Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **42**, 1-16 ; **http://www.math.u-szeged.hu/ejqtde/**

Positive solutions of singular four-point boundary value problem with p-Laplacian *

Chunmei Miao[1,2†] Huihui Pang[3] Weigao Ge[2]

[1] *College of Science, Changchun University, Changchun 130022, PR China.*

[2] *Department of Mathematics, Beijing Institute of Technology, Beijing 100081, PR China.*

[3] *College of Science, China Agricultural University, Beijing 100083, PR China.*

Email: miaochunmei@yahoo.com.cn, phh2000@163.com , gew@bit.edu.cn

Abstract

In this paper, we deal with the following singular four-point boundary value problem with p-Laplacian

$$\begin{cases} (\phi_p(u'(t)))' + q(t)f(t,u(t)) = 0, \ t \in (0,1), \\ u(0) - \alpha u'(\xi) = 0, \ u(1) + \beta u'(\eta) = 0, \end{cases}$$

where $f(t,u)$ may be singular at $u = 0$ and $q(t)$ may be singular at $t = 0$ or 1. By imposing some suitable conditions on the nonlinear term f, existence results of at least two positive solutions are obtained. The proof is based upon theory of Leray-Schauder degree and Krasnosel'skii's fixed point theorem.

Keywords. Singular, Four-point boundary value problem, Multiple positive solutions, p-Laplacian, Leray-Schauder degree, Fixed point theorem

2000 *Mathematics Subject Classification.* 34B10, 34B16, 34B18

1 Introduction

Krasnosel'skii M A, Positive solutions of operator equations Singular boundary value problems (BVPs) arise in a variety of problems in applied mathematics and physics such as gas dynamics, nuclear physics, chemical reactions, studies of atomic structures, and atomic calculations [1]. They also arise in the study of positive radial solutions of a

*Supported by NNSF of China (10671012) and SRFDP of China (20050007011).

†Corresponding author.

nonlinear elliptic equations. Such problems have been studied extensively in recent years, see, for instance, [2-8] and references therein. At the very beginning, most literature in this area concentrated on singular two-point boundary value problems. More recently, several authors begin to pay attention to singular multi-point boundary value problems [9-17]. In the existing literatures, the following multi-point boundary conditions

$$\begin{array}{ll}
u(0)=0,\ u(1)=\beta u(\eta); & u(0)=\alpha u(\xi),\ u(1)=0;\\
u(0)=0,\ u(1)=\sum\limits_{i=1}^{m-2}\beta_i u(\eta_i); & u(0)=\sum\limits_{i=1}^{m-2}\alpha_i u(\xi_i),\ u(1)=0;\\
u'(0)=0,\ u(1)=\sum\limits_{i=1}^{m-2}\beta_i u(\eta_i); & u(0)=\sum\limits_{i=1}^{m-2}\alpha_i u(\xi_i),\ u'(1)=0;\\
u'(0)=0,\ u(1)=u(\eta); & u(0)=\sum\limits_{i=1}^{m-2}\alpha_i u(\xi_i),\ u'(1)=\sum\limits_{i=1}^{m-2}\beta_i u'(\eta_i);\\
u(0)=\alpha u(\xi),\ u(1)=\beta u(\eta); & u(0)=\sum\limits_{i=1}^{m-2}\alpha_i u(\xi_i),\ u(1)=\sum\limits_{i=1}^{m-2}\beta_i u(\eta_i),
\end{array}$$

where $\alpha,\ \beta,\ \alpha_i,\ \beta_i>0,\ 0<\xi,\ \eta,\ \xi_i,\ \eta_i<1(i=1,2,\cdots,m-1)$, have been studied.

However, to our knowledge, there are few papers investigating the singular four-point boundary value problem. The aim of the present paper is to fill this gap.

In this paper, we establish sufficient conditions which guarantee the existence theory for single and multiple positive solutions to the following singular four-point BVP

$$\begin{cases}(\phi_p(u'(t)))'+q(t)f(t,u(t))=0,\ t\in(0,1),\\ u(0)-\alpha u'(\xi)=0,\ u(1)+\beta u'(\eta)=0,\end{cases}\tag{1.1}$$

where $\phi_p(s)=|s|^{p-2}s,\ p>1,\ (\phi_p)^{-1}=\phi_q,\ \frac{1}{p}+\frac{1}{q}=1,\ \alpha>0,\ \beta>0,\ 0<\xi<\eta<1.$ $f(t,u)$ may be singular at $u=0$ and $q(t)$ may be singular at $t=0$ or 1.

Existence results for one solution are obtained by using the existence principle guaranteed by the property of Leray-Schauder degree, and for two solutions by using a fixed point theorem in cones. In order to obtain the positivity of solution, $u(0)\geq 0,\ u(1)\geq 0$ is required. While note the boundary conditions $u(0)=\alpha u'(\xi),\ u(1)=-\beta u'(\eta)$, so we need to ensure $u'(\xi)\geq 0,\ u'(\eta)\leq 0$. Hence it is vital that the maximum of this solution must be achieved between ξ and η. To this end, we need to establish some suitable conditions on the nonlinear term f (see (H_4)), in course of that, we overcome more difficulties since the singularity of the nonlinear term f.

