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Positive Solutions for Singular m-Point Boundary Value Problems with Sign Changing Nonlinearities Depending on x' *

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Abstract

Using the theory of fixed point theorem in cone, this paper presents the existence of positive solutions for the singular m-point boundary value problem

$$\begin{cases} x''(t) + a(t)f(t, x(t), x'(t)) = 0, 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \end{cases}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1, \alpha_i \in [0, 1),\ i = 1,\ 2,\ \cdots,\ m-2$, with $0 < \sum_{i=1}^{m-2} \alpha_i < 1$ and f may change sign and may be singular at $x = 0$ and $x' = 0$.

Keywords: m-point boundary value problem; Singularity; Positive solutions; Fixed point theorem
Mathematics subject classification: 34B15, 34B10

1. Introduction

The study of multi-point BVP (boundary value problem) for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [3-4]. Since then, many authors studied more general nonlinear multi-point BVP, for examples [2, 5-8], and references therein. In [7], Gupta, Ntouyas, and Tsamatos considered the existence of a $C^1[0, 1]$ solution for the m-point boundary value problem

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), 0 < t < 1, \\ x'(0) = 0, x(1) = \sum_{i=1}^{m-2} a_i x(\xi_i), \end{cases}$$

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where $\xi_i \in (0,1)$, $i = 1, 2, \cdots, m-2$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $a_i \in R$, $i = 1, 2, \cdots, m-2$, have the same sign, $\sum_{i=1}^{m-2} a_i \neq 1$, $e \in L^1[0,1]$, $f : [0,1] \times R^2 \to R$ is a function satisfying Carathéodory's conditions and a growth condition of the form $| f(t,u,v) | \leq p_1(t)|u| + q_1(t)|v| + r_1(t)$ with p_1, q_1, $r_1 \in L^1[0,1]$. Recently, using Leray-Schauder continuation theorem, R.Ma and Donal O'Regan proved the existence of positive solutions of $C^1[0,1)$ solutions for the above BVP, where $f : [0,1] \times R^2 \to R$ satisfies the Carathéodory's conditions (see [8]).

Motivated by the works of [7,8], in this paper, we discuss the equation

$$\begin{cases} x''(t) + a(t)f(t, x(t), x'(t)) = 0, 0 < t < 1, \\ x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \alpha_i x(\xi_i), \end{cases} \tag{1.1}$$

where $0 < \xi_i < 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $\alpha_i \in [0,1)$ with $0 < \sum_{i=1}^{m-2} \alpha_i < 1$ and f may change sign and may be singular at $x = 0$ and $x' = 0$.

Our main features are as follows. Firstly, the nonlinearity af possesses singularity, that is, $a(t)f(t,x,x')$ may be singular at $t = 0$, $t = 1$, $x = 0$ and $x' = 0$; also the degree of singularity in x and x' may be arbitrary(i. e., if f contains $\frac{1}{x^\alpha}$ and $\frac{1}{(-x')^\gamma}$, α and γ may be big enough). Secondly, f is allowed to change sign. Finally, we discuss the maximal and minimal solutions for equations (1.1). Some ideas come from [11-12].

2. Preliminaries

Now we list the following conditions for convenience .

(H$_1$) $\beta, a, k \in C((0,1), R_+), F \in C(R_+, R_+), G \in C(R_-, R_+), ak \in L[0,1]$;

(H$_2$) F is bounded on any interval $[z, +\infty), z > 0$;

(H$_3$) $\int_{-\infty}^{-1} \frac{1}{G(y)} dy = +\infty$;

and the following conditions are satisfied

(P$_1$) $f \in C((0,1) \times R_+ \times R_-, R)$;

(P$_2$) $0 < \sum_{i=1}^{m-2} \alpha_i < 1, 0 < \xi_i < 1$ and $|f(t,x,y)| \leq k(t)F(x)G(y)$;

(P$_3$) There exists $\delta > 0$ such that $f(t,x,y) \geq \beta(t), y \in (-\delta, 0)$;

where $R_+ = (0, +\infty), R_- = (-\infty, 0), R = (-\infty, +\infty)$.

Lemma 2.1[1] Let E be a Banach space, K a cone of E, and $B_R = \{x \in E : \|x\| < R\}$, where $0 < r < R$. Suppose that F: $K \cap \overline{B_R \backslash B_r} = K_{R,r} \to K$ is a completely continuous operator and the following conditions are satisfied

(1) $\|F(x)\| \geq ||x||$ for any $x \in K$ with $\|x\| = r$.

(2) If $x \neq \lambda F(x)$ for any $x \in K$ with $\|x\| = R$ and $0 < \lambda < 1$.

Then F has a fixed point in $K_{R,r}$.

Let $C[0,1]=\{x:[0,1]\to R|x(t)$ is continuous on $[0,1]\}$ with norm $\|y\|=\max\limits_{t\in[0,1]}|y(t)|$. Then $C[0,1]$ is a Banach space.

