Psi(t)X(t))} \ | \ \leq \ \mathrm{K},$$

for all $\mathrm{t} \geq 0$.

From (15), it follows that

$$(\mathrm{I_n} \otimes \Psi(\mathrm{t}))\mathrm{z(t)}{=}(\mathrm{Y^T(t)}{\otimes}\Psi(\mathrm{t})\mathrm{X(t)})((\mathrm{Y^T(t_0)})^{-1}{\otimes}\mathrm{X^{-1}(t_0)}\Psi^{-1}(\mathrm{t_0}))(\mathrm{I_n} \otimes \Psi(\mathrm{t_0}))\mathrm{z_0}+$$

$$+\int_{\mathrm{t_0}}^{\mathrm{t}} ((\mathrm{Y^T(t)(Y^T(s))^{-1}}){\otimes}(\Psi(\mathrm{t})\mathrm{X(t)X^{-1}(s)}\Psi^{-1}(\mathrm{s})))(\mathrm{I_n}{\otimes}\Psi(\mathrm{s}))\mathrm{f(s,z(s))ds},\ \mathrm{t} \in [\mathrm{t_0,t_1})$$

and then,

$$\| \ (\mathrm{I_n} \otimes \Psi(\mathrm{t}))\mathrm{z(t)} \ \|_{\mathbb{R}^{n^2}} \leq \mathrm{K}| \ (\mathrm{Y^T(t_0)})^{-1}{\otimes}\mathrm{X^{-1}(t_0)}\Psi^{-1}(\mathrm{t_0}) \ |\| \ (\mathrm{I_n} \otimes \Psi(\mathrm{t_0}))\mathrm{z_0} \ \|_{\mathbb{R}^{n^2}} \ +$$

$$+ \int_{\mathrm{t_0}}^{\mathrm{t}} \varphi(\mathrm{s})| \ (\mathrm{Y^T(t)(Y^T(s))^{-1}}){\otimes}(\Psi(\mathrm{t})\mathrm{X(t)X^{-1}(s)}\Psi^{-1}(\mathrm{s})) \ | \cdot$$

$$\cdot \tfrac{1}{\varphi(\mathrm{s})} \ \| \ (\mathrm{I_n} \otimes \Psi(\mathrm{s}))\mathrm{f(s,z(s))} \ \|_{\mathbb{R}^{n^2}}\mathrm{ds},\ \mathrm{t} \in [\mathrm{t_0,t_1}).$$

From hypothesis c) and Lemma 6, it follows that

$$\| \ \ (\mathrm{I_n} \otimes \Psi(\mathrm{t}))\mathrm{f(t,z)} \ \ \|_{\mathbb{R}^{n^2}} \ \ \leq \ \mathrm{n}\gamma(\mathrm{t}) \ \| \ \ (\mathrm{I_n} \otimes \Psi(\mathrm{t}))\mathrm{z} \ \ \|_{\mathbb{R}^{n^2}},$$

for $\mathrm{t} \in \mathbb{R}_+$ and $\mathrm{z} \in \mathbb{R}^{\mathrm{n}^2}$.

Then, for $\mathrm{t} \in [\mathrm{t_0,t_1})$ we have

$$\| \ (\mathrm{I_n} \otimes \Psi(\mathrm{t}))\mathrm{z(t)} \ \|_{\mathbb{R}^{n^2}} \leq \mathrm{K}| \ (\mathrm{Y^T(t_0)})^{-1}{\otimes}\mathrm{X^{-1}(t_0)}\Psi^{-1}(\mathrm{t_0}) \ |\| \ (\mathrm{I_n} \otimes \Psi(\mathrm{t_0}))\mathrm{z_0} \ \|_{\mathbb{R}^{n^2}} \ +$$

$$+ \int_{\mathrm{t_0}}^{\mathrm{t}} \varphi(\mathrm{s})| \ (\mathrm{Y^T(t)(Y^T(s))^{-1}}){\otimes}(\Psi(\mathrm{t})\mathrm{X(t)X^{-1}(s)}\Psi^{-1}(\mathrm{s})) \ | \cdot$$

$$\cdot \tfrac{\mathrm{n}\gamma(\mathrm{s})}{\varphi(\mathrm{s})} \ \| \ (\mathrm{I_n} \otimes \Psi(\mathrm{s}))\mathrm{z(s)} \ \|_{\mathbb{R}^{n^2}}\mathrm{ds} \leq$$

$$\leq \mathrm{K}| \ (\mathrm{Y^T(t_0)})^{-1}{\otimes}\mathrm{X^{-1}(t_0)}\Psi^{-1}(\mathrm{t_0}) \ |\| \ (\mathrm{I_n} \otimes \Psi(\mathrm{t_0}))\mathrm{z_0} \ \|_{\mathbb{R}^{n^2}} \ +$$

$$+ \int_{\mathrm{t_0}}^{\mathrm{t}} \varphi(\mathrm{s})| \ (\mathrm{Y^T(t)(Y^T(s))^{-1}}){\otimes}(\Psi(\mathrm{t})\mathrm{X(t)X^{-1}(s)}\Psi^{-1}(\mathrm{s})) \ | \cdot$$

$$\cdot \tfrac{\mathrm{n}\gamma(\mathrm{s})}{\varphi(\mathrm{s})} \sup_{\mathrm{t_0} \leq \mathrm{s} \leq \mathrm{t}} \| \ (\mathrm{I_n} \otimes \Psi(\mathrm{s}))\mathrm{z(s)} \ \|_{\mathbb{R}^{n^2}}\mathrm{ds} \leq$$