Throughout, we always suppose the following conditions are satisfied:

(H_{1a}) $q\in\mathbb{C}(0,1)\cap\mathbb{L}^1[0,1]$ with $q(t)\geq 0,\ q(t)\not\equiv 0$ on any subinterval of $[0,1]$ and nondecreasing on $(0,1)$;

(H_{1b}) $q \in \mathbb{C}(0,1) \cap \mathbb{L}^1[0,1]$ with $q(t) \geq 0$, $q(t) \not\equiv 0$ on any subinterval of $[0,1]$ and nonincreasing on $(0,1)$;

(H_2) $f : [0,1] \times (0,+\infty) \to (0,+\infty)$ is a continuous function;

(H_3) $0 < f(t,u) \leq f_1(u) + f_2(u)$ on $[0,1] \times (0,+\infty)$ with $f_1 > 0$ continuous, nonincreasing on $(0,+\infty)$ and $\int_0^L f_1(u)du < +\infty$ for any fixed $L > 0$; $f_2 \geq 0$ is continuous on $[0,+\infty)$; $\frac{f_2}{f_1}$ nondecreasing on $(0,+\infty)$;

(H_4) There exists $R > 0$ such that

$$\int_0^\xi q(t)N(t)dt \leq \Gamma \int_\xi^\eta q(t)n(t)dt, \quad \int_\eta^1 q(t)N(t)dt \leq \Gamma \int_\xi^\eta q(t)n(t)dt,$$

where $\Gamma = (\min\{\frac{\alpha}{1-\eta} + \frac{\xi}{1-\eta}, \frac{\beta}{\xi} + \frac{1-\eta}{\xi}\})^{p-1}$, $n(t) = \inf\limits_{u\in(0,R]}\{f(t,u),\ t \in [\xi,\eta]\}$, $N(t) = \sup\limits_{u\in(0,R]}\{f(t,u),\ t \in [0,\xi] \cup [\eta,1]\}$;

(H_5) For each constant $H > 0$, there exists a function ψ_H continuous on $[0,1]$ and positive on $(0,1)$ such that $f(t,u) \geq \psi_H(t)$ on $(0,1) \times (0,H]$;

(H_6) $\int_0^\xi f_1(k_1 s)q(s)ds + \int_\eta^1 f_1(k_2(1-s))q(s)ds < +\infty$ for any $k_1 > 0, k_2 > 0$.

2 Preliminaries

Consider the Banach space $X = \mathbb{C}[0,1]$ with the maximum norm $||u|| = \max_{t\in[0,1]} |u(t)|$. By a positive solution $u(t)$ to BVP(1.1) we mean that $u(t)$ satisfies (1.1), $u \in \mathbb{C}^1[0,1]$, $(\phi_p(u'))' \in \mathbb{C}(0,1) \cap \mathbb{L}^1[0,1]$ and $u(t) > 0$ on $(0,1)$.

We suppose $F : [0,1] \times \mathbb{R} \to (0,+\infty)$ is continuous, $q \in \mathbb{C}(0,1) \cap \mathbb{L}^1[0,1]$ with $q(t) \geq 0$ on $(0,1)$ and $q(t) \not\equiv 0$ on any subinterval of $[0,1]$. For any $x \in X$, we consider the following BVP

$$\begin{cases} (\phi_p(u'(t)))' + q(t)F(t,x(t)) = 0,\ t \in (0,1), \\ u(0) - \alpha u'(\xi) = a,\ u(1) + \beta u'(\eta) = a, \end{cases} \tag{2.1}$$

where a is a fixed positive constant. Then we have

Lemma 2.1. ([18]) *For any $x \in X$, BVP (2.1) has a unique solution $u(t)$ which can be expressed as*

$$u(t) = \begin{cases} \alpha\phi_q(\int_\xi^\sigma q(s)F(s,x(s))ds) + \int_0^t \phi_q(\int_s^\sigma q(\tau)F(\tau,x(\tau))d\tau)ds + a,\ 0 \leq t \leq \sigma, \\ \beta\phi_q(\int_\sigma^\eta q(s)F(s,x(s))ds) + \int_t^1 \phi_q(\int_\sigma^s q(\tau)F(\tau,x(\tau))d\tau)ds + a,\ \sigma \leq t \leq 1, \end{cases} \tag{2.2}$$

where σ is the unique solution of the equation $\upsilon_1(t) - \upsilon_2(t) = 0,\ 0 < t < 1$, in which

$$\begin{aligned} \upsilon_1(t) &= \alpha\phi_q(\int_\xi^\sigma q(s)F(s,x(s))ds) + \int_0^t \phi_q(\int_s^\sigma q(\tau)F(\tau,x(\tau))d\tau)ds, \\ \upsilon_2(t) &= \beta\phi_q(\int_\sigma^\eta q(s)F(s,x(s))ds) + \int_t^1 \phi_q(\int_\sigma^s q(\tau)F(\tau,x(\tau))d\tau)ds. \end{aligned} \tag{2.3}$$

