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Global existence and blow-up analysis for some degenerate and quasilinear parabolic systems [1]

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Abstract. This paper deals with positive solutions of some degenerate and quasilinear parabolic systems not in divergence form: $u_{1t} = f_1(u_2)(\Delta u_1 + a_1u_1), \cdots, u_{(n-1)t} = f_{n-1}(u_n)(\Delta u_{n-1} + a_{n-1}u_{n-1})$, $u_{nt} = f_n(u_1)(\Delta u_n + a_nu_n)$ with homogenous Dirichlet boundary condition and positive initial condition, where a_i $(i = 1, 2, \cdots, n)$ are positive constants and f_i $(i = 1, 2, \cdots, n)$ satisfy some conditions. The local existence and uniqueness of classical solution are proved. Moreover, it will be proved that: (i) when $\min\{a_1, \cdots, a_n\} \le \lambda_1$ then there exists global positive classical solution, and all positive classical solutions can not blow up in finite time in the meaning of maximum norm; (ii) when $\min\{a_1, \cdots, a_n\} > \lambda_1$, and the initial datum $(u_{10}, \cdots, u_{n0})$ satisfies some assumptions, then the positive classical solution is unique and blows up in finite time, where λ_1 is the first eigenvalue of $-\Delta$ in Ω with homogeneous Dirichlet boundary condition.

Key words: quasilinear parabolic system; global existence; blow up in finite time; not in divergence form.

1 Introduction and main result

In this paper, we consider the following degenerate and quasilinear parabolic systems not in divergence form:

$$\begin{cases} u_{it} = f_i(u_{i+1})(\Delta u_i + a_iu_i), & x \in \Omega, \quad t > 0, \quad i = 1, 2, \cdots, n-1, \\ u_{nt} = f_n(u_1)(\Delta u_n + a_nu_n), & x \in \Omega, \quad t > 0, \\ u_i(x, 0) = u_{i0}(x), & x \in \Omega, \quad i = 1, 2, \cdots, n, \\ u_i(x, t) = 0, & x \in \partial\Omega, \quad t > 0, \ i = 1, 2, \cdots, n, \end{cases} \tag{1.1}$$

where a_i are positive constants, f_i, $u_{i0}(x)$, $i = 1, 2, \cdots, n$, satisfy

(H1) $u_{i0}(x) \in C^1(\bar{\Omega}), u_{i0}(x) > 0$ in Ω, ;

(H2) $u_{i0}(x) = 0$ and $\frac{\partial u_{i0}}{\partial \eta} < 0$ on $\partial\Omega$, where η is the outward normal vector on $\partial\Omega$;

(H3) $f_i \in C^1([0, \infty))$, such that $f_i > 0$ and $f_i' \ge 0$ on $[0, \infty)$;

(H4) there exists $1 \le j \le n$, such that $\liminf\limits_{s\to\infty} \frac{f_i(s)}{f_j(s)} > 0$, $i = 1, \cdots, n$.

This system can be used to describe the development of n groups in the dynamics of biological groups, where u_i are the densities of the different groups.

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It has been shown that classical positive solutions of parabolic problems of single equation

$$\begin{cases} u_t = u^p(\Delta u + u), & x \in \Omega, \quad t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, \; t > 0, \\ u(x,0) = u_0(x), & x \in \Omega \end{cases} \tag{1.2}$$

blow up at some finite time $T < \infty$ if $p \geq 1$ and the bounded smooth domain Ω is large enough such that $\lambda_1 < 1$, where λ_1 is the first eigenvalue of $-\Delta$ in Ω with homogeneous Dirichlet boundary condition. (See [1, 4, 5, 7, 8, 9, 14, 15]).

For the case $f_1(v) = v^p$, $f_2(u) = u^q$ $(p,\ q \geq 1)$, (1.1) has been discussed by many authors, see [3, 12] and the references therein. In [12], Wang and Xie discussed the following system

$$\begin{cases} u_t = v^p(\Delta u + a_1 u), & \\ v_t = u^q(\Delta v + a_2 v), & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, \; t > 0. \end{cases} \tag{1.3}$$

They proved that: (i) when $\min\{a,b\} \leq \lambda_1$ then there exists global positive classical solution, and all positive classical solutions can not blow up in finite time in the meaning of maximum norm; (ii) when $\min\{a,b\} > \lambda_1$, there is no global positive classical solution. And if in addition the initial datum (u_0, v_0) satisfies some assumptions then the positive classical solution is unique and blows up in finite time.

In [13], Wang also considered the problem

$$\begin{cases} u_t = u^p(\Delta u + a_1 v), & \\ v_t = v^q(\Delta v + a_2 u), & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, \; t > 0 \end{cases} \tag{1.4}$$

with $p, q > 0$. And he shown that all positive solutions of problem (1.4) exist globally if and only if $a_1 a_2 \leq \lambda_1^2$.

In [2], Deng and Xie promoted the problem (1.4) to the following problem

$$\begin{cases} u_t = f_1(u)(\Delta u + a_1 v), & \\ v_t = f_2(v)(\Delta v + a_2 v), & x \in \Omega, \quad t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \\ u(x,t) = v(x,t) = 0, & x \in \partial\Omega, \; t > 0. \end{cases} \tag{1.5}$$

Under the proper assumptions, they proved that: the positive solution of (1.5) blows up in finite time if and only if $\lambda_1^2 < a_1 a_2$ and $\int_0^\infty \mathrm{d}s/(sf_i(s)) < \infty$ for $i = 1, 2$.

Remark 1. *Without loss of generality, by assumptions (H3) and (H4), we may assume that $j = n$. And for any given $\delta > 0$, there exists a constant $K_0 > 0$, such that*

$$f_i(s) \geq K_0 f_n(s), \quad s \geq \delta, \ i = 1, 2, \cdots, n-1. \tag{1.6}$$

Indeed, let $K_1 = \min\limits_{1 \leq i \leq n-1} \{ \liminf\limits_{s \to \infty} \frac{f_i(s)}{f_n(s)} \}$, then $K_1 > 0$. Set $K^* = \frac{K_1}{2}$ when $K_1 < \infty$ and $K^* = 1$ when $K_1 = \infty$. It is obvious that there exists a constant $s_0 > 0$ such that

$$\frac{f_i(s)}{f_n(s)} \geq K^*, \quad i = 1, 2, \cdots, n-1, \ s > s_0.$$

Furthermore, by (H3) and (H4), denoting $K_2 = \min\limits_{1 \leq i \leq n-1} \min\limits_{\delta \leq s \leq s_0} \frac{f_i(s)}{f_n(s)}$, we have $0 < K_2 < \infty$. Hence, let $K_0 = min\{K^*, K_2\}$, which shows (1.6).

Remark 2. *By change the order of i, we may assume $a_1 \leq a_2 \leq \cdots \leq a_n$.*

Remark 3. *If $n = 2$, $f_1(s) = s^p$ and $f_2(s) = s^q$, where $p, q \geq 1$, then the assumptions (H3) and (H4) hold automatically. So our present results develop the work of [12].*

This paper is organized as follows. In section 2, we first give a Maximum Principle for degenerate parabolic systems with unbounded coefficients, which is useful in the following arguments, and the local existence of positive classical solution is proved. In sections 3 and 4, we discuss the cases $\min\{a_1, \cdots, a_n\} \leq \lambda_1$ and $\min\{a_1, \cdots, a_n\} > \lambda_1$, respectively. By above Remarks, we may assume $f_i(s) \geq f_n(s), i = 1, 2, \cdots, n-1$, and $a_1 \leq a_2 \leq \cdots \leq a_n$ throughout this paper.