Lemma 2.2 Let (H_1)-(P_3) hold. For each given natural number $n>0$, there exists $y_n\in C[0,1]$ with $y_n(t)\le -\frac{1}{n}$ such that

$$y_n(t)=-\frac{1}{n}-\int_0^t a(s)f(s,(Ay_n)(s)+\frac{1}{n},y_n(s))ds,\qquad t\in[0,1],\tag{2.1}$$

where

$$(Ay)(t)\ =\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y(\tau)d\tau-\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}-y(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i}-\int_0^t -y(\tau)d\tau,\qquad t\in[0,1].$$

Proof. For $y\in P=\{y\in C[0,1]:y(t)\le 0,t\in[0,1]\}$, define a operator as follows

$$(T_ny)(t)=-\frac{1}{n}+\min\{0,-\int_0^t a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})ds\},\qquad t\in[0,1],\tag{2.2}$$

where $n>0$ is a natural number. For $y\in P$, we have

$$\begin{aligned}
&(Ay)(t)=\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y(\tau)d\tau-\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}-y(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i}-\int_0^t -y(\tau)d\tau\\
&\ge\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y(\tau)d\tau-\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}-y(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i}-\int_0^1 -y(\tau)d\tau\\
&\ge\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y(\tau)d\tau-\frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_{m-2}}-y(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i}\\
&\ge\frac{\sum_{i=1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\int_{\xi_{m-2}}^1 -y(\tau)d\tau\\
&\ge 0,\quad t\in[0,1].
\end{aligned}$$

Let

$$\begin{aligned}
c(y(t))&=-\int_0^t a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})ds,\qquad t\in[0,1],\\
c(y_k(t))&=-\int_0^t a(s)f(s,(Ay_k)(s)+\frac{1}{n},\min\{y_k(s),-\frac{1}{n}\})ds,\qquad t\in[0,1].
\end{aligned}$$

By the equality $\min\{c,0\}=\dfrac{c-|c|}{2}$, it is easy to know

$$(T_ny)(t)=-\frac{1}{n}+\frac{c(y(t))-|c(y(t)|}{2},\qquad t\in[0,1].$$

Let $y_k,y\in P$ with $\lim_{k\to+\infty}\|y_k-y\|=0$. Then, there exists a constant $h>0$, such that $\|y_k\|\le h$ and $\|y\|\le h$. Thus, $|\min\{y_k(s),-\frac{1}{n}\}-\min\{y(s),-\frac{1}{n}\}|\to 0$, uniformly for $s\in[0,1]$ as $k\to+\infty$. Therefore, $|(Ay_k)(s)+\frac{1}{n}-((Ay)(s)+\frac{1}{n})|\to 0$ for all $s\in[0,1]$ as $k\to+\infty$. (P_1) implies that $\{a(s)f(s,(Ay_k)(s)+\frac{1}{n},\min\{y_k(s),-\frac{1}{n}\})\}\to\{a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})\}$, for $s\in(0,1)$ as $k\to+\infty$. By the Lebesgue dominated convergence theorem (the dominating function $a(s)k(s)F[\frac{1}{n},+\infty)G[-h-\frac{1}{n},-\frac{1}{n}])$, we have $\|cy_k - cy\| \to 0$, which yields that

$$\begin{aligned}\|T_n y_k - T_n y\| &= \|\frac{c(y_k)-c(y)-|c(y_k)|+|c(y)|}{2}\| \\ &\le \|\frac{c(y_k)-c(y)+|c(y_k)-c(y)|}{2}\| \\ &\le \|c(y_k)-c(y)\| \to 0, \text{ as } k\to+\infty.\end{aligned}$$

Consequently, T_n is a continuous operator.

Let C be a bounded set in P, i.e., there exists $h_1 > 0$ such that $\|y\| \le h_1$, for any $y \in C$. For any $t_1, t_2 \in [0,1], t_1 < t_2, y \in C$,

$$\begin{aligned}&|(T_n y)(t_2) - (T_n y)(t_1)| \\ &= |\frac{-\int_{t_1}^{t_2} a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\}ds}{2} \\ &+\frac{|\int_0^{t_2} a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n})ds| - |\int_0^{t_1} a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n})ds|}{2}| \\ &\le |\frac{-\int_{t_1}^{t_2} a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\}ds}{2}| \\ &+\frac{|\int_{t_1}^{t_2} a(s)f(s,(Ay)(s),\min\{y(s),-\frac{1}{n}\}ds|}{2} \\ &\le |\int_{t_1}^{t_2} a(s)k(s)ds|\sup F[\frac{1}{n},+\infty)\sup G[-h_1-\frac{1}{n},-\frac{1}{n}].\end{aligned}$$

According to the absolute continuity of the Lebesgue integral, for any $\epsilon > 0$, there exists $\delta > 0$ such that $|\int_{t_1}^{t_2} a(s)k(s)ds| < \epsilon, |t_2 - t_1| < \delta$. Therefore, $\{T_n y, y \in C\}$ is equicontinuous.

$$\begin{aligned}|(T_n y)(t)| &= |-\frac{1}{n}+\min\{0,-\int_0^t a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})ds\}| \\ &\le |\frac{1}{n}|+|\int_0^t a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})ds| \\ &\le 1+\int_0^t a(s)|f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\}|)ds \\ &\le 1+\int_0^1 a(s)k(s)ds\sup F[\frac{1}{n},+\infty)G[-h-\frac{1}{n},\frac{1}{n}], \quad t\in[0,1].\end{aligned}$$

Therefore $\{T_n y, y \in C\}$ is bounded.

Hence T_n is a completely continuous operator.

By (H_3), choose a sufficiently large $R_n > 1$ to fit $\int_{-R_n}^{-1}\frac{dy}{G(y)} > \int_0^1 a(s)k(s)ds\sup F[\frac{1}{n},+\infty)$. For $n > \frac{1}{\delta}$, we prove that

$$y(t) \ne \lambda(T_n y)(t) = \frac{-\lambda}{n}+\lambda\min\{0,-\int_0^t a(s)f(s,(Ay)(s)+\frac{1}{n},\min\{y(s),-\frac{1}{n}\})ds\}, \ t\in[0,1], \tag{2.3}$$

for any $y \in P$ with $||y|| = R_n$ and $0 < \lambda < 1$.