Lemma 2.2. *For any $x \in X$, assume that*

$$\int_0^\xi q(t)F(t,x(t))dt \le \Gamma \int_\xi^\eta q(t)F(t,x(t))dt \tag{2.4}$$

and

$$\int_\eta^1 q(t)F(t,x(t))dt \le \Gamma \int_\xi^\eta q(t)F(t,x(t))dt, \tag{2.5}$$

where $\Gamma = (\min\{\frac{\alpha}{1-\eta} + \frac{\xi}{1-\eta}, \frac{\beta}{\xi} + \frac{1-\eta}{\xi}\})^{p-1}$. If $u(t)$ is a solution of BVP (2.1), then

(i) *$u(t)$ is concave on $[0,1]$;*

(ii) *There exists $\sigma \in [\xi,\eta]$ such that $u'(\sigma) = 0$, $u(\sigma) = ||u||$; $u(t) \ge a$ for $t \in [0,1]$, $u(t)$ is nondecreasing on $[0,\xi]$ and nonincreasing on $[\eta,1]$;*

(iii) *$u(t) \ge \omega(t)||u||$ for $t \in [0,1]$, where $\omega(t) = \min\{\frac{1}{\eta}t, \frac{1}{1-\xi}(1-t)\}$.*

Proof. Suppose $u(t)$ is a solution to BVP(2.1), then

(i) $(\phi_p(u'(t)))' = -q(t)F(t,x(t)) \le 0$, so $\phi_p(u'(t))$ is nonincreasing on $[0,1]$. Therefore, $u'(t)$ is nonincreasing on $[0,1]$ which implies the concavity of $u(t)$.

(ii) From Lemma 2.1, we know that there exists $\sigma \in (0,1)$ such that $u'(\sigma) = 0$. Now we show that $\sigma \in [\xi,\eta]$. If not, then there exists $\sigma \in (0,\xi)$ such that $u'(\sigma) = 0$. By Lemma 2.1 and (2.4), we have

$$\begin{aligned} u(\sigma) &= \alpha\phi_q(\int_\xi^\sigma q(s)F(s,x(s))ds) + \int_0^\sigma \phi_q(\int_s^\sigma q(\tau)F(\tau,x(\tau))d\tau)ds \\ &< \int_0^\xi \phi_q(\int_0^\xi q(\tau)F(\tau,x(\tau))d\tau)ds \\ &\le \xi\phi_q(\Gamma\int_\xi^\eta q(\tau)F(\tau,x(\tau))d\tau) \\ &\le \beta\phi_q(\int_\xi^\eta q(\tau)F(\tau,x(\tau))d\tau) + (1-\eta)\phi_q(\int_\xi^\eta q(s)F(s,x(s))ds) \\ &\le \beta\phi_q(\int_\sigma^\eta q(s)F(s,x(s))ds) + \int_\sigma^1 \phi_q(\int_\sigma^s q(\tau)F(\tau,x(\tau))d\tau)ds \\ &= u(\sigma) \end{aligned}$$

which is a contradiction. So, $\sigma \notin (0,\xi)$. Similarly, $\sigma \notin (\eta,1)$. The concavity of $u(t)$ guarantees that $u(t)$ is nondecreasing on $[0,\xi]$ and nonincreasing on $[\eta,1]$. By the boundary conditions, we have $u(0) = \alpha u'(\xi) + a \ge a,\ u(1) = -\beta u'(\eta) + a \ge a$. Therefore, for any $t \in [0,1]$, $u(t) \ge 0$ since $u(t)$ is concave on [0,1].

(iii) Since $u(t)$ is nondecreasing on $[0,\sigma]$ and nonincreasing on $[\sigma,1]$ for $\sigma\in[\xi,\eta]$, we have

$$\frac{u(t)}{t}\geq\frac{u(\sigma)}{\sigma}\geq\frac{1}{\eta}||u||,\qquad t\in[0,\sigma],$$
$$\frac{u(t)}{1-t}\geq\frac{u(\sigma)}{1-\sigma}\geq\frac{1}{1-\xi}||u||,\quad t\in[\sigma,1].$$

Let $\omega(t)=\min\{\frac{1}{\eta}t,\frac{1}{1-\xi}(1-t)\}$, it follows that

$$u(t)\geq\omega(t)||u||,\ t\in[0,1].$$

The proof is completed. □

We shall consider the following boundary value problem

$$\begin{cases}(\phi_p(u'(t)))'+q(t)F(t,u(t))=0,\ t\in(0,1),\\ u(0)-\alpha u'(\xi)=a,\ u(1)+\beta u'(\eta)=a.\end{cases}\tag{2.6}$$