2 Local existence

We first give a maximum principle, the proof is standard and we omit it (see [12]).

Proposition 1. *Let $a_i(x,t)$, $b_i(x,t)$, $c_i(x,t)$, $i = 1, 2 \cdots, n$, be continuous functions in $\Omega \times (0,T)$. Assume that $a_i(x,t), c_i(x,t) \geq 0$ in $\Omega \times (0,T)$ and $b_i(x,t)$, $c_i(x,t)$ are bounded on $\bar{\Omega} \times [0, T_0]$ for any $T_0 < T$. If functions $u_i \in C^{2,1}(\Omega \times (0,T)) \cap C(\bar{\Omega} \times [0,T])$, $i = 1, 2, \cdots, n$, and satisfy*

$$\begin{cases} u_{it} \leq (\geq) a_i \Delta u_i + b_i u_i + c_i u_{i+1}, & i = 1, \cdots, n-1, \\ u_{nt} \leq (\geq) a_n \Delta u_n + b_n u_n + c_n u_1, & x \in \Omega, \ 0 < t < T, \\ u_i(x,0) \leq (\geq) 0, & x \in \Omega, \\ u_i(x,t) \leq (\geq) 0, & x \in \partial\Omega, \ 0 < t < T, \end{cases} \tag{2.1}$$

then

$$u_i(x,t) \leq (\geq) 0, \quad \forall (x,t) \in \bar{\Omega} \times [0,T), \quad i = 1, 2, \cdots, n.$$

Since $u_i = 0$ $(i = 1, 2, \cdots, n)$ on the boundary $\partial\Omega$, the equation of (1.1) is not strictly parabolic type. The standard parabolic theory cannot be used directly to prove the local existence of solution to problem (1.1). To overcome this difficultly we will modify both different equations and boundary conditions. For $\varepsilon > 0$, we consider the following approximate problem

$$\begin{cases} (u_{i\varepsilon})_t = g_{i\varepsilon}(u_{(i+1)\varepsilon})(\Delta u_{i\varepsilon} + a_i u_{i\varepsilon} - a_i\varepsilon), & i = 1, \cdots, n-1, \\ (u_{n\varepsilon})_t = g_{n\varepsilon}(u_{1\varepsilon})(\Delta u_{n\varepsilon} + a_n u_{n\varepsilon} - a_n\varepsilon), & x \in \Omega, \quad t > 0, \\ u_{i\varepsilon}(x, 0) = u_{i0}(x) + \varepsilon, & x \in \Omega, \\ u_{i\varepsilon}(x, t) = \varepsilon, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{2.2}$$

where

$$g_{i\varepsilon}(u_{i+1}) = \begin{cases} f_i(u_{i+1}), & u_{i+1} \geq \varepsilon,\ i = 1, \cdots, n-1, \\ f_n(u_1), & u_1 \geq \varepsilon, \\ f_i(\frac{\varepsilon}{2}), & u_{i+1} < \varepsilon, \quad i = 1, \cdots,\ n-1, \\ f_n(\frac{\varepsilon}{2}), & u_1 < \varepsilon. \end{cases}$$

By the standard parabolic theory, it is easy to prove that $u_{i\varepsilon} \geq \varepsilon,\ i = 1, \cdots,\ n$. The fact that $u_{i\varepsilon} \geq \varepsilon$ shows $g_{i\varepsilon} = f_i,\ i = 1,\ 2, \cdots,\ n$, and hence $(u_{1\varepsilon},\ u_{2\varepsilon}, \cdots,\ u_{n\varepsilon})$ solves the following problem

$$\begin{cases} (u_{i\varepsilon})_t = f_i(u_{(i+1)\varepsilon})(\Delta u_{i\varepsilon} + a_i u_{i\varepsilon} - a_i\varepsilon), & i = 1, \cdots, n-1, \\ (u_{n\varepsilon})_t = f_n(u_{1\varepsilon})(\Delta u_{n\varepsilon} + a_n u_{n\varepsilon} - a_n\varepsilon), & x \in \Omega, \quad t > 0, \\ u_{i\varepsilon}(x, 0) = u_{i0}(x) + \varepsilon, & x \in \Omega, \\ u_{i\varepsilon}(x, t) = \varepsilon, & x \in \partial\Omega, \quad t > 0, \end{cases} \tag{2.3}$$

where $\varepsilon \in (0, 1]$. By the classical parabolic theory, under hypothesis (H1)-(H4), (2.3) admits a unique positive solution $(u_{1\varepsilon}, \cdots,\ u_{n\varepsilon}) \in [C(\bar{\Omega}_T) \cap C^{2,1}(\Omega_T)]^n$ for $0 < T < T(\varepsilon)$, where $T(\varepsilon)$ is the maximal existence time.

Now, estimate the lower and upper bounds of $(u_{1\varepsilon},\ u_{2\varepsilon}, \cdots,\ u_{n\varepsilon})$.

Let $\lambda_1, \varphi(x) > 0$ be the first eigenvalue and the corresponding eigenfunction of the following eigenvalue problem

$$-\Delta\varphi = \lambda\varphi \text{ in } \Omega; \quad \varphi = 0 \text{ on } \partial\Omega \tag{2.4}$$

and think that $\max\limits_{\bar{\Omega}} \varphi(x) = 1$, then $\lambda_1 > 0$ and $\frac{\partial\varphi}{\partial\eta} < 0$ on $\partial\Omega$, where η is the outward normal vector on $\partial\Omega$. By (H_1) and (H_2), there exist positive constants k_1 and k_2 such that

$$k_1\varphi(x) \leq u_{i0}(x) \leq k_2\varphi(x), \quad x \in \bar{\Omega},\ i = 1, 2, \cdots, n. \tag{2.5}$$

Let $M = \max\limits_{1 \leq i \leq n} \max\limits_{\bar{\Omega}} u_{i0}(x)$ and $(g_1(t), \cdots,\ g_n(t))$ be the unique solution of the following ODE

$$\begin{cases} g_i' = a_i g_i f_i(g_{i+1}), & i = 1, \cdots, n-1, \\ g_n' = a_n g_n f_n(g_1), & t > 0, \\ g_i(0) = M + 1, & i = 1, 2, \cdots, n, \end{cases} \tag{2.6}$$

where $f_1, \cdots, f_n$ are given as above. Then $g_i(t) \geq M+1,\ i = 1,\ 2, \cdots,\ n$. Denote by $T^*, 0 < T^* < \infty$, its maximum existence time (note that $T^* < \infty$ must hold because that $(g_1, \cdots, g_n)$ blow up in finite time).

Applying the standard comparison principle for parabolic system, we have the following Lemmas (see [13, 14]).

