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THE CONSTRUCTIVE APPROACH ON EXISTENCE OF TIME OPTIMAL CONTROLS OF SYSTEM GOVERNED BY NONLINEAR EQUATIONS ON BANACH SPACES

JinRong Wang[1,2,3], X. Xiang[1] and W. Wei[1]

1. College of Science of Guizhou University, Guiyang, Guizhou 550025, P. R. China
2. College of Science of Kaili University, Kaili, Guizhou 556000, P. R. China
3. Policy Research Room of Qiandongnan State Committee of Chinese Communist Party, Kaili, Guizhou 556000, P. R. China

ABSTRACT. In this paper, a new approach to the existence of time optimal controls of system governed by nonlinear equations on Banach spaces is provided. A sequence of Meyer problems is constructed to approach a class of time optimal control problems. A deep relationship between time optimal control problems and Meyer problems is presented. The method is much different from standard methods.

Keywords. Time optimal control, C_0-semigroup, Existence, Transformation, Meyer approximation.

1. Introduction

The research on time optimal control problems dates back to the 1960's. Issues such as existence, necessary conditions for optimality and controllability have been discussed. We refer the reader to [6] for the finite dimensional case, and to [1, 4, 7, 9] for the infinite dimensional case. The cost functional for a time optimal control problem is the infimum of a number set. On the other hand, the cost functional for a Lagrange, Meyer or Bolza problem contains an integral term. This difference leads investigators to consider a time optimal control problem as another class of optimal control problems and use different studying framework.

Recently, computation of optimal control for Meyer problems has been extensively developed. Meyer problems can be solved numerically using methods such as dynamic programming (see [10]) and control parameterization(see [11]). However, the computation of time optimal controls is very difficult. For finite dimensional problems, one can solve a two point boundary value problems using a shooting method. However, this method is far from ideal since solving such two point boundary value problems numerically is a nontrivial task.

In this paper, we provide a new constructive approach to the existence of time optimal controls. The method is called Meyer approximation. Essentially, a sequence of Meyer problems is constructed to approximate the time optimal control problem. That is, time optimal control problem can be approximated by a sequence of optimal controls from an associated Meyer problem. Although the existence of time optimal control can be proved using other methods, the method presented here is constructive. Hence the algorithm

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E-mail: wjr9668@126.com.

based on Meyer approximation can be used to actually compute the time optimal control. This is in contrast to previously desired results.

We consider the time optimal control problem (P) of a system governed by

$$\begin{cases} \dot{z}\,(t) = Az(t) + f(t, z(t), B(t)v(t)), t \in (0, \tau), \\ z(0) = z_0 \in X,\ v \in V_{ad}, \end{cases} \tag{1.1}$$

where A is the infinitesimal generator of a C_0-semigroup $\{T\,(t)\,, t \geq 0\}$ on Banach space X and V_{ad} is the admissible control set.

Then, we construct the Meyer approximation $(\mathrm{P}_{\varepsilon_n})$ to Problem (P). Our new control system is

$$\begin{cases} \dot{x}(s) = kAx(s) + kf\big(ks, x(s), B(ks)u(s)\big), s \in (0, 1], \\ x(0) = z(0) = z_0 \in X,\ w = (u, k) \in W, \end{cases} \tag{2.1}$$

whose controls are taken from a product space. A chosen subsequence $\{\mathrm{P}_{\varepsilon_n}\}$ is the Meyer approximation to Problem (P).

By applying the family of C_0-semigroups with parameters, the existence of optimal controls for Meyer problem (P_ε) is proved. Then, we show that there exists a subsequence of Meyer Problems $(\mathrm{P}_{\varepsilon_n})$ whose corresponding sequence of optimal controls $\{w_{\varepsilon_n}\} \in W$ converges to a time optimal control of Problem (P) in some sense. In other words, in a limiting process, the sequence $\{w_{\varepsilon_n}\} \in W$ can be used to find the solution of time optimal control problem (P). The existence of time optimal controls for problem (P) is proved by this constructive approach which offers a new way to compute the time optimal control.

The rest of the paper is organized as follows. In Section 2, we formulate the time optimal control Problem (P) and Meyer problem (P_ε). In Section 3, existence of optimal controls for Meyer problems (P_ε) is proved. The last section contributes to the main result of this paper. Time optimal control can be approximated by a sequence of Meyer problems.

2. Time Optimal Control Problem (P) and Meyer problem (P_ε)

For each $\tau < +\infty$, let $I_\tau \equiv [0, \tau]$ and let $C(I_\tau, X)$ be the Banach space of continuous functions from I_τ to X with the usual supremum norm.