Define an operator $T:X\to X$ by

$$(Tu)(t)=\begin{cases}\alpha\phi_q(\int_\xi^\sigma q(s)F(s,u(s))ds)+\int_0^t\phi_q(\int_s^\sigma q(\tau)F(\tau,u(\tau))d\tau)ds+a,\ 0\leq t\leq\sigma,\\ \beta\phi_q(\int_\sigma^\eta q(s)F(s,u(s))ds)+\int_t^1\phi_q(\int_\sigma^s q(\tau)F(\tau,u(\tau))d\tau)ds+a,\ \sigma\leq t\leq 1.\end{cases}\tag{2.7}$$

We have the following result:

Lemma 2.3. ([18], Lemma 3.1) *$T:X\to X$ is completely continuous.*

Now we state an existence principle which plays an important role in our proof of existence results for one solution.

Lemma 2.4. (Existence principle) *Assume that there exists a constant $M>a$ independent of λ, such that for $\lambda\in(0,1)$, $||u||\neq M$, where $u(t)$ satisfies*

$$\begin{cases}(\phi_p(u'(t)))'+\lambda q(t)F(t,u(t))=0,\ t\in(0,1),\\ u(0)-\alpha u'(\xi)=a,\ u(1)+\beta u'(\eta)=a.\end{cases}\tag{2.7_λ}$$

Then $(2.7)_1$ has at least one solution $u(t)$ with $||u||\leq M$.

Proof. For any $\lambda\in[0,1]$, define an operator

$$N_\lambda u(t)=\begin{cases}\lambda\alpha\phi_q(\int_\xi^\sigma q(s)F(s,u(s))ds)+\int_0^t\phi_q(\int_s^\sigma q(\tau)\lambda F(\tau,u(\tau))d\tau)ds+a,\ 0\leq t\leq\sigma,\\ \lambda\beta\phi_q(\int_\sigma^\eta q(s)F(s,u(s))ds)+\int_t^1\phi_q(\int_\sigma^s q(\tau)\lambda F(\tau,u(\tau))d\tau)ds+a,\ \sigma\leq t\leq 1.\end{cases}$$

By Lemma 2.3, $N_\lambda : X \to X$ is completely continuous. It can be verified that a solution of BVP $(2.7)_\lambda$ is equivalent to a fixed point of N_λ in X. Let $\Omega = \{u \in X : ||u|| < M\}$, then Ω is an open set in X. If there exists $u \in \partial\Omega$ such that $N_1 u = u$, then $u(t)$ is a solution of $(2.7)_1$ with $||u|| \le M$. Thus the conclusion is true. Otherwise, for any $u \in \partial\Omega$, $N_1(u) \neq u$. If $\lambda = 0$, for $u \in \partial\Omega$, $(I - N_0)u(t) = u(t) - N_0 u(t) = u(t) - a \not\equiv 0$ since $||u|| = M > a$. For $\lambda \in (0, 1)$, if there is a solution $u(t)$ to BVP $(2.7)_\lambda$, by the assumption, one gets $||u|| \neq M$, which is a contradiction to $u \in \partial\Omega$.

In a word, for any $u \in \partial\Omega$ and $\lambda \in [0, 1]$, $N_\lambda u \neq u$. Homotopy invariance of Leray-Schauder degree deduce that

$$\mathrm{Deg}\{I - N_1, \Omega, 0\} = \mathrm{Deg}\{I - N_0, \Omega, 0\} = 1.$$

Hence, N_1 has a fixed point u in Ω. And BVP $(2.7)_1$ has a solution $u(t)$ with $||u|| \le M$. The proof is completed. □

To obtain two positive solutions of BVP (1.1), we need the following well-known fixed point theorem of compression and expansion of cones [19].

Theorem 2.5. (Krasnosel'skii [19, p.148]) *Let X be a Banach space and $P(\subset X)$ be a cone. Assume that are Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : \overline{\Omega}_2 \backslash \Omega_1 \cap P \to P$ be a continuous and compact operator such that either*

(i)$\|Tx\| \leqslant \|x\|$, $\forall x \in \partial\Omega_1 \cap P$ *and* $\|Tx\| \geqslant \|x\|$, $\forall x \in \partial\Omega_2 \cap P$; *or*

(ii)$\|Tx\| \leqslant \|x\|$, $\forall x \in \partial\Omega_2 \cap P$ *and* $\|Tx\| \geqslant \|x\|$, $\forall x \in \partial\Omega_1 \cap P$.

Then T has a fixed point theorem in $(\overline{\Omega}_2 \setminus \Omega_1) \cap P$.

3 Existence of positive solutions

First we give some notations.

Denote

$G(c) = \int_0^c [f_1(u) + f_2(u)]du$, $I(c) = \int_0^c \phi_q(t)dt = (\frac{p-1}{p})c^{\frac{p}{p-1}}$ for $c > 0$.

