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Resonant problem for a class of BVPs on the half-line

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Abstract

We provide an existence result for a Neumann nonlinear boundary value problem posed on the half-line. Our main tool is the multi-valued version of the Miranda Theorem.

1 Introduction

In this paper we study the following BVP

$$x'' = f(t, x, x'), \quad x'(0) = 0, \quad \lim_{t\to\infty} x'(t) = 0,$$

where $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

In recent years, the existence of solutions for boundary-value problems received wide attention. We can write down almost any nonlinear differential equation in the form $Lx = N(x)$, where L is a linear and N nonlinear operator in appropriate Banach spaces. If kernel L is nontrivial then the equation is called resonant and one can manage the problem by using the coincidence degree in that case. But, if the domain is unbounded the operator is usually non-Fredholm (like in our case). Here, for instance, we can use the perturbation method (see [2]) or we can work in the space $H^2(\mathbb{R}_+)$ (see [7]) to obtain the existence results.

The method presented below enables us to get the existence result under weaker assumptions than those mentioned in the above cited methods. Consequently, we will need some facts from set-valued analysis and multi-valued version of the Miranda Theorem to get the Theorem about the existence of at least one solution to the resonant problem.

Now, we shall set up notation and terminology.

Denote by $BC(\mathbb{R}_+, \mathbb{R})$ (we write BC) the Banach space of continuous and bounded functions with supremum norm and by $BCL(\mathbb{R}_+, \mathbb{R})$ (we write BCL) its closed subspace of continuous and bounded functions which have finite limits at $+\infty$.

The following theorem gives a sufficient condition for compactness in the space BC, ([6]).

Theorem 1. *If $K \subset BC$ satisfies following conditions:*
(1) There exists $L > 0$, that for every $x \in K$ and $t \in [0, \infty)$ we have $|x(t)| \leq L$;
(2) for each $t_0 \geq 0$, the family K is equicontinuous at t_0;
(3) for each $\varepsilon > 0$ there exists $T > 0$ and $\delta > 0$ such that if $|x(T) - y(T)| \leq \delta$, then $|x(t) - y(t)| \leq \varepsilon$ for $t \geq T$ and all $x, y \in K$,
then K is relatively compact in BC.

By a space we mean a metric space. Let X, Y be spaces. A set-valued map $\Phi : X \multimap Y$ is upper semicontinuous (written USC) if, given an open $V \subset Y$, the set $\{x \in X \mid \Phi(x) \subset V\}$ is open.

The theorem below is a generalization of the well known Miranda Theorem [5, p. 214] which gives zeros of single-valued continuous maps.

Theorem 2. *([8]) Let g be an USC map from $[-\widehat{M}, \widehat{M}]$ into convex and compact subsets of $\mathbb{R}$ and satisfying the conditions:*

$$\text{if } d \in g(\widehat{M}), \text{ then } d \geq 0 \tag{1}$$

and

$$\text{if } d \in g(-\widehat{M}), \text{ then } d \leq 0. \tag{2}$$

Then there exists $\widetilde{x} \in [-\widehat{M}, \widehat{M}]$ such that $0 \in g(\widetilde{x})$.

The result, we shall present in this paper, is a generalization of the sublinear case considered in [8]. Therefore, the following Lemma will be of crucial importance for our reason.

Lemma 1. *([4])(Bihari's lemma) Let $b : [0, +\infty) \longrightarrow [0, +\infty)$ be a continuous function. Assume that the function $y \in C([0, +\infty), \mathbb{R})$ is non-negative and satisfies the following inequality*

$$y(t) \leq C + \int_0^t b(s)\omega(y(s))\, ds, \quad t \geq 0,$$

where $C \geq 0$ is a real constant and $\omega(s)$ is a continuous, positive and non-decreasing function such that $W(+\infty) = +\infty$, where

$$W(t) = \int_0^t \frac{ds}{\omega(s)}.$$

Then, one has for all $t \geq 0$ that

$$y(t) \leq W^{-1}\Big(W(C) + \int_0^t b(s)ds\Big).$$

2 Resonant problem

Let us consider an asymptotic BVP

$$x'' = f(t, x, x'), \quad x'(0) = 0, \quad \lim_{t\to\infty} x'(t) = 0, \tag{3}$$

where $f : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

The following assumption will be needed in this paper:

(i) $|f(t, x, y)| \leq b(t)\omega(|y|)+c(t)$, where $b, c \in L^1(0, \infty)$, b is continuous and $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nondecreasing function such that $W(+\infty) = +\infty$, where
$$W(t) = \int_0^t \frac{ds}{\omega(s)}.$$

(ii) there exists $M > 0$ such that $xf(t, x, y) \geq 0$ for $t \geq 0$, $y \in \mathbb{R}$ and $|x| \geq M$.