Consider the following nonlinear control system

$$\begin{cases} \dot{z}\,(t) = Az(t) + f\big(t, z(t), B(t)v(t)\big), t \in (0, \tau), \\ z(0) = z_0 \in X,\ v \in V_{ad}. \end{cases} \tag{1.1}$$

We make the following assumptions:

[A] A is the infinitesimal generator of a C_0-semigroup $\{T\,(t)\,, t \geq 0\}$ on X with domain $D(A)$.

[F] $f : I_\tau \times X \times X \to X$ is measurable in t on I_τ and for each $\rho > 0$, there exists a constant $L\,(\rho) > 0$ such that for almost all $t \in I_\tau$ and all z_1, z_2, y_1, $y_2 \in X$, satisfying $\|z_1\|\,, \|z_2\|\,, \|y_1\|$, $\|y_2\| \leq \rho$, we have

$$\|f\,(t, z_1, y_1) - f\,(t, z_2, y_2)\| \leq L\,(\rho)\,(\|z_1 - z_2\| + \|y_1 - y_2\|)\,.$$

For arbitrary $(t, y) \in I_\tau \times X$, there exists a positive constant $M > 0$ such that

$$\|f\,(t, z, y)\| \leq M(1 + \|z\|).$$

[B] Let E be a reflexive Banach space. Operator $B \in L_\infty(I_\tau, L(E, X))$, $\|B\|_\infty$ stands for the norm of operator B on Banach space $L_\infty\big(I_\tau, L(E, X)\big)$. $B : L_p(I_\tau, E) \to L_p(I_\tau, X)(1 < p < +\infty)$ is strongly continuous.

[U] Multivalued maps $\Gamma(\cdot) : I_\tau \to 2^E \backslash \{\emptyset\}$ has closed, convex and bounded values. $\Gamma(\cdot)$ is graph measurable and $\Gamma(\cdot) \subseteq \Omega$ where Ω is a bounded set of E.

Set

$$V_{ad} = \{v(\cdot) \mid I_\tau \to E \text{ measurable}, v(t) \in \Gamma(t) \text{ a.e.}\}.$$

Obviously, $V_{ad} \neq \emptyset$ (see Theorem 2.1 of [12]) and $V_{ad} \subset L_p(I_\tau, E)(1 < p < +\infty)$ is bounded, closed and convex.

By standard process (see Theorem 5.3.3 of [2]), one can easily prove the following existence of mild solutions for system (1.1).

Theorem 2.1: Under the assumptions [A], [B], [F] and [U], for every $v \in V_{ad}$, system (1.1) has a unique mild solution $z \in C(I_\tau, X)$ which satisfies the following integral equation

$$z(t) = T(t)z_0 + \int_0^t T(t-\theta) f\big(\theta, z(\theta), B(\theta)v(\theta)\big)d\theta.$$

Definition 2.1: (**Admissible trajectory**) Take two points z_0, z_1 in the state space X. Let z_0 be the initial state and let z_1 be the desired terminal state with $z_0 \neq z_1$. Denote $z(v) \equiv \{z(t,v) \in X \mid t \geq 0\}$ be the state trajectory corresponding to the control $v \in V_{ad}$. A trajectory $z(v)$ is said to be admissible if $z(0,v) = z_0$ and $z(t,v) = z_1$ for some finite $t > 0$.

Set

$$V_0 = \big\{v \in V_{ad} \mid z(v) \text{ is an admissible trajectory}\big\} \subset V_{ad}.$$

For given z_0, $z_1 \in X$ and $z_0 \neq z_1$, if $V_0 \neq \emptyset$ (i.e., There exists at least one control from the admissible class that takes the system from the given initial state z_0 to the desired target state z_1 in the finite time.), we say the system (1.1) can be controlled. Let

$$\tau(v) \equiv \inf\{t \geq 0 \mid z(t,v) = z_1\}$$

denote the transition time corresponding to the control $v \in V_0 \neq \emptyset$ and define

$$\tau^* = \inf\{\tau(v) \geq 0 \mid v \in V_0\}.$$

Then, the time optimal control problem can be stated as follows:

Problem (P): Take two points z_0, z_1 in the state space X. Let z_0 be the initial state and let z_1 be the desired terminal state with $z_0 \neq z_1$. Suppose that there exists at least one control from the admissible class that takes the system from the given initial state z_0 to the desired target state z_1 in the finite time. The time optimal control problem is to find a control $v^* \in V_0$ such that

$$\tau(v^*) = \tau^* = \inf\{\tau(v) \geq 0 \mid v \in V_0\}.$$

For fixed $\hat{v} \in V_{ad}$, $\hat{T} = \tau(\hat{v}) > 0$. Now we introduce the following linear transformation

$$t = ks,\ 0 \leq s \leq 1 \text{ and } k \in [0, \hat{T}].$$

Through this transformation system (1.1) can be replaced by

$$\begin{cases} \dot{x}(s) = kAx(s) + kf\big(ks, x(s), B(ks)u(s)\big), s \in (0,1] \\ x(0) = z(0) = z_0 \in X,\ w = (u,k) \in W, \end{cases} \tag{2.1}$$

where $x(\cdot) = z(k\cdot)$, $u(\cdot) = v(k\cdot)$ and define

$$W = \Big\{(u,k) \mid u(s) = v(ks), 0 \leq s \leq 1, v \in V_{ad}, k \in [0, \hat{T}]\Big\}.$$

By Theorem 2.1, one can verify that

Theorem 2.2: Under the assumptions [A], [B], [F] and [U], for every $w \in W$, system (2.1) has a unique mild solution $x \in C([0,1], X)$ which satisfies the following integral equation

$$x(s) = T_k(s)z_0 + \int_0^s T_k(s-\theta)\, kf\big(k\theta, x(\theta), B(k\theta)u(\theta)\big)d\theta,$$

where kA is the generator of a C_0-semigroup $\{T_k(t), t \geq 0\}$ (see Lemma 3.1).

For the controlled system (2.1), we consider

Meyer problem ($\mathbf{P}_\varepsilon$): Minimize the cost functional given by

$$J_\varepsilon(w) = \frac{1}{2\varepsilon}\|x(w)(1) - z_1\|^2 + k$$

over W, where $x(w)$ is the mild solution of (2.1) corresponding to control w.
i.e., Find a control $w_\varepsilon = (u_\varepsilon, k_\varepsilon)$ such that the cost functional $J_\varepsilon(w)$ attains its minimum on W at w_ε.

3. Existence of optimal controls for Meyer Problem (P_ε)

In this section, we discuss the existence of optimal controls for Meyer Problem (P_ε). First, in order to study system (2.1), we have to deal with family of C_0-semigroups with parameters which are widely used in this paper.

Lemma 3.1: If the assumption [A] holds, then

(1) For given $k \in [0, \hat{T}]$, kA is the infinitesimal generator of C_0-semigroup $\{T_k(t), t \geq 0\}$ on X.

(2) There exist constants $C \geq 1$ and $\omega \in (-\infty, +\infty)$ such that

$$\|T_k(t)\| \leq Ce^{\omega kt} \text{ for all } t \geq 0.$$

(3) If $k_n \to k_\varepsilon$ in $[0, \hat{T}]$ as $n \to \infty$, then for arbitrary $x \in X$ and $t \geq 0$,

$$T_{k_n}(t) \xrightarrow{\tau_s} T_{k_\varepsilon}(t) \text{ as } n \to \infty \quad (\tau_s \text{ denotes strong operator topology})$$

uniformly in t on some closed interval of $[0, \hat{T}]$ in the strong operator topology sense.

Proof. (1) By the famous Hille-Yosida theorem (see Theorem 2.2.8 of [2]),

$$\begin{cases} (i)\ A \text{ is closed and } \overline{D(A)} = X; \\ (ii)\ \rho(A) \supset (\omega, +\infty) \text{ and } \|R(\lambda, A)\| \leq (\lambda - \omega)^{-1}, \text{ for } \lambda > \omega. \end{cases}$$

It is obvious that for fixed $k \in [0, \hat{T}]$,

$$\begin{cases} (\dot{i})\ kA \text{ is also closed and } \overline{D(kA)} = X; \\ (\dot{i}\dot{i})\ \rho(kA) \supset (k\omega, +\infty) \text{ and } \|R(\lambda, kA)\| = k^{-1}\|R(k^{-1}\lambda, A)\| \leq (\lambda - k\omega)^{-1}, \text{ for } \lambda > k\omega. \end{cases}$$

Using Hille-Yosida theorem again, one can complete it.

(2) By virtue of (1) and Theorem 1.3.1 of [2], one can verify it easily.