Lemma 1. *Assume that* $u_i(x,t) \in C\Big(\bar{\Omega} \times [0, T(\varepsilon))\Big) \cap C^{2,1}\Big(\Omega \times (0, T(\varepsilon))\Big) (i = 1, \cdots,\ n)$ *is a lower (or upper) solution of* (2.3)*, then* $(u_1, \cdots,\ u_n) \leq (\geq)(u_{1\varepsilon}, \cdots,\ u_{n\varepsilon})$ *on* $\bar{\Omega} \times [0, T(\varepsilon))$.

Lemma 2. *If* $\varepsilon_1 < \varepsilon_2$*, then* $(u_{1\varepsilon_1}(x,t), \cdots,\ u_{n\varepsilon_1}(x,t)) \leq (u_{1\varepsilon_2}(x,t), \cdots,\ u_{n\varepsilon_2}(x,t))$ *on* $\bar{\Omega} \times [0, T(\varepsilon_2))$ *and* $T(\varepsilon_1) \geq T(\varepsilon_2)$.

Lemma 3. *Let* $\varepsilon < 1,\ (u_{1\varepsilon}, \cdots,\ u_{n\varepsilon})$ *be the solution of* (2.3), then for any fixed $T:\ 0 < T < \min\{T(\varepsilon), T^*\}$,

$$u_{i\varepsilon}(x,t) \leq g_i(t), \qquad \forall\ (x,t) \in \bar{\Omega} \times [0, T],\ i = 1, 2, \cdots, n,$$

which implies that $T(\varepsilon) \geq T^*$ for all $\varepsilon < 1$.

Proof. Set $w_i(x,t) = g_i(t) - u_{i\varepsilon}(x,t)$, then we have

$$\begin{aligned}
w_{1t} &= g_1' - (u_{1\varepsilon})_t \\
&= a_1 g_1 f_1(g_2) - f_1(u_{2\varepsilon})(\Delta u_{1\varepsilon} + a_1 u_{1\varepsilon} - a_1\varepsilon) \\
&= a_1 g_1 f_1(g_2) - f_1(u_{2\varepsilon})(-\Delta w_1 + a_1 g_1 - a_1 w_1 - a_1 \varepsilon) \\
&= f_1(u_{2\varepsilon})(\Delta w_1 + a_1 w_1) + a_1 g_1 (f_1(g_2) - f_1(u_{2\varepsilon})) + a_1 \varepsilon f_1(u_{2\varepsilon}) \\
&> f_1(u_{2\varepsilon})(\Delta w_1 + a_1 w_1) + a_1 g_1 \Big(\int_0^1 f_1'(u_{2\varepsilon} + s(g_2 - u_{2\varepsilon}))\mathrm{d}s \Big) w_2, \\
& \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \\
w_{(n-1)t} &> f_{n-1}(u_{n\varepsilon})(\Delta w_{n-1} + a_{n-1} w_{n-1}) + a_{n-1} g_{n-1} \Big(\int_0^1 f_{n-1}'(u_{n\varepsilon} + s(g_n - u_{n\varepsilon}))\mathrm{d}s \Big) w_n, \\
w_{nt} &> f_n(u_1\varepsilon)(\Delta w_n + a_n w_n) + a_n g_n \Big(\int_0^1 f_n'(u_{1\varepsilon} + s(g_1 - u_{1\varepsilon}))\mathrm{d}s \Big) w_1, \quad x \in \Omega,\ 0 < t \leq T, \\
w_i(x,0) &= M + 1 - u_{i0} - \varepsilon > 0, \quad x \in \Omega,\ \ i = 1, 2, \cdots, n, \\
w_i(x,t) &= g_i(t) - \varepsilon > 0, \quad x \in \partial\Omega,\ \ 0 < t \leq T,\ \ i = 1, 2, \cdots, n.
\end{aligned}$$

Proposition 1 implies that $w_i \geq 0$ $(i = 1, 2, \cdots, n)$ and hence the result of this Lemma holds. □

In the following we denote $T_* = T^*/2$.

Lemma 4. *Let* $\varepsilon < 1,\ (u_{1\varepsilon},\ \cdots u_{n\varepsilon})$ *be the solution of* (2.3), and the positive constant k_1 satisfies (2.5), then we have the following estimates:
(i) if $\lambda_1 \leq a_1 \leq \cdots \leq a_n$ then

$$u_{i\varepsilon}(x,t) \geq k_1 \varphi(x) + \varepsilon, \quad \forall (x,t) \in \bar{\Omega} \times [0, T(\varepsilon)),\ i = 1,\ 2, \cdots,\ n;$$

(ii) if there exists $j:\ 1 \le j \le n-1$, such that $a_j < \lambda_1 \le a_{j+1}$, then we have, for $\rho = \max\{(\lambda_1 - a_1)f_1(g_2(T_*)), \cdots, (\lambda_1 - a_j)f_j(g_{j+1}(T_*)\}$,

$$u_{i\varepsilon}(x,t) \ge \begin{cases} k_1\varphi(x)e^{-\rho t} + \varepsilon, & 1 \le i \le j,\ \forall (x,t) \in \bar{\Omega} \times [0, T_*], \\ k_1\varphi(x) + \varepsilon, & j+1 \le i \le n,\ \forall (x,t) \in \bar{\Omega} \times [0, T_*]; \end{cases}$$

(iii) if $a_1 \le \cdots \le a_n < \lambda_1$, then we have, for $r = \max\big\{(\lambda_1 - a_1)f_1(g_2(T_*)), \cdots, (\lambda_1 - a_{n-1})f_{n-1}(g_n(T_*)),\ (\lambda_1 - a_n)f_n(g_1(T_*))\big\}$,

$$u_{i\varepsilon}(x,t) \ge k_1\varphi(x)e^{-rt} + \varepsilon, \quad \forall (x,t) \in \bar{\Omega} \times [0, T_*],\ i = 1, 2 \cdots,\ n.$$

Proof. (i) when $\lambda_1 \le a_1 \le \cdots \le a_n$, set

$$w_i(x,t) = u_{i\varepsilon} - (k_1\varphi(x) + \varepsilon), \quad i = 1,\ 2, \cdots,\ n.$$

Then we have, by (2.3),

$$\begin{aligned} w_{1t} &= u_{1\varepsilon t} = f_1(u_{2\varepsilon})(\Delta u_{1\varepsilon} + a_1 u_{1\varepsilon} - a_1\varepsilon) \\ &= f_1(u_{2\varepsilon})(\Delta w_1 + k_1\Delta\varphi + a_1 w_1 + a_1 k_1 \varphi) \\ &= f_1(u_{2\varepsilon})(\Delta w_1 + a_1 w_1) + (a_1 - \lambda_1)k_1\varphi f_1(u_{2\varepsilon}), \\ &\ge f_1(u_{2\varepsilon})(\Delta w_1 + a_1 w_1), \quad x \in \Omega,\ 0 < t < T(\varepsilon), \\ w_1(x,0) &= u_{10}(x) - k_1\varphi(x) \ge 0, \quad x \in \Omega, \\ w_1(x,t) &= 0, \quad x \in \partial\Omega,\ 0 < t < T(\varepsilon). \end{aligned}$$

Applying Proposition 1 we see that $w_1(x,t) \ge 0$, i.e. $u_{1\varepsilon} \ge k_1\varphi + \varepsilon$. Similarly we have $u_{i\varepsilon} \ge k_1\varphi + \varepsilon,\ i = 2, \cdots,\ n$.