Now, we can formulate our main result.

Theorem 3. *Under assumption **(i)** and **(ii)** problem* (3) *has at least one solution.*

The proof will be divided into a sequence of Lemmas.

First, we consider problem

$$y' = f(t, c + \int_0^t y, y), \quad y(0) = 0, \tag{4}$$

for fixed $c \in \mathbb{R}$. Observe that (4) is equivalent to an initial value problem

$$x'' = f(t, x, x'), \quad x(0) = c, \quad x'(0) = 0. \tag{5}$$

Since f is continuous, then by the Local Existence Theorem we get that problem (5) has at least one local solution. We can write (4) as

$$y_c(t) = \int_0^t f\Big(s, c + \int_0^s y_c(u)du, y_c(s)\Big)ds \tag{6}$$

Set

$$B := \int_0^\infty b(s)ds, \quad C := \int_0^\infty c(s)ds. \tag{7}$$

By (i) and (7) we get

$$|y_c(t)| \leq \int_0^t (b(s)\omega(|y_c(s)|) + c(s))ds \leq C + \int_0^t b(s)\omega(|y_c(s)|)ds. \tag{8}$$

Now, due to Bihari's Lemma (see Lemma 1), we have

$$|y_c(t)| \leq W^{-1}\Big(W(C) + \int_0^t b(s)ds\Big).$$

Hence, by the Theorem on a Priori Bounds [5, p. 146], (5) has a global solution for $t \geq 0$. We obtain that (4) has a global solution for $t \geq 0$. Moreover, by assumption (i) and (7), we have

$$\begin{aligned} |y_c(t)| &\leq W^{-1}\Big(W(C) + \int_0^t b(s)ds\Big) \leq W^{-1}\Big(W(C) + \int_0^\infty b(s)ds\Big) \\ &= W^{-1}\Big(W(C) + B\Big) < \infty \end{aligned} \tag{9}$$

Hence all global solutions are bounded for $t \geq 0$.

The function $t \mapsto f(t, c + \int_0^t y_c(u)du, y_c(t))$ is absolutely integrable; i.e.,

$$\forall_{\varepsilon > 0} \quad \exists_{M > 0} \quad \Big| \int_M^\infty f\Big(t, c + \int_0^t y_c(u)du, y_c(t)\Big)dt\Big| < \varepsilon.$$

In particular, there exists a limit $\lim_{t\to\infty} y_c(t)$, for every c. Thus all solutions of (4) have finite limits at $+\infty$.

Let us consider the nonlinear operator A, $\mathbb{R} \times BCL \ni (c, x) \to A_c(x) \in BCL$, given by

$$A_c(x)(t) = \int_0^t f(s, c + \int_0^s x(u)du, x(s))ds. \tag{10}$$

It is easy to see that A is well defined. By using the Lebesgue Dominated Convergence Theorem one can prove the continuity of A.

Lemma 2. *Under assumption (i) operator A is completely continuous.*

Proof. We shall show that the image of $K := \{(c, x) \in \mathbb{R} \times BCL \mid \|(c, x)\|_{\mathbb{R}\times BC} \le R\}$ under A is relatively compact. Condition (1) of Theorem 1 is satisfied, since $|A_c(x)(t)| \le C + \omega(R)B$.

Now, we prove condition (2). We show that for any $t_0 \ge 0$ and $\varepsilon > 0$ there exists $\delta > 0$ that for each $x \in K$ if $|t - t_0| < \delta$, then $|A_c(x)(t) - A_c(x)(t_0)| < \varepsilon$. Let us choose an arbitrary $\varepsilon > 0$. By (i) there exist δ_1, $\delta_2 > 0$ such that

$$\text{if } |t - t_0| < \delta_1, \text{ then } \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} b(s)ds < \frac{\varepsilon}{2\omega(R)},$$
$$\text{if } |t - t_0| < \delta_2, \text{ then } \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} c(s)ds < \frac{\varepsilon}{2}.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, for $|t - t_0| < \delta$, we get

$$\begin{aligned} |A_c(x)(t) - A_c(x)(t_0)| &\le \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} |f(s, c + \int_0^s x(u)du, x(s))|ds \\ &\le \omega(R) \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} b(s)ds + \int_{\min\{t_0,t\}}^{\max\{t_0,t\}} c(s)ds \\ &< \omega(R)\frac{\varepsilon}{2\omega(R)} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