Clearly, $G(c)$ is increasing in c, $I^{-1}(c)$ exists and $I^{-1}(c) = (\frac{p}{p-1})^{\frac{p-1}{p}} c^{\frac{p-1}{p}}$. Since $p > 1$, $(\frac{p}{p-1})^{\frac{p-1}{p}} > 1$, we have $I^{-1}(uv) \le I^{-1}(u)I^{-1}(v)$ for any $u > 0,\ v > 0$. For $c < 0$, it is easy to see that $I(-c) = I(c)$.

We now give the main results for BVP (1.1) in this paper.

Theorem 3.1. *Suppose* (H_{1a})-(H_6) *hold. Furthermore, we assume that*

(H$_7$) *there exists* $r > 0$ *such that*

$$\frac{r}{M_{\beta,\eta}(r)} > 1,$$

where

$$M_{\beta,\eta}(r) = \phi_q(I^{-1}(G(r)))[\beta\phi_q(I^{-1}(q(\eta))) + \int_0^1 \phi_q(I^{-1}(q(t)))dt].$$

Then BVP (1.1) *has a positive solution* $u(t)$ *with* $||u|| \leq r$.

Proof. From (H$_7$), we choose $r > 0$ and $0 < \varepsilon < r$ such that

$$\frac{r}{\varepsilon + M_{\beta,\eta}(r)} > 1. \tag{3.1}$$

Let $n_0 \in \{1, 2, 3, \cdots\}$ satisfying that $\frac{1}{n_0} \leq \varepsilon$. Set $\mathbb{N}_0 = \{n_0, n_0 + 1, n_0 + 2, \cdots\}$.

In what follows, we show that the following BVP

$$\begin{cases} (\phi_p(u'(t)))' + q(t)f(t, u(t)) = 0, \ t \in (0, 1), \\ u(0) - \alpha u'(\xi) = \dfrac{1}{m}, \ u(1) + \beta u'(\eta) = \dfrac{1}{m} \end{cases} \tag{3.2}$$

has a solution for each $m \in \mathbb{N}_0$.

In order to obtain a solution of BVP (3.2) for each $m \in \mathbb{N}_0$, we consider the following BVP

$$\begin{cases} (\phi_p(u'(t)))' + q(t)f^*(t, u(t)) = 0, \ t \in (0, 1), \\ u(0) - \alpha u'(\xi) = \dfrac{1}{m}, \ u(1) + \beta u'(\eta) = \dfrac{1}{m}, \end{cases} \tag{$3.2)_m$}$$

where

$$f^*(t, u) = \begin{cases} f(t, u), & u \geq \dfrac{1}{m}, \\ f(t, \dfrac{1}{m}), & u \leq \dfrac{1}{m}. \end{cases}$$

Clearly, $f^* \in \mathbb{C}([0, 1] \times \mathbb{R}, (0, +\infty))$.

To obtain a solution of BVP $(3.2)_m$ for each $m \in \mathbb{N}_0$, by applying Lemma 2.4, we consider the family of BVPs

$$\begin{cases} (\phi_p(u'(t)))' + \lambda q(t)f^*(t, u(t)) = 0, \ t \in (0, 1), \\ u(0) - \alpha u'(\xi) = \dfrac{1}{m}, \ u(1) + \beta u'(\eta) = \dfrac{1}{m}, \end{cases} \tag{$3.2)_m^\lambda$}$$

where $\lambda \in [0, 1]$. Let $u(t)$ be a solution of $(3.2)_m^\lambda$. From (H$_4$) and Lemma 2.2, we observe that $u(t)$ is concave, $u(t) \geq \frac{1}{m}$ on [0,1] and there exists $\hat{\sigma} \in [\xi, \eta]$ such that $u(\hat{\sigma}) = ||u||$, $u'(\hat{\sigma}) = 0$, $u'(t) \geq 0, \ t \in [0, \hat{\sigma}]$ and $u'(t) \leq 0, \ t \in [\hat{\sigma}, 1]$.

For $t \in [\hat{\sigma}, 1]$ and $\lambda \in (0, 1)$, in view of (H$_3$), we have

$$0 \leq -(\phi_p(u'(t)))' = \lambda q(t)f^*(t, u(t)) = \lambda q(t)f(t, u(t)) \leq q(t)[f_1(u(t)) + f_2(u(t))]. \tag{3.3}$$

Multiplying (3.3) by $-u'(t)$, for $t \in [\hat{\sigma}, 1]$, it follows that

$$(\phi_p(u'(t)))'\phi_q(\phi_p(u'(t))) \leq q(t)[f_1(u(t)) + f_2(u(t))](-u'(t)). \tag{3.4}$$