(ii) when there exists $j:\ 1 \le j \le n-1$, such that $a_j < \lambda_1 \le a_{j+1}$, set

$$w_i(x,t) = u_{i\varepsilon} - (k_1\varphi(x)e^{-\rho t} + \varepsilon),\ 1 \le i \le j.$$

A routine calculation yields

$$\begin{aligned} w_{1t} &= u_{1\varepsilon t} + \rho k_1 \varphi e^{-\rho t} = f_1(u_{2\varepsilon})(\Delta u_{1\varepsilon} + a_1 u_{1\varepsilon} - a_1\varepsilon) + \rho k_1\varphi e^{-\rho t} \\ &= f_1(u_{2\varepsilon})(\Delta w_1 + k_1\Delta\varphi e^{-\rho t} + a_1 w_1 + a_1 k_1\varphi e^{-\rho t}) + \rho k_1 \varphi e^{-\rho t} \\ &= f_1(u_{2\varepsilon})(\Delta w_1 + a_1 w_1) + (a_1 - \lambda_1) f_1(u_{2\varepsilon}) k_1 \varphi e^{-\rho t} + \rho k_1 \varphi e^{-\rho t} \\ &= f_1(u_{2\varepsilon})(\Delta w_1 + a_1 w_1) + k_1\varphi\big((a_1 - \lambda_1)f_1(u_{2\varepsilon}) + \rho\big)e^{-\rho t}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ w_{jt} &= f_j(u_{(j+1)\varepsilon})(\Delta w_j + a_j w_j) + k_1\varphi\big((a_j - \lambda_1)f_j(u_{(j+1)\varepsilon}) + \rho\big)e^{-\rho t}. \end{aligned}$$

Since $u_{i\varepsilon}(x,t) \le g_i(t) \le g_i(T_*),\ 1 \le i \le n$ for all $(x,t) \in \bar{\Omega} \times [0, T_*]$, it follows that

$$(a_i - \lambda_1)f_i(u_{(i+1)\varepsilon}) + \rho \ge \rho - (\lambda_1 - a_i)f_i(g_{i+1}(T_*)) \ge 0,\ 1 \le i \le j.$$

Therefore

$$w_{it} \ge f_i(u_{(i+1)\varepsilon})(\Delta w_i + a_i w_i),\ 1 \le i \le j.$$

It is obvious that

$$w_i(x,0) = u_{i0}(x) - k_1\varphi(x) \geq 0, \quad x \in \Omega,\ i = 1, 2, \cdots, j,$$
$$w_i(x,t) = 0, \qquad x \in \partial\Omega,\ 0 < t \leq T_*,\ i = 1,\ 2, \cdots,\ j.$$

Proposition 1 asserts that $w_i \geq 0,\ 1 \leq i \leq j$, i.e. $u_{i\varepsilon} \geq k_1\varphi(x)e^{-\rho t} + \varepsilon,\ 1 \leq i \leq j$. The proof of the second one is the same as that of (i).

(iii) when $a_1 \leq \cdots \leq a_n < \lambda_1$, the proof is the same as that of (ii). The proof is completed. □

It follows from Lemma 2 that $T_1 := T(1) < T(\varepsilon)$ for all $\varepsilon \in (0,\ 1)$, and there exists $u_i(x,t) \in L^\infty(\Omega \times (0, T_1))$ such that $(u_{1\varepsilon}, \cdots, u_{n\varepsilon}) \to (u_1, \cdots, u_n)$ as $\varepsilon \to 0^+$. Furthermore, making use of the fact that on any smooth sub-domain $\Omega' \subset\subset \Omega$, by Lemma 4 we obtain that $u_i(x,t)$ $(i = 1, 2, \cdots, n)$ has a positive lower bound in $\Omega' \times [0,\ T_1]$ which is independent of ε.

From the fact just proved it follows in turn by the interior parabolic Hölder estimates, which can obtained by the same argument as that of Theorem VII 3.1 in [6] that for each $\tau > 0$ and $\Omega' \subset\subset \Omega$, there is $\alpha > 0$ such that

$$\|u_i\|_{C^{\alpha,\ \alpha/2}(\Omega' \times [\tau,\ T_1])} \leq C(\Omega',\ \tau)$$

and then the local Hölder continuity of f_i $(i = 1, 2, \cdots, n)$ on $(0,\ \infty)$ together with Schauder estimates and diagonal methods we have that there exist subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ and $u_i \in C_{loc}^{2+\alpha,\ 1+\alpha/2}(\Omega \times (0, \tilde{T}])$, where $\tilde{T} := \min\{T_*,\ T_1\}$, such that

$$(u_{1\varepsilon'}, \cdots, u_{n\varepsilon'}) \longrightarrow (u_1, \cdots,\ u_n) \ \ in \ \ [C^{2+\alpha,\ 1+\alpha/2}(\bar{\Omega}' \times [\tau,\ \tilde{T}])]^n \ \ as \ \ \varepsilon' \to 0^+$$

for any $\Omega' \subset\subset \Omega$ and $0 < \tau < \tilde{T}$. And hence $(u_1, \cdots,\ u_n)$ satisfies the problem (1.1).

Fix $\varepsilon_0 :\ 0 < \varepsilon_0 \ll 1$. For any $\Omega' \subset\subset \Omega$ and $0 < \varepsilon' < \varepsilon_0$, thanks to Lemma 4 and

$$u_{i\varepsilon'}(x,t) \leq g_i(t) \ \ on \ \bar{\Omega}' \times [0,\ \tilde{T}],$$

the L^p theory and imbedding theorem show that the $C^{\alpha,\ \alpha/2}(\bar{\Omega}' \times [0,\ \tilde{T}])$ norms of $u_{i\varepsilon'}$ $(i = 1, 2, \cdots, n)$ are uniformly bounded for all $\varepsilon' < \varepsilon_0$. And hence

$$(u_{1\varepsilon'}, \cdots,\ u_{n\varepsilon'}) \longrightarrow (u_1, \cdots,\ u_n) \ in \ [C^{\beta,\ \beta/2}(\bar{\Omega}' \times [0,\ \tilde{T}])]^n \ (0 < \beta < \alpha) \ as \ \varepsilon' \to 0^+,$$

which implies that $u_i \in C(\Omega \times [0,\ \tilde{T}]), i = 1, 2, \cdots, n$. Similar to the arguments of [4, 15] we can prove that $(u_1, \cdots, u_n)$ is continuous on $\partial\Omega \times (0, \tilde{T}]$. Using the initial and boundary condition of (2.3) we see that $(u_1, \cdots, u_n)$ satisfies the initial and boundary conditions of (1.1), i.e.

$$(u_1, \cdots, u_n) \in \Big[C_{loc}^{2+\beta, 1+\beta/2}(\Omega \times (0, \tilde{T}]) \cap C(\bar{\Omega} \times [0, \tilde{T}])\Big]^n$$

is a classical solution of (1.1).