It remains to prove condition (3). By assumption (i) for every $\varepsilon > 0$ there exist t_1, t_2 large enough that

$$\int_{t_1}^{\infty} b(s)ds < \frac{\varepsilon}{6\omega(R)}, \quad \int_{t_2}^{\infty} c(s)ds < \frac{\varepsilon}{6}.$$

Let $T = \max\{t_1, t_2\}$ and $\delta := \varepsilon/3$. If $|A_c(x)(T) - A_c(y)(T)| \leq \delta$, then for $t \geq T$ we get

$$\begin{aligned}
&|A_c(x)(t) - A_c(y)(t)| \leq \\
&\leq |A_c(x)(T) - A_c(y)(T)| + \int_T^\infty \big|f(s, c + \int_0^s x(u)du, x(s))\big|ds \\
&\quad + \int_T^\infty \big|f(s, c + \int_0^s y(u)du, y(s))\big|ds \\
&\leq |A_c(x)(T) - A_c(y)(T)| + 2\int_T^\infty \omega(R)b(s)ds + 2\int_T^\infty c(s)ds \\
&\leq \frac{\varepsilon}{3} + 2\omega(R)\frac{\varepsilon}{6\omega(R)} + 2\frac{\varepsilon}{6} = \varepsilon.
\end{aligned}$$

The proof is complete. □

Note that the solutions of (4) are fixed points of operator A defined by (10). Let $\operatorname{fix} A_c(\cdot)$ denote the set of fixed points of operator A_c, where c is given. Let us consider the multifunction $g : \mathbb{R} \multimap \mathbb{R}$ given by

$$g(c) := \{\lim_{t\to\infty} y_c(t) : y_c \in \operatorname{fix} A_c(\cdot)\}. \tag{11}$$

Now, let $\varphi : BCL \to \mathbb{R}$ be a single-valued function such that

$$\varphi(y_c) = \lim_{t\to\infty} y_c(t). \tag{12}$$

It is easily seen that function φ is continuous.

Set

$$\Phi : \mathbb{R} \ni c \multimap \mathrm{FixA_c}(\cdot) \subset \mathrm{BCL} \tag{13}$$

and notice that $g = \varphi \circ \Phi$.

Lemma 3. *Let assumption **(i)** holds. Then the set-valued map Φ is USC with compact values.*

Proof. The set-valued map Φ is upper semicontinuous with compact values if given a sequence (c_n) in $\mathbb{R}^k$, $c_n \to c_0$ and $(x_n) \in \Phi(c_n)$, (x_n) has a convergent subsequence to some $x_0 \in \Phi(x_0)$. Taking any sequence (c_n), $c_n \to c_0$ and $(x_n) \in \Phi(c_n)$ we have

$$x_n = A_{c_n}(x_n). \tag{14}$$

By (9), we get that the solutions of (4) are equibounded for any c. Hence both sequences (x_n) and (c_n) are bounded. Proposition 2 yields that operator A is completely continuous. Then, by (14), (x_n) is relatively compact. Hence, passing to a subsequence if necessary, we may assume that $x_n \to x_0$ in BCL. The continuity of A implies that

$$x_0 = A_{c_0}(x_0).$$

Hence, $x_0 \in \Phi(c_0)$ and the proof is complete. □

Lemma 4. *Let assumption (i) hold. Then the set-valued map Φ has connected values.*

Proof. We shall show that $\operatorname{fix} A_c(\cdot)$ is connected in BCL. On the contrary, suppose that the set is not connected. Since $\operatorname{fix} A_c(\cdot)$ is compact, there exist compact sets A and B such that $A, B \neq \emptyset$, $A \cap B = \emptyset$ and $A \cup B = \operatorname{fix} F(c, \cdot)$. Let $\varepsilon := \operatorname{dist}(A, B)$, $\varepsilon > 0$. Then

$$\forall_{y \in A, z \in B} \quad \|y - z\| \geq \varepsilon. \tag{15}$$

By (9) there exists $T > 0$ such that for any $y \in \operatorname{fix} A_c(\cdot)$ we get

$$\int_T^\infty \left|f\left(t, c + \int_0^t y(u)du, y(t)\right)\right| dt < \frac{1}{3}\varepsilon. \tag{16}$$