(3) Since $\{k_n\}$ is a bounded sequence of $[0, \hat{T}]$ and $k_n > 0$, due to continuity theorem of real number, there exists a subsequence of $\{k_n\}$, denoted by $\{k_n\}$ again such that $k_n \to k_\varepsilon$ in $[0, \hat{T}]$ as $n \to \infty$. For arbitrary $x \in X$ and $\lambda > k_n\omega$, we have

$$R(\lambda, k_nA)x = (\lambda I - k_nA)^{-1}x \to (\lambda I - k_\varepsilon A)^{-1}x = R(\lambda, k_\varepsilon A)x, \quad \text{as } n \to \infty.$$

Using Theorem 4.5.4 of [2], then

$$T_{k_n}(t)x \to T_{k_\varepsilon}(t)x \quad \text{as} \quad n \to \infty.$$

Further,

$$T_{k_n}(t) \xrightarrow{\tau_s} T_{k_\varepsilon}(t) \quad \text{as} \quad n \to \infty$$

uniformly in t on some closed interval in the strong operator topology sense. □

Lemma 3.2: For each $g \in L_p([0,T_0],X)$ with $1 \le p < +\infty$,

$$\lim_{h\to 0}\int_0^{T_0} \|g(t+h)-g(t)\|^p dt = 0$$

where $g(s)=0$ for s does not belong to $[0,T_0]$.

Proof. See details on problem 23.9 of [12]. □

We show that Meyer problem (P_ε) has a solution $w_\varepsilon = (u_\varepsilon, k_\varepsilon)$ for fixed $\varepsilon > 0$.

Theorem 3.A: Assumptions [A], [B], [F] and [U] hold. Meyer problem (P_ε) has a solution.

Proof. Let $\varepsilon > 0$ be fixed. Since $J_\varepsilon(w) \ge 0$, there exists $\inf\{J_\varepsilon(w), w \in W\}$. Denote $m_\varepsilon \equiv \inf\{J_\varepsilon(w), w \in W\}$ and choose $\{w_n\} \subseteq W$ such that

$$J_\varepsilon(w_n) \to m_\varepsilon$$

where

$$w_n = (u_n, k_n) \in W = V_{ad} \times [0,\hat{T}].$$

By assumption [U], there exists a subsequence $\{u_n\} \subseteq V_{ad}$ such that

$$u_n \xrightarrow{w} u_\varepsilon \text{ in } V_{ad} \text{ as } n \to \infty,$$

and V_{ad} is closed and convex, thanks to Mazur Lemma, $u_\varepsilon \in V_{ad}$.
By assumption [B], we have

$$Bu_n \xrightarrow{s} Bu_\varepsilon \text{ in } L_p([0,1],X) \text{ as } n \to \infty.$$

Since k_n is bounded and $k_n > 0$, there also exists a subsequence $\{k_n\}$ denoted by $\{k_n\} \subseteq [0,\hat{T}]$ again, such that

$$k_n \to k_\varepsilon \text{ in } [0,\hat{T}] \text{ as } n \to \infty.$$

Let x_n and x_ε be the mild solutions of system (2.1) corresponding to $w_n = (u_n, k_n) \in W$ and $w_\varepsilon = (u_\varepsilon, k_\varepsilon) \in W$ respectively. Then we have

$$\begin{aligned} x_n(s) &= T_n(s)z_0 + \int_0^s T_n(s-\theta)k_n F_n(\theta)d\theta, \\ x_\varepsilon(s) &= T_\varepsilon(s)z_0 + \int_0^s T_\varepsilon(s-\theta)k_\varepsilon F_\varepsilon(\theta)d\theta, \end{aligned}$$

where

$$\begin{aligned} T_n(\cdot) &\equiv T_{k_n}(\cdot); \quad F_n(\cdot) \equiv f\big(k_n\cdot, x_n(\cdot), B(k_n\cdot)u_n(\cdot)\big); \\ T_\varepsilon(\cdot) &\equiv T_{k_\varepsilon}(\cdot); \quad F_\varepsilon(\cdot) \equiv f\big(k_\varepsilon\cdot, x_\varepsilon(\cdot), B(k_\varepsilon\cdot)u_\varepsilon(\cdot)\big). \end{aligned}$$

By assumptions [F], [B], [U] and Gronwall Lemma, it is easy to verify that there exists a constant $\rho > 0$ such that

$$\|x_\varepsilon\|_{C([0,1],X)} \le \rho \quad \text{and} \quad \|x_n\|_{C([0,1],X)} \le \rho.$$

Further, there exists a constant $M_\varepsilon > 0$ such that

$$\|F_\varepsilon\|_{C([0,1],X)} \le M_\varepsilon(1+\rho).$$

Denote

$$
\begin{aligned}
R_1 &= \|T_n(s)z_0 - T_\varepsilon(s)z_0\|, \\
R_2 &= \left\| \int_0^s T_n(s-\theta)k_n F_n(\theta)d\theta - \int_0^s T_n(s-\theta)k_n F_n^\varepsilon(\theta)d\theta \right\|, \\
R_3 &= \left\| \int_0^s T_n(s-\theta)k_n F_n^\varepsilon(\theta)d\theta - \int_0^s T_\varepsilon(s-\theta)k_\varepsilon F_\varepsilon(\theta)d\theta \right\|,
\end{aligned}
$$

where

$$
F_n^\varepsilon(\theta) \equiv f\big(k_n\theta, x_\varepsilon(\theta), B(k_\varepsilon\theta)u_\varepsilon(\theta)\big).
$$