Theorem 1. *Problem (1.1) has a positive classical solution* $(u_1, \cdots,\ u_n) \in \Big[C_{loc}^{2+\beta,1+\beta/2}(\Omega \times (0, T]) \cap C(\bar{\Omega} \times [0, T])\Big]^n$ *for some* $\beta : 0 < \beta < 1$ *and* $T < \infty$.

3 The case $a_1 \le \lambda_1$: global existence

In this section we will prove the global existence of positive classical solution.

Theorem 2. *If $a_1 \le \lambda_1$ then problem (1.1) has positive global classical solution $(u_1, \cdots, u_n)$. Moreover, all positive classical solutions must satisfy the following estimates.*

(i) if $a_n \le \lambda_1$, then

$$u_i(x,t) \le k_2\varphi(x) \ on\ \bar\Omega \times [0,\infty),\ i = 1, \cdots, n;$$

(ii) if there exists $j:\ 1 \le j \le n-1$ such that $a_j \le \lambda_1 < a_{j+1}$, then

$$u_i(x,t) \le \begin{cases} k_2\varphi(x), & i = 1, \cdots, j, \\ k_2 e^{a_i f_i(F_{i+1})t}, & i = j+1, \cdots, n, \end{cases}$$

where $F_{n+1} = k_2, F_i = k_2 e^{a_i f_i(F_{i+1})t}, j+1 \le i \le n$, and the positive constant k_2 is given by (2.5).

Proof. For any given $\varepsilon : 0 < \varepsilon < 1$, let $(u_{1\varepsilon}, \cdots, u_{n\varepsilon})$ be the unique positive classical solution of (2.3) which is defined on $\bar\Omega \times [0, T(\varepsilon))$ with $T(\varepsilon) \le \infty$, and the positive constants k_1 and k_2 are given by (2.5).

Step 1: upper bounds of $(u_{1\varepsilon}, \cdots, u_{n\varepsilon})$.

(i) If $a_n \le \lambda_1$, let $w_i(x,t) = k_2\varphi(x) + \varepsilon - u_{i\varepsilon}(x,t),\ i = 1, \cdots, n$, then we have

$$\begin{aligned} w_{1t} &= -f_1(u_{2\varepsilon})(\Delta u_{1\varepsilon} + a_1 u_{1\varepsilon} - a_1\varepsilon) \\ &= f_1(u_{2\varepsilon})(\Delta w_1 + a_1 w_1) + k_2 f_1(u_{2\varepsilon})(\lambda_1 - a_1)\varphi \\ &\ge f_1(u_{2\varepsilon})(\Delta w_1 + a_1 w_1) \qquad (x,\ t) \in \Omega \times (0,\ T(\varepsilon)), \\ w_1(x,t) &= 0, \qquad (x,\ t) \in \partial\Omega \times (0,\ T(\varepsilon)), \\ w_1(x,0) &= k_2\varphi(x) - u_0(x) \ge 0, \qquad x \in \Omega. \end{aligned}$$

Proposition 1 implies that $w_1 \ge 0$, i.e. $u_{1\varepsilon}(x,t) \le k_2\varphi(x) + \varepsilon$ *on* $\bar\Omega \times (0, T(\varepsilon))$. By the same way we can prove that $u_{i\varepsilon}(x,t) \le k_2\varphi(x) + \varepsilon$ *on* $\bar\Omega \times (0, T(\varepsilon)),\ 2 \le i \le n$.

(ii) if there exists $j:\ 1 \le j \le n-1$ such that $a_j \le \lambda_1 < a_{j+1}$, then similar to (i), $u_{i\varepsilon}(x,t) \le k_2\varphi(x) + \varepsilon$ on $\bar\Omega \times (0, T(\varepsilon)),\ i = 1, \cdots, j$. Let

$$w_i = k_2 e^{a_i f_i(F_{(i+1)\varepsilon})t} + \varepsilon - u_{i\varepsilon},\quad i = j+1, \cdots, n,$$

where $F_{(n+1)\varepsilon} = k_2 + \varepsilon,\ F_{n\varepsilon} = k_2 e^{a_n f_n(k_2+\varepsilon)t} + \varepsilon,\ F_{i\varepsilon} = k_2 e^{a_i f_i(F(i+1)\varepsilon)t} + \varepsilon,\ j+1 \le i \le n-1$. in view of $u_{i\varepsilon}(x,t) \le k_2\varphi(x) + \varepsilon \le k_2 + \varepsilon,\ 1 \le i \le j$, we have that

$$\begin{aligned} w_{nt} &= k_2 a_n f_n(k_2+\varepsilon) e^{a_n f_n(k_2+\varepsilon)t} - (u_{n\varepsilon})_t \\ &= k_2 a_n f_n(k_2+\varepsilon) e^{a_n f_n(k_2+\varepsilon)t} - f_n(u_{1\varepsilon})(\Delta u_{n\varepsilon} + a_n u_{n\varepsilon} - a_n\varepsilon) \\ &= k_2 a_n f_n(k_2+\varepsilon) e^{a_n f_n(k_2+\varepsilon)t} - f_n(u_{1\varepsilon})(-\Delta w_n + a_n k_2 e^{a_n f_n(k_2+\varepsilon)t} - a_n w_n) \\ &= f_n(u_{1\varepsilon})(\Delta w_n + a_n w_n) + a_n k_2 e^{a_n f_n(k_2+\varepsilon)t}(f_n(k_2+\varepsilon) - f_n(u_{1\varepsilon})) \\ &\ge f_n(u_{1\varepsilon})(\Delta w_n + a_n w_n), \quad (x,t) \in \Omega \times (0, T(\varepsilon)), \\ w_n(x,t) &= k_2 e^{a_n f_n(k_2+\varepsilon)t} + \varepsilon - \varepsilon > 0, \qquad (x,t) \in \partial\Omega \times (0, T(\varepsilon)), \\ w_n(x,0) &= k_2 - u_{n0} > 0, \quad x \in \Omega. \end{aligned}$$

Proposition 1 implies that $w_n \geq 0$, i.e. $u_{n\varepsilon} \leq k_2 e^{a_n f_n(k_2+\varepsilon)t} + \varepsilon$ on $\bar{\Omega} \times (0, T(\varepsilon))$. Similarly, we have

$$\begin{aligned} w_{(n-1)t} &= f_{n-1}(u_{n\varepsilon})(-\Delta w_{n-1} + a_{n-1} w_{n-1}) + a_{n-1} k_2 e^{a_{n-1} f_{n-1}(F_{n\varepsilon})t}(f_{n-1}(F_{n\varepsilon}) - f_{n-1}(u_{n\varepsilon})) \\ &\geq f_{n-1}(u_{n\varepsilon})(-\Delta w_{n-1} + a_{n-1} w_{n-1}), \qquad (x,t) \in \Omega \times (0, T(\varepsilon)), \\ w_{n-1}(x,t) &= k_2 e^{a_{n-1} f_{n-1}(F_{n\varepsilon})t} > 0, \quad (x,t) \in \partial\Omega \times (0, T(\varepsilon)), \\ w_{n-1}(x,0) &= k_2 - u_{(n-1)0} > 0, \quad x \in \Omega. \end{aligned}$$

Proposition 1 also implies that $w_{n-1} \geq 0$, i.e. $u_{(n-1)\varepsilon} \leq k_2 e^{a_{n-1} f_{n-1}(F_{n\varepsilon})t} + \varepsilon$ on $\bar{\Omega} \times (0, T(\varepsilon))$. Similarly, we can get $u_{i\varepsilon} \leq k_2 e^{a_i f_i(F_{i+1})t} + \varepsilon$ on $\bar{\Omega} \times (0, T(\varepsilon))$, $j+1 \leq i \leq n-2$.