Let $y \in A$, i $z \in B$. Now, consider the functions y and z cut to the compact set $[0, T]$ and set $y|_{[0,T]}$ and $z|_{[0,T]}$. By Kneser's Theorem [3, p. 413], the set $\operatorname{fix} A_c(\cdot)|_{[0,T]}$ is connected in $C([0, T], \mathbb{R})$. From this, there exist $x_1, \ldots, x_k$ in $\operatorname{fix} A_c(\cdot)|_{[0,T]}$ such that $x_1 = y\,|_{[0,T]}$, $x_k = z\,|_{[0,T]}$ and

$$\|x_i - x_{i+1}\|_{C([0,T],\mathbb{R})} < \frac{1}{3}\varepsilon. \tag{17}$$

Hence, at least two sequel x_i, x_{i+1} in $\operatorname{fix} A_c(\cdot)|_{[0,T]}$ are such that $x_i \in A$ i $x_{i+1} \in B$. Moreover, $x_i, x_{i+1} \in \operatorname{fix} A_c(\cdot)$. By (15), (16) and (17) we get a contradiction. Indeed

$$\begin{aligned}
\varepsilon &\leq \|x_i - x_{i+1}\|_{BCL} \\
&\leq \max\{\|x_i - x_{i+1}\|_{C([0,T],\mathbb{R})}, \sup_{t \geq T} |x_i(t) - x_{i+1}(t)|\} \\
&\leq \|x_i - x_{i+1}\|_{C([0,T],\mathbb{R})} + \sum_{j=i}^{i+1} \int_T^\infty |f(t, c + \int_0^t x_j(u)du, x_j(t))| dt \\
&< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon.
\end{aligned}$$

Hence, $\operatorname{fix} A_c(\cdot)$ is connected in BCL, which proves the Lemma. $\square$

Lemma 5. *Let assumption (i) hold. Then the multifunction g is USC map from $\mathbb{R}$ into its compact intervals.*

Proof. By Proposition 4 we know that Φ has compact and connected values. Now, by continuity of φ, the set

$$\{\lim_{t\to\infty} y_c(t) \mid y_c \in \operatorname{fix} A_c(\cdot)\}$$

is compact and connected subset of $\mathbb{R}$ for every c. From this, we conclude that multifunction $g(c) = (\varphi \circ \Phi)(c)$ maps $\mathbb{R}$ into its compact intervals. Moreover, g is USC as a superposition of set-valued map with compact values and continuous function [1, p. 47]. $\square$

By the above Lemmas, we get the proof of Theorem 3.

The proof of Theorem 3.
Let $y_c \in \operatorname{fix} A_c(\cdot)$ be the bounded global solution of (4) and g be given by (11).

Observe that $x(t) = c + \int_0^t y_c(s)ds$ is a solution of (3) if there exists an $c \in \mathbb{R}$ such that $0 \in g(c)$.

We shall show that g satisfies assumptions of the multi-valued version of Miranda Theorem (see Theorem 2).

By Lemma 5 we get that g is USC map from $\mathbb{R}$ into its compact (so convex) intervals.

Let $c = M + 1$, where M is as in (ii). We will show that $y_c(t) \geq 0$ for $t \geq 0$ and all $y_c \in \operatorname{fix} A_c(\cdot)$.

By (4) we have $y_c(0) = 0$. Assume that for some t and $y_c \in \operatorname{fix} A_c(\cdot)$ we have $y_c(t) < 0$. Then there exist $t_* := \inf\{t \mid y_c(t) < 0\}$ such that $y_c(t_*) = 0$ and $y_c(t) \geq 0$ for $t < t_*$ (if $t_* \neq 0$). By continuity of $y_c(t)$ there exists $t_1 > t_*$ such that $\int_{t_*}^{t_1} |y_c(t)|dt \leq 1$. Hence, we get

$$x(t) = c + \int_{t_*}^{t} y_c(s)ds = M + 1 + \int_{t_*}^{t} y_c(s)ds \geq M \quad \text{for} \quad t \in [t_*, t_1]\,.$$

Now, by (ii) we have

$$x(t)f(t, x(t), y(t)) = x(t)y_c'(t) \geq 0.$$

Hence $y_c'(t) \geq 0$ for $t \in [t_*, t_1]$. It means that $y_c(t)$ is nondecreasing on $[t_*, t_1]$. Since $y_c(t_*) = 0$ we get a contradiction. Hence $y_c(t) \geq 0$ for $t \geq 0$. In consequence, if $d \in g(c) = \lim_{t\to\infty} y_c(t)$, then $d \geq 0$. Hence, the condition (1) is satisfied for $\widehat{M} := M + 1$

To prove that if $d \in g(-\widehat{M})$, then $d \leq 0$ we proceed analogously. Hence, by multi-valued version of Miranda Theorem, there exists an $\tilde{c} \in [-\widehat{M}, \widehat{M}]$ such that, $0 \in g(\tilde{c})$. This completes the proof.

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