In fact, if there exists $y \in P$ with $\|y\| = R_n$ and $0 < \lambda < 1$ such that

$$y(t) = \frac{-\lambda}{n} + \lambda \min\{0, -\int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds\}, \ t \in [0,1]. \quad (2.4)$$

$y(0) = \dfrac{-\lambda}{n}$. Since $n > \dfrac{1}{\delta}$, we have $-\delta < y(0) < 0$, which implies there exists $\delta_0 > 0$ such that $y(t) > -\delta, t \in (0, \delta_0)$. ($P_3$) implies

$$\int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds > 0, \quad t \in [0,1].$$

Let $t^* = \sup\{s \in [0,1] | \int_0^t a(\tau)f(\tau, (Ay)(\tau) + \frac{1}{n}, \min\{y(\tau), -\frac{1}{n}\})d\tau > 0, 0 \le t \le s\}$.

We show that $t^* = 1$. If $t^* < 1$, we have

$$\begin{cases} \int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds > 0, \ t \in (0, t^*), \\ \int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds = 0, \ t = t^*, \end{cases}$$

$$y(t) = \frac{-\lambda}{n} - \lambda \int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds, \ t \in (0, t^*], \quad (2.5)$$

$$y(t^*) = \frac{-\lambda}{n} > -\delta. \quad (2.6)$$

(2.6) and (P_3) imply there exists $r > 0$ such that $f(t, x, y) \ge \beta(t), t \in (t^* - r, t^*)$. So

$$\begin{aligned} y(t^*) &= \frac{-\lambda}{n} - \lambda \int_0^{t^*} a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds \\ &\le \frac{-\lambda}{n} - \lambda \int_0^{t^*-r} a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds - \lambda \int_{t^*-r}^{t^*} a(s)\beta(s)ds, \end{aligned}$$

$$\int_0^{t^*-r} a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds + \int_{t^*-r}^{t^*} a(s)\beta(s)ds < 0,$$

which is a contradiction. Then, $t^* = 1$. Hence,

$$y(t) = \frac{-\lambda}{n} - \lambda \int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, \min\{y(s), -\frac{1}{n}\})ds, \ t \in [0,1]. \quad (2.7)$$

Since $\|y\| = R_n > 1$ and $y \in P$, there exists a $t_0 \in (0,1)$ with $y(t_0) = -R_n < -1$ and a $t_1 \in (0,1)$ such that $y(t) < -1 < -\frac{1}{n}, t \in (t_0, t_1]$, which together with (2.7) implies that

$$y(t) = \frac{-\lambda}{n} - \lambda \int_0^t a(s)f(s, (Ay)(s) + \frac{1}{n}, y(s))ds, \ t \in (t_0, t_1]. \quad (2.8)$$

Differentiating (2.8) and using (H_2), we obtain

$$-y'(t) = \lambda a(t)f(t, (Ay)(t) + \frac{1}{n}, y(t)) \le a(t)F((Ay)(t) + \frac{1}{n})G(y(t)), \ t \in (t_0, t_1].$$

And then

$$\frac{-y'(t)}{G(y(t))} \leq a(t)k(t)\sup F[(Ay)(t)+\frac{1}{n},+\infty) \leq a(t)k(t)\sup F[\frac{1}{n},+\infty),\ t\in(t_0,t_1). \quad (2.9)$$

Integrating for (2.9) from t_0 to t_1, we have

$$\int_{y(t_0)}^{y(t_1)}\frac{dy}{G(y)} \leq \int_{t_0}^{t_1} a(s)k(s)ds\sup F[\frac{1}{n},+\infty),\ t\in(t_0,t_1). \quad (2.10)$$

Then

$$\int_{-R_n}^{-1}\frac{dy}{G(y)} \leq \int_{-R_n}^{y(t_1)}\frac{dy}{G(y)} \leq \int_{t_0}^{t_1} a(s)k(s)ds\sup F[\frac{1}{n},+\infty) \leq \int_0^1 a(s)k(s)ds\sup F[\frac{1}{n},+\infty),$$

which contradicts

$$\int_{-R_n}^{-1}\frac{dy}{G(y)} > \int_0^1 a(s)k(s)ds\sup F[\frac{1}{n},+\infty).$$

Hence(2.3) holds. Then put $r=\frac{1}{n}$, Lemma 2.1 leads to the desired result. This completes the proof.

Lemma 2.3[10] Let $\{x_n(t)\}$ be an infinite sequence of bounded variation function on $[a,b]$ and $\{x_n(t_0)\}(t_0\in[a,b])$ and $\{V(x_n)\}$ be bounded($V(x)$ denotes the total variation of x). Then there exists a subsequence $\{x_{n_k}(t)\}$ of $\{x_n(t)\}, i\neq j, n_i\neq n_j$, such that $\{x_{n_k}(t)\}$ converges everywhere to some bounded variation function $x(t)$ on $[a,b]$.

Lemma 2.4[9](Zorn) If X is a partially ordered set in which every chain has an upper bound, then X has a maximal element.

3. Main results

Theorem 3.1 Let (H$_1$)-(P$_3$) hold. Then the m-point boundary value problem (1.1) has at least one positive solution.