Integrating (3.4) from $\hat{\sigma}$ to t ($t \geq \hat{\sigma}$), by (H_1) and the fact that $G(c)$ is increasing on c, we obtain

$$\begin{aligned}\int_0^{\phi_p(u'(t))} \phi_q(z)dz &\leq q(t)\int_{u(t)}^{u(\hat{\sigma})} [f_1(z) + f_2(z)]dz \\ &= q(t)[G(u(\hat{\sigma})) - G(u(t))] \\ &\leq q(t)G(u(\hat{\sigma})),\end{aligned}$$

this implies

$$I(-\phi_p(u'(t))) = I(\phi_p(u'(t))) \leq q(t)G(u(\hat{\sigma})),$$

$$0 \leq -u'(t) \leq \phi_q(I^{-1}(q(t)))\phi_q(I^{-1}(G(u(\hat{\sigma})))), \tag{3.5}$$

therefore,

$$0 \leq -u'(\eta) \leq \phi_q(I^{-1}(q(\eta)))\phi_q(I^{-1}(G(u(\hat{\sigma})))).$$

Integrating (3.5) from $\hat{\sigma}$ to 1, by the boundary condition of $(3.2)^{\lambda}_{m}$, we have

$$\begin{aligned}u(\hat{\sigma}) &\leq \frac{1}{m} + \phi_q(I^{-1}(G(u(\hat{\sigma}))))[\beta\phi_q(I^{-1}(q(\eta))) + \int_0^1 \phi_q(I^{-1}(q(t)))dt] \\ &\leq \varepsilon + \phi_q(I^{-1}(G(u(\hat{\sigma}))))[\beta\phi_q(I^{-1}(q(\eta))) + \int_0^1 \phi_q(I^{-1}(q(t)))dt].\end{aligned}$$

Hence,

$$\frac{u(\hat{\sigma})}{\varepsilon + \phi_q(I^{-1}(G(u(\hat{\sigma}))))[\beta\phi_q(I^{-1}(q(\eta))) + \int_0^1 \phi_q(I^{-1}(q(t)))dt]} \leq 1,$$

i.e.,

$$\frac{u(\hat{\sigma})}{\varepsilon + M_{\beta,\eta}(u(\hat{\sigma}))} \leq 1. \tag{3.6}$$

Due to (3.1) and (3.6), we have $u(\hat{\sigma}) = ||u|| \neq r$. Further, Lemma 2.4 implies that $(3.2)_m$ has at least a solution $u^m \in \mathbb{C}^1[0, 1]$ and $(\phi_p((u^m)'))' \in \mathbb{C}(0, 1) \cap \mathbb{L}^1[0, 1]$ with $||u^m|| \leq r$ (independent of m) for any fixed m. From Lemma 2.2, we note that $u^m(t) \geq \frac{1}{m} > 0$. So $f^*(t, u^m(t)) = f(t, u^m(t))$. Therefore, $u^m(t)$ is a solution to BVP (3.2).

Using Arzelà-Ascoli theorem, we shall show that BVP (1.1) has at least a positive solution $u(t)$ which satisfies $\lim\limits_{m\to\infty} u^m(t) = u(t)$ for $t \in (0, 1)$. Note that

$$0 < \frac{1}{m} \leq u^m(t) \leq r, \quad t \in [0, 1].$$

(H_5) implies that there is a continuous function $\psi_r : (0, 1) \to (0, +\infty)$(independent of m) satisfying

$$f(t, u^m(t)) \geq \psi_r(t),\ t \in (0, 1),$$

hence,

$$-(\phi_p(u^m(t))')' \geq \psi_r(t)q(t),\ t \in (0,1). \tag{3.7}$$

For any $m \in \mathbb{N}_0$, by Lemma 2.2, there exists $t^m \in [\xi, \eta]$ such that $u^m(t^m) = ||u^m||,\ (u^m)'(t^m) = 0,\ (u^m)'(t) \geq 0$ for $t \in [0, t^m]$ and $(u^m)'(t) \leq 0$ for $t \in [t^m, 1]$.

If $t \in [0, \xi]$, integrating (3.7) from t to t^m, we obtain

$$\phi_p((u^m)'(t)) \geq \int_t^{t^m} q(s)\psi_r(s)ds. \tag{3.8}$$

Integrating (3.8) from 0 to t, we have

$$\begin{aligned} u^m(t) &\geq \frac{1}{m} + \int_0^t \phi_q(\int_s^{t^m} q(\tau)\psi_r(\tau)d\tau)ds \\ &\geq \int_0^t \phi_q(\int_s^{\xi} q(\tau)\psi_r(\tau)d\tau)ds, \end{aligned}$$

further,

$$u^m(\xi) \geq \int_0^{\xi} \phi_q(\int_s^{\xi} q(\tau)\psi_r(\tau)d\tau)ds = \theta_1 > 0.$$

Since $u^m(t)$ is concave in $[0, \xi]$, we have

$$\frac{u^m(t)}{t} \geq \frac{u^m(\xi)}{\xi} \Rightarrow u^m(t) \geq \frac{u^m(\xi)}{\xi}t \geq \frac{\theta_1}{\xi}t. \tag{3.9}$$