Step 2: lower bounds of $(u_{1\varepsilon}, \cdots, u_{n\varepsilon})$.

(i) if $a_1 = \lambda_1$, similar to the proof of (i) of Lemma 4 we have

$$u_{i\varepsilon}(x,t) \geq k_1 \varphi(x) + \varepsilon, \quad (x,t) \in \Omega \times (0,\ T(\varepsilon));$$

(ii) if there exists $j:\ 1 \leq j \leq n-1$, such that $a_j < \lambda_1 \leq a_{j+1}$, then

$$u_{i\varepsilon}(x,t) \geq \begin{cases} k_1\varphi(x) \exp\{-\int_0^t (\lambda_1 - a_i) f_i(k_2 e^{a_{i+1} f_{i+1}(F_{(i+2)\varepsilon})s} + \varepsilon)\mathrm{d}s\} + \varepsilon, & (x,t) \in \overline{\Omega} \times [0, T(\varepsilon)), \quad 1 \leq i \leq j, \\ k_1\varphi(x) + \varepsilon, \quad (x,t) \in [0, T(\varepsilon)), \quad j+1 \leq i \leq n. \end{cases}$$

In fact, similar to the proof of (ii) of Lemma 4 we know that $u_{i\varepsilon} \geq k_1\varphi(x) + \varepsilon$ on $\bar{\Omega} \times [0, T_\varepsilon)$, $j+1 \leq i \leq n$. Denote

$$h_i(t) = (\lambda_1 - a_i) f_i(k_2 e^{a_{i+1} f_{i+1}(F_{(i+2)\varepsilon})t} + \varepsilon),\ i = 1, 2, \cdots, j,$$

where $F_{(n+1)\varepsilon} = k_2 + \varepsilon$, $F_{i\varepsilon} = k_2 e^{a_i f_i(F(i+1)\varepsilon)t} + \varepsilon$, $i = 1, 2, \cdots, j$. And let

$$w_i = u_{i\varepsilon} - k_1\varphi(x) \exp\Big\{ - \int_0^t h_i(s)\mathrm{d}s \Big\} - \varepsilon,\ i = 1, 2, \cdots, j.$$

Note that $u_{(j+1)\varepsilon} \leq k_2 e^{a_{j+1} f_{j+1}(F_{(j+2)\varepsilon})t} + \varepsilon$ on $\bar{\Omega} \times [0, T_\varepsilon)$, and by the direct computations we can see that

$$\begin{aligned} w_{jt} &= (u_{j\varepsilon})_t + k_1\varphi(x) h_j(t) e^{-\int_0^t h_j(s)\mathrm{d}s} \\ &= f_j(u_{(j+1)\varepsilon})(\Delta w_j + a_j w_j) + k_1\varphi(x)(a_j - \lambda_1) f_j(u_{(j+1)\varepsilon}) e^{-\int_0^t h_j(s)\mathrm{d}s} + k_1\varphi(x) h_j(t) e^{-\int_0^t h_j(s)\mathrm{d}s} \\ &\geq f_j(u_{(j+1)\varepsilon})(\Delta w_j + a_j w_j) + k_1\varphi(x)(h_j(t) - (\lambda_1 - a_j) f_j(u_{(j+1)\varepsilon})) e^{-\int_0^t h_j(s)\mathrm{d}s} \\ &\geq f_j(u_{(j+1)\varepsilon})(\Delta w_j + a_j w_j), \quad (x,t) \in \Omega \times (0, T(\varepsilon)), \\ w_j(x,t) &= \varepsilon - \varepsilon = 0, \quad (x,t) \in \partial\Omega \times (0, T(\varepsilon)), \\ w_j(x,0) &= u_{j0} - k_1\varphi(x) \geq 0, \quad x \in \Omega. \end{aligned}$$

By Proposition 1 we have $w_j \geq 0$, i.e. $u_{j\varepsilon} \geq k_1\varphi(x) e^{-\int_0^t h_j(s)\mathrm{d}s} + \varepsilon$ on $\bar{\Omega} \times (0, T(\varepsilon))$.

If $1 \leq i \leq j$, similar as above we also can prove

$$u_{i\varepsilon} \geq k_1\varphi(x) e^{-\int_0^t h_i(s)\mathrm{d}s} + \varepsilon, \ \text{ on } \bar{\Omega} \times [0, T(\varepsilon)), \quad 1 \leq i \leq j.$$

(iii) If $a_n \le \lambda_1$, then for $i = 1, 2, \cdots, n$, then

$$u_{i\varepsilon}(x,t) \ge k_1\varphi(x)\exp\Big\{-\int_0^t (\lambda_1 - a_i) f_i(k_2 e^{a_{i+1}f_{i+1}(F_{(i+2)\varepsilon})s} + \varepsilon)\mathrm{d}s\Big\} + \varepsilon, \quad (x,t) \in \bar{\Omega} \times [0, T_\varepsilon),$$

where $f_{n+1} = f_1$ and F_i given above.

The proof is similar as (ii).

Step 3: the upper bounds of $(u_{1\varepsilon}, \cdots, u_{n\varepsilon})$ obtained by Step 1 show that $(u_{1\varepsilon}, \cdots, u_{n\varepsilon})$ exists globally, i.e. $T(\varepsilon) = \infty$ for all $0 < \varepsilon < 1$. For any $\Omega_n \subset\subset \Omega$ and $0 < \tau < T_n < \infty$, Step 1 and 2 show that there exist positive constants $\sigma(n,\tau)$ and $M(n,\tau)$ such that

$$\sigma(n,\tau) \le u_{i\varepsilon} \le M(n,\tau) \text{ on } \bar{\Omega}_n \times [\tau, T_n],\ i = 1, 2, \cdots, n$$

for all $0 < \varepsilon < 1$. Applying the standard local Schauder estimates and diagonal method we have that there exist subsequence $\{\varepsilon'\}$ of $\{\varepsilon\}$ and $u_i \in C^{2+\alpha,1+\alpha/2}_{loc}(\Omega \times (0,\infty))$ such that

$$(u_{1\varepsilon'}, \cdots,\ u_{n\varepsilon'}) \longrightarrow (u_1, \cdots, u_n) \ \ in\ \ [C^{2+\alpha,1+\alpha/2}_{loc}(\bar{\Omega}_* \times [\tau,\ T_0])]^n \ as\ \varepsilon' \to 0^+$$

for any $\Omega_* \subset\subset \Omega$ and $0 < \tau < T_0 < \infty$. And hence $(u_1, \cdots,\ u_n)$ satisfies the problem (1.1) in $\Omega \times (0,\infty)$.

Similar to the arguments of §2 we see that

$$(u_1, \cdots,\ u_n) \in \Big[C^{2+\alpha,1+\alpha/2}_{loc}(\Omega \times (0,\infty)) \cap C(\bar{\Omega} \times [0,\infty))\Big]^n$$

is a classical solution of (1.1).