Proof. Put $M_n=\min\{y_n(t): t\in[0,\xi_{m-2}]\}$, (H$_1$) implies $\gamma=\sup\{M_n\}<0$. In fact, if $\gamma=0$, there exists $n_k>N>0$ such that $M_{n_k}\to 0$ and $-\delta<y_{n_k}<0$. (H$_1$) implies

$$\begin{aligned} y_{n_k}(t) &= -\frac{1}{n}-\int_0^t a(s)f(s,(Ay_{n_k})(s)+\frac{1}{n},y_{n_k}(s))ds \\ &< -\frac{1}{n}-\int_0^t a(s)\beta(s)ds \\ &< -\int_0^t a(s)\beta(s)ds,\quad t\in[0,\xi_{m-2}]. \end{aligned}$$

Then $y_{n_k}(\xi_{m-2})<-\int_0^{\xi_{m-2}} a(s)\beta(s)ds$, which contradicts to $M_{n_k}\to 0$.

Set $\tau=\max\{\gamma,-\delta,-\int_0^{\xi_{m-2}} a(s)\beta(s)\}$. In the remainder of the proof, assume $n>-\frac{1}{\tau}$
.

1). First, we prove there exists a $t_n\in(0,\xi_{m-2}]$ with $y_n(t_n)=\tau$. In fact, since $y_n(0)=-\frac{1}{n}>\tau$, there exists $\delta_0>0$ such that $y_n(t)>\tau, t\in(0,\delta_0)$. Let $t_n=\sup\{t|s\in$

$[0,t], y_n(s) > \tau\}$ .Then $y_n(t_n) = \tau$. If $t_n > \xi_{m-2}$, we have $y_n(t) > \tau > -\delta, t \in [0, \xi_{m-2}]$. (H_1) shows that

$$\begin{aligned} y_n(t) &= -\frac{1}{n} - \int_0^t a(s) f(s, (Ay_n)(s) + \frac{1}{n}, y_n(s)) ds \\ &\le -\frac{1}{n} - \int_0^t a(s)\beta(s) ds \\ &\le -\int_0^t a(s)\beta(s) ds, \quad t \in [0, \xi_{m-2}]. \end{aligned}$$

Then $\tau < y_n(\xi_{m-2}) \le -\int_0^{\xi_{m-2}} a(s)\beta(s) ds < \tau$, which is a contradiction.

Second, we prove

$$y_n(t) \le \tau, \quad t \in [t_n, 1]. \tag{3.1}$$

In fact, if there exists a $t \in (t_n, 1]$ such that $y_n(t) > \tau$, and we choose $t', t'' \in [t_n, 1], t' < t''$ to fit $y_n(t') = \tau, \tau < y_n(t) < -\dfrac{1}{n}, t \in (t', t'']$, from (2.1)

$$0 < \int_{t'}^{t''} a(s) f(s, (Ay_n)(s) + \frac{1}{n}, y_n(s)) ds = y_n(t') - y_n(t'') < 0.$$

This contradiction implies that (3.1) holds. Then

$$\begin{cases} y_n(t) \le -\int_0^t a(s)\beta(s) ds, & t \in [0, t_n], \\ y_n(t) \le \tau, & t \in [t_n, 1]. \end{cases}$$

Let $W(t) = \max\{-\int_0^t a(s)\beta(s) ds, \tau\}, t \in (0,1)$. Obviously, $W(t)$ is bounded on $[\dfrac{1}{3k}, 1 - \dfrac{1}{3k}]$ and $y_n(t) \le W(t), t \in [0,1]$.

2). $\{y_n(t)\}$ is equicontinuous on $[\dfrac{1}{3k}, 1 - \dfrac{1}{3k}]$($k \ge 1$ is a natural number) and uniformly bounded on $[0,1]$.

Notice that

$$\begin{aligned} (Ay_n)(t) + \frac{1}{n} &= \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_n(\tau) d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_n(\tau) d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_n(\tau) d\tau + \frac{1}{n} \\ &> \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_\xi^1 -y_n(\tau) d\tau \ge \frac{\sum_{i=1}^{m-2} \alpha_i}{1 - \sum_{i=1}^{m-2} \alpha_i} (-\tau)(1 - \xi) = \Theta, t \in [0,1]. \end{aligned}$$

We know from (2.9)

$$\int_{y_n(t)}^{-\frac{1}{n}} \frac{dy_n}{G(y_n)} \le \int_0^t a(s) k(s) ds \sup F[\Theta, +\infty), t \in [0,1]. \tag{3.2}$$

Now (H_3) and (3.2) show that $\omega(t) = \inf\{y_n(t)\} > -\infty$ is bounded on $[0,1]$. On the other hand, it follows from (2.1) and (3.1) that

$$|y_n'(t)| \le k(t) a(t) \sup F[\Theta, +\infty) \sup G[\omega_k, \max\{\tau, W(\frac{1}{3k})\}], \ (n \ge k), \tag{3.3}$$