If $t \in [\eta, 1]$, integrating (3.7) from t^m to t, we obtain

$$-\phi_p((u^m)'(t)) \geq \int_{t^m}^t q(s)\psi_r(s)ds. \tag{3.10}$$

Integrating (3.10) from t to 1, we have

$$\begin{aligned} u^m(t) &\geq \frac{1}{m} + \int_t^1 \phi_q(\int_{t^m}^s q(\tau)\psi_r(\tau)d\tau)ds \\ &\geq \int_t^1 \phi_q(\int_{t^m}^s q(\tau)\psi_r(\tau)d\tau)ds, \end{aligned}$$

therefore,

$$u^m(\eta) \geq \int_\eta^1 \phi_q(\int_\eta^s q(\tau)\psi_r(\tau)d\tau)ds = \theta_2 > 0.$$

Since $u^m(t)$ is concave in $[\eta, 1]$, we get

$$\frac{u^m(t)}{1-t} \geq \frac{u^m(\eta)}{1-\eta} \Rightarrow u^m(t) \geq \frac{u^m(\eta)}{1-\eta}(1-t) \geq \frac{\theta_2}{1-\eta}(1-t). \tag{3.11}$$

If $t \in [\xi, \eta]$, in view of concavity of $u^m(t)$, we have

$$u^m(t) \geq \min\{u^m(\xi), u^m(\eta)\} \geq \theta, \tag{3.12}$$

where $\theta = \min\{\theta_1, \theta_2\}$.

Let

$$\delta(t)=\begin{cases}\frac{\theta}{\xi}t, & t\in[0,\xi],\\ \theta, & t\in[\xi,\eta],\\ \frac{\theta}{1-\eta}(1-t), & t\in[\eta,1],\end{cases} \tag{3.13}$$

where $\theta=\min\{\theta_1,\theta_2\}$.

By (3.9), (3.11), (3.12) and (3.13), for any $m\in\mathbb{N}_0$, we have

$$u^m(t)\geq\delta(t),\ t\in[0,1].$$

At the same time, it follows from (H_3) and (H_6) that

$$\begin{aligned}0\leq-(\phi_p(u^m)'(t))'=q(t)f(t,u^m(t))&\leq q(t)[f_1(u^m(t))+f_2(u^m(t))]\\ &\leq q(t)f_1(\delta(t))+\max_{0\leq\tau\leq r}f_2(\tau)q(t).\end{aligned}$$

Thus,

$$|\phi_p((u^m)'(t))|\leq\int_0^1 q(s)f_1(\delta(s))ds+\max_{0\leq\tau\leq r}f_2(\tau)\int_0^1 q(s)ds$$

and

$$|(u^m)'(t)|\leq\phi_q(\int_0^1 q(s)f_1(\delta(s))ds+\max_{0\leq\tau\leq r}f_2(\tau)\int_0^1 q(s)ds)<+\infty. \tag{3.14}$$

Therefore, $\{u^m(t)\}_{m\in\mathbb{N}_0}$ is equi-continuous on [0,1]. Furthermore, from the fact that

$$0<u^m(t)\leq r,\ t\in[0,1],$$

we have $\{u^m(t)\}_{m\in\mathbb{N}_0}$ is uniformly bounded on [0,1].

The Arzelà-Ascoli theorem guarantees that there is a subsequence $\mathbb{N}^*\subset\mathbb{N}_0$, a function $u\in\mathbb{C}^1[0,1]$ satisfying $u^m(t)\to u(t)$ uniformly on $[0,1]$ and $t^m\to\sigma$ as $m\to+\infty$ in $\mathbb{N}^*$. From the definition of $u^m(t)$, we have

$$u^m(t)=\begin{cases}\alpha\phi_q(\int_\xi^{t^m}q(s)f(s,u^m(s))ds)+\int_0^t\phi_q(\int_s^{t^m}q(\tau)f(\tau,u^m(\tau))d\tau)ds+\frac{1}{m},\ 0\leq t\leq t^m,\\ \beta\phi_q(\int_{t^m}^{\eta}q(s)f(s,u^m(s))ds)+\int_t^1\phi_q(\int_{t^m}^{s}q(\tau)f(\tau,u^m(\tau))d\tau)ds+\frac{1}{m},\ t^m\leq t\leq 1.\end{cases} \tag{3.15}$$