Estimates (i) and (ii) can be proved in the similar way to that of Step 1. The proof is completed.

4 The case $a_1 > \lambda_1$: blow up result

In this section we will prove the blow up result of problem (1.1). Let G be a bounded domain of R^N, $\lambda_1(G)$ be the first eigenvalue of $-\Delta$ on G with homogeneous boundary condition. And we consider the following initial-boundary problem:

$$\begin{cases} w_t = Af(w)(\Delta w + Bw), & s \in G,\ t > 0, \\ w(x,0) = C, & x \in G, \\ w(x,t) = C, & x \in \partial\Omega,\ t \ge 0. \end{cases} \tag{4.1}$$

where constants $A, B, C > 0$ and $f(w)$ satisfy (H3). By the standard method (see [10]), it follows that (4.1) has a unique classical solution $w(x,t)$ and $w(x,t) \ge C$. And we can get following Lemma (see [2]):

Lemma 5. *Let f satisfy* (H3)*, then for any $A > 0$ and $C > 0$, the unique (local) solution $w(x,t)$ of* (4.1) *blow up in finite time T if $\int_0^\infty 1/(sf(s))\mathrm{d}s < \infty$ and $\lambda_1 < B$, while the solution $w(x,t)$ exists globally if $\int_0^\infty 1/(sf(s))\mathrm{d}s = \infty$.*

Theorem 3. *Assume that $a_1 > \lambda_1$, the initial datum $(u_{10}, \cdots,\ u_{n0}) \in [C^4(\bar{\Omega})]^n$ and* (H1) – (H4) *hold. If $\Delta u_{i0} + a_i u_{i0} \ge 0$ in Ω and $\Delta u_{i0} = 0$ on $\partial\Omega$, $i = 1, \cdots,\ n$, then the positive classical solution of (1.1) is unique and blow up in the finite time.*

Proof. **Step 1**: monotonicity of $(u_1, \cdots, u_n)$ in t.
Let $\varepsilon \in (0,1)$ and $(u_{1\varepsilon}, \cdots, u_{n\varepsilon})$ be the solution of (2.3), then

$$(u_{1\varepsilon}, \cdots, u_{n\varepsilon}) \in \Big[C^{2+\alpha,1+\alpha/2}(\Omega \times (0,T(\varepsilon))) \times C^{2,1}(\bar\Omega \times [0,T(\varepsilon)))\Big]^n$$

since $\Delta u_{i0} = 0,\ i = 1,2,\cdots,n$ on $\partial\Omega$ (see Theorem 7.1 of chap.7 of [6]). Let $w_i = u_{i\varepsilon t},\ i = 1,2,\cdots,n$, then we have,

$$\begin{aligned}
w_{1t} &= \Big(f_1(u_{2\varepsilon})(\Delta u_{1\varepsilon} + a_1 u_{1\varepsilon} - a_{1\varepsilon})\Big)_t \\
&= f_1(u_{2\varepsilon})(\Delta w_1 + a_1 w_1) + f_{1t}(u_{2\varepsilon}) w_2 (\Delta u_{1\varepsilon} + a_1 u_{1\varepsilon} - a_1\varepsilon) \\
&= f_1(u_{2\varepsilon})(\Delta w_1 + a_1 w_1) + \big[\frac{f_{1t}(u_{2\varepsilon})}{f_1(u_{2\varepsilon})} w_2\big] w_1, \quad (x,t) \in \Omega \times (0,T(\varepsilon)), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,, \\
w_{(n-1)t} &= f_{n-1}(u_{n\varepsilon})(\Delta w_{n-1} + a_{n-1} w_{n-1}) + \big[\frac{f_{(n-1)t}(u_{n\varepsilon})}{f_{n-1}(u_{n\varepsilon})} w_n\big] w_{n-1},\ (x,t) \in \Omega \times (0,T(\varepsilon)), \\
w_{nt} &= f_n(u_{1\varepsilon})(\Delta w_n + a_n w_n) + \big[\frac{f_{nt}(u_{1\varepsilon})}{f_n(u_{1\varepsilon})} w_1\big] w_n, \quad (x,t) \in \Omega \times (0,T(\varepsilon)), \\
w_i(x,0) &= f_i(u_{i0}(x) + \varepsilon)(\Delta u_{i0}(x) + a_i u_{i0}(x)) \geq 0, \qquad x \in \Omega,\ i = 1,\cdots,\ n, \\
w_i(x,t) &= 0, \qquad (x,t) \in \Omega \times (0,T(\varepsilon)),\ i = 1,\cdots,\ n.
\end{aligned}$$

In view of $u_{i\varepsilon} \geq \varepsilon$ and $w_i \in C(\bar\Omega \times [0,T(\varepsilon))),\ i = 1,\cdots,\ n$, the L^p-theory and Schauder-Theory implies that $(w_1, \cdots,\ w_n)$ is a classical solution, i.e. $w_i \in C(\bar\Omega \times [0,T(\varepsilon))) \cap C^{2,1}(\Omega \times (0,T(\varepsilon)))$. Proposition 1 shows that $w_i \geq 0$, i.e. $u_{i\varepsilon t} \geq 0,\ i = 1,2,\cdots,n.$ since

$$(u_{1\varepsilon}, \cdots,\ u_{n\varepsilon}) \longrightarrow (u_1, \cdots,\ u_n) \ \ in\ \ [C^{2+\alpha,1+\alpha/2}_{loc}(\Omega \times (0,T)]^n \ \ \text{as}\ \ \varepsilon \to 0^+,$$

we know that $u_{it} \geq 0,\ i = 1,\cdots,\ n$ and hence $u_i \geq u_{i0}(x),\ i = 1,\cdots,\ n$, in $\Omega \times (0,T)$.

Step 2: the uniqueness.

Let $(u_1,\ \cdots,\ u_n)$ be the solution of (1.1) obtained by §2, then by step 1 $u_{it} \geq 0,\ i = 1,\cdots,\ n$, which implies $\Delta u_i + a_i u_i \geq 0$. Let $(\tilde u_1, \cdots, \tilde u_n)$, which defined on $\bar\Omega \times [0,\ \tilde T)$, be another positive solution of (1.1) with the same initial datum $(u_{10}, \cdots,\ u_{n0})$, and set $w_i = \tilde u_i - u_i,\ i = 1,\cdots,\ n$, then we have, for any $0 < T_0 < \min\{T,\ \tilde T\}$,