where $\omega_k = \inf\{\omega(t), t \in [\frac{1}{3k}, 1-\frac{1}{3k}]\}$. Thus (3.3) and the absolute continuity of Lebesgue integral show that $\{y_n(t)\}$ is equicontinuous on $[\frac{1}{3k}, 1-\frac{1}{3k}]$. Now the Arzela-Ascoli theorem guarantees that there exists a subsequence of $\{y_n^{(k)}(t)\}$, which converges uniformly on $[\frac{1}{3k}, 1-\frac{1}{3k}]$. When $k=1$, there exists a subsequence $\{y_n^{(1)}(t)\}$ of $\{y_n(t)\}$, which converges uniformly on $[\frac{1}{3}, \frac{2}{3}]$.When $k=2$, there exists a subsequence $\{y_n^{(2)}(t)\}$ of $\{y_n^{(1)}(t)\}$, which converges uniformly on $[\frac{1}{6}, \frac{5}{6}]$. In general, there exists a subsequence $\{y_n^{(k+1)}(t)\}$ of $\{y_n^{(k)}(t)\}$, which converges uniformly on $[\frac{1}{3(k+1)}, 1-\frac{1}{3(k+1)}]$. Then the diagonal sequence $\{y_k^{(k)}(t)\}$ converges pointwise in $(0,1)$ and it is easy to verify that $\{y_k^{(k)}(t)\}$ converges uniformly on any interval $[c,d] \subseteq (0,1)$. Without loss of generality, let $\{y_k^{(k)}(t)\}$ be itself of $\{y_n(t)\}$ in the rest. Put $y(t) = \lim_{n\to\infty} y_n(t), t \in (0,1)$. Then $y(t)$ is continuous on $(0,1)$ and since $y_n(t) \le W(t) < 0$, we have $y(t) \le 0, t \in (0,1)$.

3) Now (3.2) shows

$$\sup\{\max\{-y_n(t), t \in [0,1]\}\} < +\infty.$$

We have

$$\lim_{t\to 0^+} \sup\{\int_0^t -y_n(s)ds\} = 0, \quad \lim_{t\to 1^-} \sup\{\int_t^1 -y_n(s)ds\} = 0, \quad t \in [0,1], \tag{3.4}$$

and

$$\begin{aligned}
(Ay_n)(t) &= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_n(\tau)d\tau - \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} -y_n(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i} - \int_0^t -y_n(\tau)d\tau \\
&< \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_n(\tau)d\tau \\
&< +\infty, \ t \in [0,1].
\end{aligned} \tag{3.5}$$

Since (3.4) and (3.5) hold, the Fatou theorem of the Lebesgue integral implies $(Ay)(t) < +\infty$, for any fixed $t \in (0,1)$.

4) $y(t)$ satisfies the following equation

$$y(t) = -\int_0^t a(s)f(s,(Ay)(s),y(s))ds, \quad t \in (0,1). \tag{3.6}$$

Since $y_n(t)$ converges uniformly on $[a,b] \subset (0,1)$, (3.4) implies that $(Ay_n)(s)$ converges to $(Ay)(s)$ for any $s \in (0,1)$. For fixed $t \in (0,1)$ and any d , $0 < d < t$, we have

$$y_n(t) - y_n(d) = -\int_d^t a(s)f(s,(Ay_n)(s)+\frac{1}{n}, y_n(s))ds. \tag{3.7}$$

for all $n > k$. Since $y_n(s) \le \max\{\tau, W(d)\}$, $(Ay_n)(s) + \frac{1}{n} \ge \Theta$, $s \in [d,t]$, $\{(Ay_n)(s)\}$ and $\{y_n(s)\}$ are bounded and equicontinuous on $[d,t]$

$$y(t) - y(d) = -\int_d^t a(s)f(s,(Ay)(s),y(s))ds. \tag{3.8}$$

Putting $t = d$ in (3.2), we have

$$\int_{y_n(d)}^{-\frac{1}{n}} \frac{dy_n}{G(y_n)} \leq \int_0^d a(s)k(s)ds \sup F[\Theta, +\infty). \tag{3.9}$$

Letting $n \to \infty$ and $d \to 0^+$, we obtain

$$y(0^+) = \lim_{d \to 0^+} y(d) = 0.$$

Letting $d \to 0^+$ in (3.8), we have

$$y(t) = -\int_0^t a(s)f(s, (Ay)(s), y(s))ds, \quad t \in (0, 1), \tag{3.10}$$

and

$$(Ay)(1) = \sum_{i=1}^{m-2} \alpha_i (Ay)(\xi_i).$$

Hence $x(t) = (Ay)(t)$ is a positive solution of (1.1). □

Theorem 3.2 Suppose that (H_1)-(P_3) hold. Then the set of positive solutions of (1.1) is compact in $C^1[0, 1]$.

Proof Let $M = \{y \in C[0, 1]$: $(Ay)(t)$ is a positive solution of equation (1.1) $\}$. We show that

(1) M is not empty;

(2) M is relatively compact(bounded, equicontinuous);

(3) M is closed.

Obviously, Theorem 3.1 implies M is not empty.

First, we show that $M \subset C[0, 1]$ is relatively compact. For any $y \in M$, differentiating (3.10) and using (H_2), we obtain

$$\begin{aligned} -y'(t) &= a(t)f(t, (Ay)(t), y(t)) \\ &\leq a(t)|f(t, (Ay)(t), y(t))| \\ &\leq a(t)k(t)F[\Theta, +\infty)G(y(t)), \ t \in (0, 1), \end{aligned}$$

$$\begin{aligned} \frac{-y'(t)}{G(y(t))} &\leq a(t)k(t) \sup F[(Ay)(t), +\infty) \\ &\leq a(t)k(t) \sup F[\Theta, +\infty), t \in [0, 1]. \end{aligned} \tag{3.11}$$

Integrating for (3.11) from 0 to t, we have

$$\int_{y(t)}^0 \frac{dy}{G(y)} \leq \int_0^1 a(s)k(s)ds \sup F[\Theta, +\infty), t \in [0, 1]. \tag{3.12}$$

Now (H_3) and (3.12) show that for any $y \in M$, there exists $K > 0$ such that $|y(t)| < K$,$\forall\ t \in [0, 1]$. Then M is bounded.