By assumption [F],

$$
\begin{aligned}
R_2 &\le C_{k_n}k_n \int_0^s \|F_n(\theta) - F_n^\varepsilon(\theta)\|d\theta \\
&\le C_{k_n}k_n L(\rho) \int_0^s \|x_n(\theta) - x_\varepsilon(\theta)\|d\theta \\
&\quad + C_{k_n}k_n L(\rho) \int_0^s \|B(k_n\theta)u_n(\theta) - B(k_\varepsilon\theta)u_\varepsilon(\theta)\|d\theta \\
&\le R_{21} + R_{22} + R_{23}
\end{aligned}
$$

where

$$
\begin{aligned}
C_{k_n} &\equiv Ce^{\omega k_n}, \\
R_{21} &\equiv C_{k_n}k_n L(\rho) \int_0^s \|x_n(\theta) - x_\varepsilon(\theta)\|d\theta, \\
R_{22} &\equiv C_{k_n}k_n L(\rho) \int_0^s \|B(k_n\theta)u_\varepsilon(\theta) - B(k_\varepsilon\theta)u_\varepsilon(\theta)\|d\theta, \\
R_{23} &\equiv C_{k_n}k_n L(\rho) \int_0^s \|B(k_n\theta)u_n(\theta) - B(k_n\theta)u_\varepsilon(\theta)\|d\theta.
\end{aligned}
$$

and

$$
\begin{aligned}
R_3 &\le \int_0^s \|k_n T_n(s-\theta)F_n^\varepsilon(\theta) - k_\varepsilon T_n(s-\theta)F_\varepsilon(\theta)\|d\theta \\
&\quad + k_\varepsilon \int_0^s \|T_n(s-\theta)F_\varepsilon(\theta) - T_\varepsilon(s-\theta)F_\varepsilon(\theta)\|\, d\theta \\
&\le R_{31} + R_{32} + R_{33}
\end{aligned}
$$

where

$$
\begin{aligned}
R_{31} &\equiv C_{k_n} \int_0^s \|k_n F_n^\varepsilon(\theta) - k_n F_\varepsilon(\theta)\|d\theta, \\
R_{32} &\equiv C_{k_n} \int_0^s \|k_n F_\varepsilon(\theta) - k_\varepsilon F_\varepsilon(\theta)\|d\theta, \\
R_{33} &\equiv k_\varepsilon M_\varepsilon(1+\rho) \int_0^s \|T_n(s-\theta) - T_\varepsilon(s-\theta)\|d\theta.
\end{aligned}
$$

By Lemma 3.1, one can obtain $R_1 \to 0$ and $R_{33} \to 0$ as $n \to \infty$ immediately.

It follows from

$$
\begin{aligned}
R_{22} &\le C_{k_n}k_n L(\rho) \left(\int_0^1 \|B(k_n\theta) - B(k_\varepsilon\theta)\|^p d\theta \right)^{\frac{1}{p}} \left(\int_0^1 \|u_\varepsilon(\theta)\|^q d\theta \right)^{\frac{1}{q}}, \\
R_{23} &\le C_{k_n}k_n L(\rho) \int_0^s \|B(k_\varepsilon\theta)u_n(\theta) - B(k_\varepsilon\theta)u_\varepsilon(\theta)\|d\theta \\
&\quad + C_{k_n}k_n L(\rho) \left(\int_0^1 \|B(k_n\theta) - B(k_\varepsilon\theta)\|^p d\theta \right)^{\frac{1}{p}} \left(\int_0^1 \|u_n(\theta) - u_\varepsilon(\theta)\|^q d\theta \right)^{\frac{1}{q}}, \\
R_{31} &\le C_{k_n}k_n \int_0^s \|F_n^\varepsilon(\theta) - F_\varepsilon(\theta)\|d\theta,
\end{aligned}
$$

Lemma 3.2 and assumption [B] that $R_{22} \to 0$, $R_{23} \to 0$ and $R_{31} \to 0$, as $n \to \infty$.