$$\begin{aligned}
w_{1t} &= \tilde u_{1t} - u_{1t} = f_1(\tilde u_2)(\Delta \tilde u_1 + a_1 \tilde u_1) - f_1(u_2)(\Delta u_1 + a_1 u_1) \\
&= f_1(\tilde u_2)(\Delta w_1 + \Delta u_1 + a_1 w_1 + a_1 u_1) - f_1(u_2)(\Delta u_1 + a_1 u_1) \\
&= f_1(\tilde u_2)(\Delta w_1 + a_1 w_1) + \big(f_1(\tilde u_2) - f_1(u_2)\big)(\Delta u_1 + a_1 u_1) \\
&= f_1(\tilde u_2)(\Delta w_1 + a_1 w_1) + \Big(\int_0^1 f_1'\big(u_2 + s(\tilde u_2 - u_2)\big)\mathrm{d}s(\Delta u_1 + a_1 u_1)\Big) w_2,\ (x,t) \in \Omega \times (0,T_0), \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\,, \\
w_{(n-1)t} &= f_{n-1}(\tilde u_n)(\Delta w_{n-1} + a_{n-1} w_{n-1}) \\
&\quad + \Big(\int_0^1 f_1'\big(u_n + s(\tilde u_n - u_n)\big)\mathrm{d}s(\Delta u_{n-1} + a_{n-1} u_{n-1})\Big) w_n,\ (x,t) \in \Omega \times (0,T_0), \\
w_{nt} &= f_n(\tilde u_1)(\Delta w_n + a_n w_n) + \Big(\int_0^1 f_1'\big(u_1 + s(\tilde u_1 - u_1)\big)\mathrm{d}s(\Delta u_n + a_n u_n)\Big) w_1, \quad (x,t) \in \Omega \times (0,T_0), \\
w_i &= 0, \qquad (x,t) \in \partial\Omega \times (0,T_0) \cup \Omega \times \{0\},\ i = 1,2\cdots,n.
\end{aligned}$$

Since $\Delta u_i + a_i u_i \geq 0, i = 1, 2 \cdots, n$, Proposition 1 implies that $w_i(x,\ t) \equiv 0, i = 1, 2, \cdots, n$, i.e. $\tilde{u_i}(x,t) \equiv u_i(x,t),\ i = 1, 2, \cdots, n$. The uniqueness is proved.

Step 3: $(u_1, \cdots,\ u_n)$ blow up in the finite time.

Since $a_1 > \lambda_1$, $\exists \Omega' : \Omega' \subset\subset \Omega$, s.t. the first eigenvalue λ_1' of $-\Delta$ in Ω' with homogeneous Dirichlet boundary condition satisfies $a_1 > \lambda_1'$. Applying $u_i \geq u_{i0} > 0$ $(i = 1, 2, \cdots, n)$ in Ω we know that

$$u_i(x,t) \geq \sigma,\ i = 1, 2, \cdots, n \ \text{ on } \ \bar{\Omega}' \times (0,T)$$

for some positive constant σ. Next, we consider the following system

$$\begin{cases} \underline{u}_{it} = f_i(\underline{u}_{i+1})(\Delta \underline{u}_i + a_i \underline{u}_i), & i = 1, \cdots, n-1, \\ \underline{u}_{nt} = f_n(\underline{u}_1)(\Delta \underline{u}_n + a_n \underline{u}_n), & x \in \Omega',\ t > 0, \\ \underline{u}_i(x,t) = \sigma, & x \in \partial\Omega',\ \ i = 1, 2, \cdots, n, \\ \underline{u}_i(x,0) = \sigma, & x \in \Omega',\ \ i = 1, 2, \cdots, n. \end{cases} \tag{4.2}$$

By the classical parabolic theory, there exists a nonnegative classical solution $(\underline{u}_1, \cdots,\ \underline{u}_n)$ for $(x,t) \in \Omega' \times (0, T')$, where T' denotes the maximal existence time of (4.2). The standard comparison principle for parabolic system implies that $T' \geq T$ and

$$u_i(x,t) \geq \underline{u}(x,t), \qquad (x,t) \in \bar{\Omega}' \times [0,T),\quad i = 1, 2, \cdots, n.$$

If we can prove that the solution $(\underline{u}_1(x,t), \cdots,\ \underline{u}_n(x,t))$ of (4.2) blow up in finite time, So does $(u_1, \cdots u_n)$.

Since the initial data $(\sigma, \cdots,\ \sigma)$ is a lower solution of (4.2), the standard upper and lower solutions method asserts that $\underline{u}_{it} \geq 0,\ i = 1, 2, \cdots, n$, which implies that $\Delta \underline{u}_i + a_i \underline{u}_i \geq 0,\ i = 1, 2, \cdots, n$. And hence

$$\underline{u}_i(x,t) \geq \sigma,\ i = 1, 2, \cdots, n, \quad (x,t) \in \bar{\Omega}' \times [0, T').$$

Furthermore, $(\underline{u}_1, \cdots \underline{u}_n)$ satisfies, by (1.6),

$$\begin{cases} \underline{u}_{it} \ \geq K_0 f_n(\underline{u}_{i+1})(\Delta \underline{u}_i + a_i \underline{u}_i), & i = 1, 2, \cdots, n-1, \\ \underline{u}_{nt} \ = f_n(\underline{u}_1)(\Delta \underline{u}_n + a_n \underline{u}_n), & x \in \Omega',\ 0 < t < T'. \end{cases} \tag{4.3}$$

Choose $k = \min\{1, K_0\}$, and denote by $z(x,t)$ the unique positive solution of the following problem:

$$\begin{cases} z_t = k f_n(z)(\Delta z + a_1 z), & x \in \Omega',\ t > 0, \\ z(x,0) = \sigma, & x \in \bar{\Omega}', \\ z(x,t) = \sigma, & x \in \partial\Omega',\ t > 0. \end{cases} \tag{4.4}$$

By Lemma 5 it comes that $z(x,t)$ blow up in finite time $T_0 < \infty$. Moreover $z_t \geq 0$, i.e. $\Delta z + a_1 z \geq 0$, since the initial data is a lower solution of (4.4). Next we will prove $z(x,t) \leq \underline{u}_i(x,t),\ i = 1, 2, \cdots, n$. Let

$$w_i(x,t) = \underline{u}_i(x,t) - z(x,t),\ i = 1, 2, \cdots, n,$$

then

$$
\begin{aligned}
w_{1t} &= \underline{u}_{1t} - z_t \geq kf_n(\underline{u}_2)(\Delta \underline{u}_1 + a_1\underline{u}_1) - kf_n(z)(\Delta z + a_1 z) \\
&\geq kf_n(\underline{u_2})(\Delta w_1 + \Delta z + a_1 w_1 + a_1 z) - kf_n(z)(\Delta z + a_1 z) \\
&= kf_n(\underline{u_2})(\Delta w_1 + a_1 w_1) + k(\Delta z + a_1 z)(f_n(\underline{u}_2) - f_n(z)) \\
&= kf_n(\underline{u_2})(\Delta w_1 + a_1 w_1) + [k(\Delta z + a_1 z)\int_0^1 f_n'(z + s(\underline{u}_2 - z))\mathrm{d}s]w_2, \qquad x \in \Omega',\ 0 < t < T_0, \\
w_1(x,0) &= 0, \qquad\qquad x \in \bar{\Omega}', \\
w_1(x,t) &= 0, \qquad\qquad x \in \partial\bar{\Omega}',\ \ 0 < t < T_0.
\end{aligned}
$$

Proposition 1 implies $w_1 \geq 0$, i.e. $\underline{u}_1(x,t) \geq z(x,t)$. similarly we can prove $\underline{u}_i(x,\ t) \geq z(x,t),\ i = 2,\cdots,n$. which also implies that $(\underline{u}_1,\cdots,\ \underline{u}_n)$ blow up in finite time, and so does the solution $(u_1,\cdots,\ u_n)$ of (1.1). The proof is completed.

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