For any $y \in M$, we obtain from (3.11)

$$\begin{aligned} -y'(t) &= a(t)f(t, (Ay)(t), y(t)) \\ &\leq a(t)|f(t, (Ay)(t), y(t))| \\ &\leq a(t)k(t)F[\Theta, +\infty)G(y(t)), \ t \in (0, 1), \end{aligned}$$

and

$$\begin{aligned} y'(t) &= -a(t)f(t,(Ay)(t),y(t)) \\ &\leq a(t)|f(t,(Ay)(t),y(t))| \\ &\leq a(t)k(t)F[\Theta,+\infty)G(y(t)),\ t\in(0,1), \end{aligned}$$

which yields

$$\frac{-y'(t)}{G(y(t))+1} \leq a(t)k(t)\sup F[\Theta,+\infty),\ t\in(0,1), \tag{3.13}$$

and

$$\frac{y'(t)}{G(y(t))+1} \leq a(t)k(t)\sup F[\Theta,+\infty),\ t\in(0,1). \tag{3.14}$$

Notice that the rights are always positive in (3.13) and (3.14). Let $I(y(t)) = \int_0^{y(t)} \frac{dy}{G(y)+1}$. For any $t_1, t_2 \in [0,1]$, integrating for (3.13) and (3.14) from t_1 to t_2 ,we obtain

$$|I(y(t_1)) - I(y(t_2))| \leq \int_{t_1}^{t_2} a(t)k(t)F[\Theta,+\infty)dt. \tag{3.15}$$

Since I^{-1} is uniformly continuous on $[I(-K),0]$, for any $\overline{\epsilon}>0$, there is a $\epsilon'>0$ such that

$$|I^{-1}(s_1) - I^{-1}(s_2)| < \overline{\epsilon}, \forall |s_1-s_2|<\epsilon', s_1,s_2\in[I(-K),0]. \tag{3.16}$$

And (3.15) guarantees that for $\epsilon'>0$, there is a $\delta'>0$ such that

$$|I(y(t_1)) - I(y(t_2))| < \epsilon', \forall |t_1-t_2|<\delta', t_1,t_2\in[0,1]. \tag{3.17}$$

Now (3.16) and (3.17) yield that

$$|y(t_1)-y(t_2)| = |I^{-1}(I(y(t_1)) - I^{-1}(I(y(t_2))| < \overline{\epsilon},\ t_1,t_2\in[0,1], \tag{3.18}$$

which means that M is equicontinuous. So M is relatively compact.

Second, we show that M is closed. Suppose that $\{y_n\}\subseteq M$ and $\lim\limits_{n\to+\infty}\max\limits_{t\in[0,1]}|y_n(t)-y_0(t)|=0$. Obviously $y_0\in C[0,1]$ and $\lim\limits_{n\to+\infty}(Ay_n)(t)=(Ay_0)(t)$, $t\in[0,1]$. Moreover,

$$\begin{aligned} (Ay_n)(t) &= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_n(\tau)d\tau - \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i}-y_n(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i} - \int_0^t -y_n(\tau)d\tau \\ &< \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_n(\tau)d\tau \\ &< \frac{K}{1-\sum_{i=1}^{m-2}\alpha_i}, t\in[0,1]. \end{aligned} \tag{3.19}$$

For $y_n\in M$, from (3.10) we obtain

$$y_n(t) = -\int_0^t a(s)f(s,(Ay_n)(s),y_n(s))ds,\quad t\in(0,1). \tag{3.20}$$

For fixed $t\in(0,1)$, there exists $0<d<t$ such that

$$y_n(t)-y_n(d) = -\int_d^t a(s)f(s,(Ay_n)(s),y_n(s))ds. \tag{3.21}$$

Since $y_n(s) \le \max\{\tau, W(d)\}, (Ay_n)(s) \ge \Theta, s \in [d,t]$, the Lebesgue Dominated Convergence Theorem yields that

$$y_0(t) - y_0(d) = -\int_d^t a(s)f(s,(Ay_0)(s),y_0(s))ds, \;\; t \in (0,1). \tag{3.22}$$

From (3.10), we have

$$\begin{aligned} -y_n'(t) &= a(t)f(t,(Ay_n)(s),y_n(s)) \\ &\le a(t)k(t)F[\Theta,+\infty)G(y_n(t)), \quad t \in (0,1), \end{aligned}$$

which yields

$$\frac{-y_n'(t)}{G(y_n(t))} \le a(t)k(t)\sup F[\Theta,+\infty), \;\; t \in (0,1).$$

Integrating from 0 to d

$$\int_{y_n(d)}^0 \frac{dy_n}{G(y_n)} \le \int_0^d a(s)k(s)ds \sup F[\Theta,+\infty). \tag{3.23}$$

Letting $n \to \infty$ and $d \to 0^+$, we obtain

$$y_0(0^+) = \lim_{d \to 0^+} y_0(d) = 0.$$

Letting $d \to 0^+$ in (3.22), we have

$$y_0(t) = -\int_0^t a(s)f(s,(Ay_0)(s),y_0(s))ds, \quad t \in (0,1), \tag{3.24}$$

and

$$(Ay_0)(1) = \sum_{i=1}^{m-2} \alpha_i (Ay_0)(\xi_i).$$

Then $x_0(t) = (Ay_0)(t)$ is a positive solution of (1.1). So $y_0 \in M$ and M is a closed set.