Since $k_n \to k_\varepsilon$ as $n \to \infty$,

$$R_{32} \to 0 \text{ as } n \to \infty.$$

Then, we obtain that

$$\begin{aligned}\|x_n(s) - x_\varepsilon(s)\| &\leq R_1 + R_2 + R_3 \\ &\leq \sigma_\varepsilon + C_{k_n} k_n L(\rho) \int_0^s \|x_n(\theta) - x_\varepsilon(\theta)\| d\theta\end{aligned}$$

where

$$\sigma_\varepsilon = R_1 + R_{22} + R_{23} + R_{31} + R_{32} \to 0 \text{ as } n \to \infty.$$

By Gronwall Lemma, we obtain

$$x_n \xrightarrow{s} x_\varepsilon \text{ in } C([0,1], X) \text{ as } n \to \infty.$$

Thus, there exists a unique control $w_\varepsilon = (u_\varepsilon, k_\varepsilon) \in W$ such that

$$m_\varepsilon = \lim_{n\to\infty} J_\varepsilon(w_n) = J_\varepsilon(w_\varepsilon) \geq m_\varepsilon.$$

This shows that $J_\varepsilon(w)$ attains its minimum at $w_\varepsilon \in W$, and hence x_ε is the solution of system (2.1) corresponding to control w_ε. □

4. Meyer Approximation

In this section, we will show the main result of Meyer approximation of the time optimal control problem (P). In order to make the process clear we divide it into three steps.

Step 1: By Theorem 3.A, there exists $w_\varepsilon = (u_\varepsilon, k_\varepsilon) \in W$ such that $J_\varepsilon(w)$ attains its minimum at $w_\varepsilon \in W$, i.e.,

$$J_\varepsilon(w_\varepsilon) = \frac{1}{2\varepsilon} \|x(w_\varepsilon)(1) - z_1\|^2 + k_\varepsilon = \inf_{w\in W} J_\varepsilon(w).$$

By controllability of problem (P), $V_0 \neq \emptyset$. Take $\tilde{v} \in V_0$ and let $\tau(\tilde{v}) = \tilde{\tau} < +\infty$ then $z(\tilde{v})(\tilde{\tau}) = z_1$. Define $\tilde{u}(s) = \tilde{v}(\tilde{\tau}s)$, $0 \leq s \leq 1$ and $\tilde{w} = (\tilde{u}, \tilde{\tau}) \in W$. Then $\tilde{x}(\cdot) = z(\tilde{v})(\tilde{\tau}\cdot)$ is the mild solution of system (2.1) corresponding to control $\tilde{w} = (\tilde{u}, \tilde{\tau}) \in W$. Of course we have $\tilde{x}(1) = z_1$.

For any $\varepsilon > 0$ submitting $\tilde{w}$ to J_ε, we have

$$J_\varepsilon(\tilde{w}) = \tilde{\tau} \geq J_\varepsilon(w_\varepsilon) = \frac{1}{2\varepsilon} \|x(w_\varepsilon)(1) - z_1\|^2 + k_\varepsilon.$$

This inequality implies that

$$\begin{cases} 0 \leq k_\varepsilon \leq \tilde{\tau}, \\ \|x(w_\varepsilon)(1) - z_1\|^2 \leq 2\varepsilon\tilde{\tau} \text{ hold for all } \varepsilon > 0. \end{cases}$$

We can choose a subsequence $\{\varepsilon_n\}$ such that $\varepsilon_n \to 0$ as $n \to \infty$ and

$$\begin{cases} k_{\varepsilon_n} \to k^0 \text{ in } [0, \hat{T}], \\ x(w_{\varepsilon_n})(1) \equiv x_{\varepsilon_n}(1) \to z_1 \text{ in } X, & \text{as } n \to \infty, \\ u_{\varepsilon_n} \xrightarrow{w} u^0 \text{ in } V_{ad},\ w_{\varepsilon_n} = (u_{\varepsilon_n}, k_{\varepsilon_n}) \in W. \end{cases}$$

Since V_{ad} is closed and convex, thanks to Mazur Lemma again, $u^0 \in V_{ad}$.
Further, by assumption [B], we obtain

$$\begin{cases} k_{\varepsilon_n} \to k^0 \text{ in } [0, \hat{T}], \\ x(w_{\varepsilon_n})(1) \equiv x_{\varepsilon_n}(1) \to z_1 \text{ in } X, & \text{as } n \to \infty, \\ Bu_{\varepsilon_n} \xrightarrow{s} Bu^0 \text{ in } L_p([0,1], X). \end{cases}$$