$$\leq \mathrm{K}| \ (\mathrm{Y^T(t_0)})^{-1}{\otimes}\mathrm{X^{-1}(t_0)}\Psi^{-1}(\mathrm{t_0}) \ |\| \ (\mathrm{I_n} \otimes \Psi(\mathrm{t_0}))\mathrm{z_0} \ \|_{\mathbb{R}^{n^2}} \ +$$

$$+ \ \mathrm{nqM} \sup_{\mathrm{t_0} \leq \mathrm{s} \leq \mathrm{t}} \| \ (\mathrm{I_n} \otimes \Psi(\mathrm{s}))\mathrm{z(s)} \ \|_{\mathbb{R}^{n^2}}$$

and hence

$$\sup_{t_0\le s\le t} \parallel (I_n \otimes \Psi(s))z(s) \parallel_{\mathbb{R}^{n^2}} \le$$

$$\le (1-nqM)^{-1}K|\ (Y^T(t_0))^{-1}\otimes X^{-1}(t_0)\Psi^{-1}(t_0)\ |\| (I_n \otimes \Psi(t_0))z_0 \parallel_{\mathbb{R}^{n^2}} .$$

Thus, for $t \in [t_0,t_1)$ we have

$$\parallel (I_n \otimes \Psi(t))z(t) \parallel_{\mathbb{R}^{n^2}} \le$$

$$\le (1-nqM)^{-1}K|\ (Y^T(t_0))^{-1}\otimes X^{-1}(t_0)\Psi^{-1}(t_0)\ |\| (I_n \otimes \Psi(t_0))z_0 \parallel_{\mathbb{R}^{n^2}} .$$

This inequality shows that $t_1 = +\infty$ and hence, the solution z is defined on $[t_0,+\infty)$.

From the inequality

$$\parallel (I_n \otimes \Psi(t))z(t) \parallel_{\mathbb{R}^{n^2}} \le$$

$$\le (1-nqM)^{-1}K|\ (Y^T(t_0))^{-1}\otimes X^{-1}(t_0)\Psi^{-1}(t_0)\ |\| (I_n \otimes \Psi(t_0))z_0 \parallel_{\mathbb{R}^{n^2}},\ t \ge t_0,$$

it follows that the trivial solution of the system (3) is $I_n \otimes \Psi-$stable on $\mathbb{R}_+$.

From Lemma 7, it follows that the trivial solution of the equation (1) is $\Psi-$stable on $\mathbb{R}_+$.

The proof is now complete.

Theorem 8. Suppose that:

a). the hypothesis (H) is satisfied;

b). there exist a continuous function $\varphi : \mathbb{R}_+ \longrightarrow (0,\infty)$ and a positive constant M such that the fundamental matrix X of the system (5) satisfies the inequality

$$\int_0^t \varphi(s)\ |\ \ \Psi(t)X(t)X^{-1}(s)\Psi^{-1}(s)\ \ |\ ds\ \le\ M,$$

for all $t \ge 0$;

c). the matrix function F satisfies the inequality

$$|\ \ \Psi(t)F(t,Z)\ \ |\ \ \le\ \gamma(t)\ |\ \ \Psi(t)Z\ \ |,$$

for all $t \in \mathbb{R}_+$ and for all $Z \in \mathbb{M}_{n\times n}$, where $\gamma : \mathbb{R}_+ \longrightarrow [0,+\infty)$ is a continuous function such that

$$q = \sup_{t\ge 0} \frac{|\ \ B(t)\ \ |\ +\gamma(t)}{\varphi(t)}\ <\ \frac{1}{nM}.$$

Then, the trivial solution of the equation (1) is $\Psi-$stable on $\mathbb{R}_+$.

Proof. It is similar to the proof of the above Theorem 7.

Acknowledgments. The author would like to thank the anonymous referees for careful reading, valuable comments, suggestions and advice for improvement of previous version of this paper.

References

[1] Bellman, R. $-$ *Introduction to Matrix Analysis,* McGraw-Hill Book Company, Inc. New York 1960 (translated in Romanian).

[2] Coppel, W. A. $-$ *Stability and Asymptotic Behavior of Differential Equations,* Heath, Boston, 1965.

[3] Diamandescu, A. $-$ *Existence of* $\Psi-$*bounded solutions for a system of differential equations,* Electronic Journal of Differential Equations, Vol. 2004(2004), No. 63, pp. $1-6$. URL: http://www.emis.de/journals/EJDE/

[4] Diamandescu, A. $-$ *On the* $\Psi-$*stability of a nonlinear Volterra integro-differential system,* Electronic Journal of Differential Equations, Vol. 2005(2005), No. 56, pp. $1-14$. URL: http://www.emis.de/journals/EJDE/

[5] Diamandescu, A. $-$ $\Psi-$*bounded solutions for a Lyapunov matrix differential equation,* Electronic Journal of Qualitative Theory Differential Equations, 2009, No. 17, $1-11$. URL: http://www.math.u-szeged.hu/ejqtde/

[6] Massera, J.L. and Schäffer, J.J. $-$ *Linear differential equations and functional analysis,* I, Ann. of math. 67(1958) $517-573$.

[7] Murty M.S.N. and Suresh Kumar G. $-$ *On* $\Psi-$*boundedness and* $\Psi-$*stability of matrix Lyapunov systems,* J. Appl. Math. Comput(2008) 26: $67-84$.

[8] Perron, O. $-$ *Die Stabilitätsfrage bei Differentialgleichungen,* Math. Z., 32(1930), $703-728$.

(Received June 14, 2009)

Aurel Diamandescu
University of Craiova,
Department of Applied Mathematics,
13, "Al. I. Cuza" st., 200585 Craiova, Romania.
E - mail address: adiamandescu@central.ucv.ro