Hence $\{Ay, y \subseteq M\} \in C^1[0,1]$ is compact.

Theorem 3.3 Suppose (H_1)-(P_3) hold. Then (1.1) has a minimal positive solution and a maximal positive solution in $C^1[0,1]$.

Proof. Let $\Omega = \{x(t) : x(t)$ is a $C^1[0,1]$ positive solution of (1.1)$\}$. Theorem 3.1 implies that is nonempty. Define a partially ordered $\le$ in $\Omega : x \le y$ iff $x(t) \le y(t)$ for any $t \in [0,1]$. We prove only that any chain in $< \Omega, \le >$ has a lower bound in Ω. The rest is obtained from Zorn's lemma. Let $\{x_\alpha(t)\}$ be a chain in $< \Omega, \le >$. Since $C[0,1]$ is a separable Banach space, there exists countable set at most $\{x_n(t)\}$, which is dense in $\{x_\alpha(t)\}$. Without loss of generality, we may assume that $\{x_n(t)\} \subseteq \{x_\alpha(t)\}$. Put $z_n(t) = \min\{x_1(t), x_2(t), \cdots, x_n(t)\}$. Since $\{x_\alpha(t)\}$ is a chain, $z_n(t) \in \Omega$ for any n (in fact, $z_n(t)$ equals one of $x_n(t)$) and $z_{n+1}(t) \le z_n(t)$ for any n. Put $z(t) = \lim_{m \to +\infty} z_n(t)$. We prove that $z(t) \in \Omega$.

By Lemma 2.2, there exists $y_n(t)$ (e.g., $y_n(t)$ may be $z_n'(t)$), which is a solution of

$$(Ty)(t) = -\int_0^t a(s)f(s,(Ay)(s),y(s))ds \qquad t \in [0,1],$$

such that

$$z_n(t) \;=\; \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_n(\tau)d\tau - \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} -y_n(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i} - \int_0^t -y_n(\tau)d\tau.$$

(3.2) imply that $\{\|y_n\|\}$ is bounded. From Lemma 2.3, there exists a subsequence $\{y_{n_k}(t)\}$ of $\{y_n(t)\}$, $i \neq j, n_i \neq n_j$, which converges everywhere on [0, 1]. Without loss of generality, let $\{y_{n_k}(t)\}$ be itself of $\{y_n(t)\}$. Put $y_0(t) = \lim\limits_{m\to+\infty} y_n(t), t \in [0,1]$. Use $y_n(t), y_0(t)$, and 0 in place of $y_n(t), y(t)$, and $1/n$ in Theorem 3.1, respectively. A similar argument to show Theorem 3.1 yields that $y_0(t)$ is a solution of

$$y(t) = -\int_0^t a(s)f(s,(Ay)(s),y_n(s))ds, \qquad t \in [0,1].$$

The boundedness of $\{\|y_n\|\}$ leads to

$$\begin{aligned} z(t) \;&=\; \lim_{m\to+\infty} z_n(t) \\ &= \lim_{m\to+\infty}[\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_n(\tau)d\tau - \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} -y_n(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i} - \int_0^t -y_n(\tau)d\tau] \\ &= \frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\int_0^1 -y_0(\tau)d\tau - \frac{\sum_{i=1}^{m-2}\alpha_i\int_0^{\xi_i} -y_0(\tau)d\tau}{1-\sum_{i=1}^{m-2}\alpha_i} - \int_0^t -y_0(\tau)d\tau. \end{aligned}$$

Hence $z \in \Omega$. By Lemma 2.2, for any $x \in \{x_\alpha\}$, there exists $\{x_{n_k}\} \subseteq \{x_n\}$ such that $\|x_{n_k} - x\| \to 0$. Notice that $x_{n_k}(t) \geq z_{n_k}(t) \geq z(t), t \in [0,1]$. Letting $k \to +\infty$, we have $x(t) \geq z(t)$, $t \in [0,1]$; i.e.,$\{x_\alpha\}$ has lower boundedness in Ω. Zorn's lemma shows that (1.1) has a minimal $C^1[0,1]$ positive solution. By a similar proof, we can get the a maximal $C^1[0,1]$ positive solution. The proof is complete.

Theorem 3.4 Suppose that (H_1)-(P_3) hold , $f(t,x,z)$ is decreasing in x for all $(t,z) \in [0,1]\times R_-$, $a(0)f(0,x,z) \neq 0$ and $\lim\limits_{t\to 0} f(t,x,y) \neq +\infty$. Then (1.1) has an unique positive solution in $C^1[0,1]$.

Proof. Assume that x_1 and x_2 are two positive different solutions to (1.1), i.e., there exists $t_0 \in (0,1]$ such that $x_1(t_0) \neq x_2(t_0)$. Without loss of generality, assume that $x_1(t_0) > x_2(t_0)$. Let $\varphi(t) = x_1(t) - x_2(t)$ for all $t \in [0,1]$. Obviously, $\varphi \in C[0,1]\cap C^1(0,1]$ with $\varphi(t_0) > 0$.

Let $t_* = \inf\{0 < t < t_0 | \varphi(s) > 0$ for all $s \in t \in [t,t_0]\}$ and $t^* = \sup\{t_0 < t < 1 | \varphi(s) > 0$ for all $s \in t \in [t_0,t]\}$. It is easy to see that $\varphi(t) > 0$ for all $t \in (t_*,t^*)$ and φ has maximum in $[t_*,t^*]$. Let t' satisfying that $\varphi(t') = \max_{t\in[t_*,t^*]}\varphi(t)$. There are three cases: (1) $t' \in (t_*,t^*)$; (2) $t' = t^* = 1$;(3) $t' = 0$.