Step 2: Let x_{ε_n} and x^0 be the mild solutions of system (2.1) corresponding to $w_{\varepsilon_n}=(u_{\varepsilon_n}, k_{\varepsilon_n}) \in W$ and $w^0 = (u^0, k^0) \in W$ respectively. Then we have

$$\begin{aligned} x_{\varepsilon_n}(s) &= T_{\varepsilon_n}(s)z_0 + \int_0^s T_{\varepsilon_n}(s-\theta)k_{\varepsilon_n}F_{\varepsilon_n}(\theta)d\theta, \\ x^0(s) &= T_0(s)z_0 + \int_0^s T_0(s-\theta)k^0F^0(\theta)d\theta, \end{aligned}$$

where

$$\begin{aligned} T_{\varepsilon_n}(s) &\equiv T_{k_{\varepsilon_n}}(s)\ ; \quad F_{\varepsilon_n}(\theta) \equiv f\big(k_{\varepsilon_n}\theta, x_{\varepsilon_n}(\theta), B(k_{\varepsilon_n}\theta)u_{\varepsilon_n}(\theta)\big); \\ T_0(s) &\equiv T_{k^0}(s); \qquad F^0(\theta) \equiv f\big(k^0\theta, x^0(\theta), B(k^0\theta)u^0(\theta)\big). \end{aligned}$$

By assumptions [F], [B], [U] and Gronwall Lemma, it is easy to verify that there exists a constant $\rho > 0$ such that

$$\|x_{\varepsilon_n}\|_{C([0,1],X)} \leq \rho \quad \text{and} \quad \|x^0\|_{C([0,1],X)} \leq \rho.$$

Further, there exists a constant $M^0 > 0$ such that

$$\|F^0\|_{C([0,1],X)} \leq M^0(1+\rho).$$

Denote

$$\begin{aligned} L_1 &= \|T_{\varepsilon_n}(s)z_0 - T_0(s)z_0\|, \\ L_2 &= \left\| \int_0^s T_{\varepsilon_n}(s-\theta)k_{\varepsilon_n}F_{\varepsilon_n}(\theta)d\theta - \int_0^s T_{\varepsilon_n}(s-\theta)k_{\varepsilon_n}F^0_{\varepsilon_n}(\theta)d\theta \right\|, \\ L_3 &= \left\| \int_0^s T_{\varepsilon_n}(s-\theta)k_{\varepsilon_n}F^0_{\varepsilon_n}(\theta)d\theta - \int_0^s T_0(s-\theta)k^0F^0(\theta)d\theta \right\|. \end{aligned}$$

where

$$F^0_{\varepsilon_n}(\theta) \equiv f\big(k_{\varepsilon_n}\theta, x^0(\theta), B(k^0\theta)u^0(\theta)\big).$$

By assumption [F],

$$\begin{aligned} L_2 &\leq C_{k_{\varepsilon_n}}k_{\varepsilon_n}\int_0^s \|F_{\varepsilon_n}(\theta) - F^0_{\varepsilon_n}(\theta)\|d\theta, \\ &\leq C_{k_{\varepsilon_n}}k_{\varepsilon_n}L(\rho)\int_0^s \|x_{\varepsilon_n}(\theta) - x^0(\theta)\|d\theta \\ &\quad + C_{k_{\varepsilon_n}}k_{\varepsilon_n}L(\rho)\int_0^s \|B(k_{\varepsilon_n}\theta)u_{\varepsilon_n}(\theta) - B(k^0\theta)u^0(\theta)\|d\theta, \\ &\leq L_{21} + L_{22} + L_{23}, \end{aligned}$$

where

$$\begin{aligned} C_{k_{\varepsilon_n}} &\equiv Ce^{\omega k_{\varepsilon_n}}, \\ L_{21} &\equiv C_{k_{\varepsilon_n}}k_{\varepsilon_n}L(\rho)\int_0^s \|x_{\varepsilon_n}(\theta) - x^0(\theta)\|d\theta, \\ L_{22} &\equiv C_{k_{\varepsilon_n}}k_{\varepsilon_n}L(\rho)\int_0^s \|B(k_{\varepsilon_n}\theta)u^0(\theta) - B(k^0\theta)u^0(\theta)\|d\theta, \\ L_{23} &\equiv C_{k_{\varepsilon_n}}k_{\varepsilon_n}L(\rho)\int_0^s \|B(k_{\varepsilon_n}\theta)u_{\varepsilon_n}(\theta) - B(k_{\varepsilon_n}\theta)u^0(\theta)\|d\theta. \end{aligned}$$

and

$$\begin{aligned} L_3 &\leq \int_0^s \|T_{\varepsilon_n}(s-\theta)k_{\varepsilon_n}F^0_{\varepsilon_n}(\theta) - T_{\varepsilon_n}(s-\theta)k^0F^0(\theta)\|d\theta \\ &\quad + \int_0^s \big\|T_{\varepsilon_n}(s-\theta)k^0F^0(\theta) - T_0(s-\theta)k^0F^0(\theta)\big\|\, d\theta \\ &\leq L_{31} + L_{32} + L_{33} \end{aligned}$$