(1) $t' \in (t_*,t^*)$. It is easy to see that $\varphi''(t') \leq 0$ and $\varphi'(t') = 0$. Then $\varphi''(t') = x_1''(t') - x_2''(t')$

$$= -a(t')f(t',x_1(t'),x_1'(t')) + a(t')f(t',x_2(t'),x_2'(t')) > 0,$$

a contradiction.

(2) $t' = t^* = 1$. Since $t' = t^* = 1$, we have $\sum\limits_{i=1}^{m-2} \alpha_i \max\{\varphi(\xi_i)\} > \sum\limits_{i=1}^{m-2} \alpha_i \varphi(\xi_i) = \varphi(1)$, a contradiction to $0 < \sum\limits_{i=1}^{m-2} \alpha_i < 1$.

(3) $t' = 0$. Since $t' = 0$ and x_1 and x_2 are solutions, the proof of lemma 2.2 implies that there exist $x_{n,1}$ and $x_{n,2}$ such that

$$\|x_{n,1} - x_1\| < \frac{\varphi(0)}{2}, \quad \|x_{n,2} - x_2\| < \frac{\varphi(0)}{2}$$

where

$$x_{n,1}(t) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_{n,1}(\tau)d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_{n,1}(\tau)d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_{n,1}(\tau)d\tau, \qquad t \in [0,1],$$

$$x_{n,2}(t) = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i} \int_0^1 -y_{n,2}(\tau)d\tau - \frac{\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} -y_{n,2}(\tau)d\tau}{1 - \sum_{i=1}^{m-2} \alpha_i} - \int_0^t -y_{n,2}(\tau)d\tau, \qquad t \in [0,1],$$

and

$$y_{n,1}(t) = -\frac{1}{n} - \int_0^t a(s) f(s, x_{n,1}(s) + \frac{1}{n}, y_{n,1}(s))ds, \quad t \in [0,1],$$
$$y_{n,2}(t) = -\frac{1}{n} - \int_0^t a(s) f(s, x_{n,2}(s) + \frac{1}{n}, y_{n,2}(s))ds, \quad t \in [0,1],$$

$y_{n,1}(t) \le -\frac{1}{n}$, $y_{n,2}(t) \le -\frac{1}{n}$ for all $t \in [0,1]$.

By a similar proof with above, there exists $t_1 \in (0,1]$ such that $x_{n,1}(t_1) \neq x_{n,2}(t_1)$. Without loss of generality, assume that $x_{n,1}(t_1) > x_{n,2}(t_1)$. Let $\varphi_n(t) = x_{n,1}(t) - x_{n,2}(t)$ for all $t \in [0,1]$. Obviously, $\varphi_n \in C[0,1] \cap C^1(0,1]$ with $\varphi_n(t_1) > 0$. Let $t_* = \inf\{0 < t < t_1 | \varphi_n(s) > 0 \text{ for all } s \in t \in [t, t_1]\}$ and $t^* = \sup\{t_1 < t < 1 | \varphi_n(s) > 0 \text{ for all } s \in t \in [t_1, t]\}$. It is easy to see that $\varphi_n(t) > 0$ for all $t \in (t_{1*}, t^{1*})$ and φ_n has maximum in $[t_{1*}, t^{1*}]$. Let t'' satisfying that $\varphi(t'') = \max_{t \in [t_{1*}, t^{1*}]} \varphi(t)$. There are three cases: 1) $t'' \in (t_{1*}, t^{1*})$; 2) $t'' = t^* = 1$; 3) $t'' = 0$.

The proof of 1) and 2) are similar with (1) and (2).

3) $t'' = 0$. We have $\varphi_n(t) < \varphi_n(0)$, $t \in (0,1]$, $\varphi_n'(0) = 0$, $\varphi_n'(t_\xi) < 0$, $t_\xi \in (0,1)$. Then

$$\underline{\lim}_{t_\xi \to 0+} \varphi_n''(t) = \underline{\lim}_{t_\xi \to 0+} \frac{\varphi_n'(t_\xi) - \varphi_n'(0)}{t_\xi - 0} \le 0.$$

On the other hand, since $\varphi_n''(0) = x_{n,1}''(0) - x_{n,2}''(0)$

$$= -a(0) f(0, x_{n,1}(0) + \frac{1}{n}, x_{n,1}'(0)) + a(0) f(0, x_{n,2}(0) + \frac{1}{n}, x_{n,2}'(0)) > 0,$$

a contradiction. Then (1.1) has at most one solution. The proof is complete.

Example 3.1. In (1.1), let $f(t,x,y) = k(t)[1 + x^{-\gamma} + (-y)^{-\sigma} - (-y)\ln(-y)]$, $a(t) = t^{-\frac{1}{3}}$, and

$$k(t) = t^{-\frac{1}{2}}, \quad 0 < t < 1,$$

where $\gamma > 0, \sigma < -2$, and let $F(x) = 1 + x^{-\gamma}, G(y) = 1 + (-y)^{-\sigma} - (-y)\ln(-y)$. Then

$$f(t,x,y) \leq k(t)F(x)G(y), \quad \delta = 1, \quad \beta(t) = k(t),$$

and

$$\int_{-\infty}^{-1} \frac{dy}{G(y)} = +\infty.$$

By Theorem 3.1, (1.1) at least has a positive solution and Corollary 3.1 implies the set of solutions is compact.

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