where

$$\begin{aligned}
L_{31} &\equiv C_{k_{\varepsilon_n}} \int_0^s \|k_{\varepsilon_n} F^0_{\varepsilon_n}(\theta) - k_{\varepsilon_n} F^0(\theta)\| d\theta, \\
L_{32} &\equiv C_{k_{\varepsilon_n}} \int_0^s \|k_{\varepsilon_n} F^0(\theta) - k^0 F^0(\theta)\| d\theta, \\
L_{33} &\equiv k^0 M^0 (1+\rho) \int_0^s \|T_{\varepsilon_n}(s-\theta) - T_0(s-\theta)\| d\theta.
\end{aligned}$$

Similar to the proof in Theorem 3.A, one can obtain

$$\begin{aligned}
\|x_{\varepsilon_n}(s) - x^0(s)\| &\le L_1 + L_2 + L_3 \\
&\le \sigma^0 + C_{k_{\varepsilon_n}} k_{\varepsilon_n} L(\rho) \int_0^s \|x_{\varepsilon_n}(\theta) - x^0(\theta)\| d\theta
\end{aligned}$$

where

$$\sigma^0 = L_1 + L_{22} + L_{23} + L_{31} + L_{32} \to 0 \text{ as } n \to \infty.$$

Using Gronwall Lemma again, we obtain

$$x_{\varepsilon_n} \xrightarrow{s} x^0 \text{ in } C([0,1], X) \text{ as } n \to \infty.$$

Step 3: It follows from Step 1 and Step 2,

$$\begin{cases} \|x_{\varepsilon_n}(1) - z_1\| \le \sqrt{2\varepsilon_n \tilde{\tau}} \longrightarrow 0, \text{ as } n \to \infty, \\ \|x_{\varepsilon_n}(1) - x^0(1)\| \longrightarrow 0, \text{ as } n \to \infty, \end{cases}$$

and

$$\|x^0(1) - z_1\| \le \|x_{\varepsilon_n}(1) - z_1\| + \|x_{\varepsilon_n}(1) - x^0(1)\| \longrightarrow 0, \text{ as } n \to \infty,$$

that

$$x^0(1) = z_1.$$

It is very clear that $k^0 \neq 0$ unless $z_0 = z_1$. This implies that

$$k^0 > 0.$$

Define $v^0(\cdot) = u^0(\cdot/k^0)$. In fact, $z^0(\cdot) = x^0(\cdot/k^0)$ is the mild solution of system (1.1) corresponding to control $v^0 \in V_0$, then

$$z^0(k^0) = x^0(1) = z_1 \text{ and } \tau(v^0) = k^0 > 0.$$

By the definition of $\tau^* = \inf\{\tau(v) \ge 0 \mid v \in V_0\}$,

$$k^0 \ge \tau^*.$$

For any $v \in V_0$,

$$\tau(v) \ge J_\varepsilon(w_\varepsilon) = \frac{1}{2\varepsilon} \|x(w_\varepsilon)(1) - z_1\|^2 + k_\varepsilon.$$

Thus,

$$\tau(v) \ge k_\varepsilon.$$

Further,

$$\tau(v) \ge k_{\varepsilon_n} \text{ for all } \varepsilon_n > 0.$$

Since k^0 is the limit of k_{ε_n} as $n \to \infty$,

$$\tau(v) \ge \tau(v^0) = k^0 \text{ for all } v \in V_0.$$

Hence,

$$k^0 \le \tau^*.$$

Thus,

$$0 < \tau(v^0) = k^0 = \tau^*.$$

The equality implies that v^0 is an optimal control of Problem (P) and $k^0 > 0$ is just optimal time.

The following conclusion can be seen from the discussion above.

Conclusion: Under the above assumptions, there exists a sequence of Meyer problems (P_{ε_n}) whose corresponding sequence of optimal controls $\{w_{\varepsilon_n}\} \in W$ can approximate the time optimal control problem (P) in some sense. In other words, by limiting process, the sequence of the optimal controls $\{w_{\varepsilon_n}\} \in W$ can be used to find the solution of time optimal control problem (P).

Theorem 4.A: Assumptions [A], [B], [F] and [U] hold. Problem (P) has a solution.

Remark 1: If $B(t) \equiv B$, then Theorem 3.A and Theorem 4.A also hold.

Remark 2: If $B(t)$ does not have strong continuity and semigroup is compact, we will carry out the full details as well as some related problems in a forthcoming paper.

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