Let $m\to+\infty$ in $\mathbb{N}^*$ in (3.15), by the continuity of f and Lebesgue's dominated convergence theorem, we get

$$u(t)=\begin{cases}\alpha\phi_q(\int_\xi^{\sigma}q(s)f(s,u(s))ds)+\int_0^t\phi_q(\int_s^{\sigma}q(\tau)f(\tau,u(\tau))d\tau)ds,\ 0\leq t\leq\sigma,\\ \beta\phi_q(\int_{\sigma}^{\eta}q(s)f(s,u(s))ds)+\int_t^1\phi_q(\int_{\sigma}^{s}q(\tau)f(\tau,u(\tau))d\tau)ds,\ \sigma\leq t\leq 1,\end{cases}$$

hence,

$$\begin{cases} (\phi_p(u'(t)))' + q(t)f(t,u(t)) = 0, \ t \in (0,1), \\ u(0) - \alpha u'(\xi) = 0, \ u(1) + \beta u'(\eta) = 0. \end{cases}$$

From $\delta(t) \leq u^m(t) \leq r, \ t \in [0,1]$, we have $\delta(t) \leq u(t) \leq r, \ t \in [0,1]$. So $u(t) > 0, \ t \in (0,1)$ and $-(\phi_p(u'(t)))' = q(t)f(t,u) \leq q(t)f_1(\delta(t)) + \max_{0\leq\tau\leq r} f_2(\tau)q(t) \in \mathbb{L}^1[0,1]$. Therefore, $(\phi_p(u'(t)))' \in \mathbb{L}^1[0,1]$ which means $u(t)$ is a positive solution to BVP (1.1). The proof is completed. □

Theorem 3.2. *Suppose* (H_{1b})-(H_6) *hold. Furthermore, we assume that*

(H_8) *there exists* $r > 0$ *such that*

$$\frac{r}{M_{\alpha,\xi}(r)} > 1,$$

where

$$M_{\alpha,\xi}(r) = \phi_q(I^{-1}(G(r)))[\alpha\phi_q(I^{-1}(q(\xi))) + \int_0^1 \phi_q(I^{-1}(q(t)))dt].$$

Then BVP (1.1) *has a positive solution* $u(t)$ *with* $||u|| \leq r$.

Theorem 3.3. *Suppose* (H_{1a})-(H_7) *hold. Furthermore, we assume that*

(H_9) *there exists a fixed constant* $\delta \in (0, \frac{1}{2})$ *with* $[\xi,\eta] \subset [\delta, 1-\delta]$, *and* $\mu \in \mathbb{C}[\delta, 1-\delta]$ *with* $\mu > 0$ *on* $[\delta, 1-\delta]$ *such that*

$$q(t)f(t,u) \geq \mu(t)[f_1(u) + f_2(u)] \ \text{on} \ [\delta, 1-\delta] \times (0, +\infty);$$

(H_{10}) *there exists* $R > r$ *such that*

$$\frac{2R\phi_q(f_1(\omega_1(\delta)R))}{\phi_q[f_1(R)f_1(\omega_1(\delta)R) + f_1(R)f_2(\omega_1(\delta)R)]} \leq b_0,$$

where

$$b_0 = \min\{\alpha,\beta\}\min\{1, 2^{2-q}\}\phi_q(\int_\xi^\eta \mu(s)ds),$$
$$\omega_1(\delta) = \min\{\frac{\delta}{\eta}, \frac{\delta}{1-\xi}\}.$$

Then BVP (1.1) *has a positive solution* $u_1(t)$ *with* $r < ||u_1|| \leq R$.

Proof. Let

$$K = \{u \in X : u(t) \geq 0, \ u(t) \text{ is concave}, \ u(t) \geq \omega(t)||u||, \ t \in [0,1]\}.$$

Obviously, K is a cone of X. Since $r < R$, denote open subsets Ω_1 and Ω_2 of X:

$$\Omega_1 = \{u \in X : ||u|| < r\}, \quad \Omega_2 = \{u \in X : ||u|| < R\}.$$

Let $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to X$ be defined by

$$(Au)(t) = \begin{cases} \alpha\phi_q(\int_\xi^\sigma q(s)f(s,u(s))ds) + \int_0^t \phi_q(\int_s^\sigma q(\tau)f(\tau,u(\tau))d\tau)ds, \ 0 \le t \le \sigma, \\ \beta\phi_q(\int_\sigma^\eta q(s)f(s,u(s))ds) + \int_t^1 \phi_q(\int_\sigma^s q(\tau)f(\tau,u(\tau))d\tau)ds, \ \sigma \le t \le 1. \end{cases} \tag{3.16}$$

It is easy to verify that the fixed points of operator A are the positive solutions of BVP (1.1). So it suffices to show that A has at least one fixed point.

First we prove $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$. If $u \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$, differentiating (3.16) for t, we obtain

$$(Au)'(t) = \begin{cases} \phi_q(\int_t^\sigma q(\tau)f(\tau,u(\tau))d\tau), \ 0 \le t \le \sigma, \\ -\phi_q(\int_\sigma^t q(\tau)f(\tau,u(\tau))d\tau), \ \sigma \le t \le 1 \end{cases}$$

and

$$(Au)''(t) = \begin{cases} -(q-1)(\int_t^\sigma q(\tau)f(\tau,u(\tau))d\tau)^{q-2}q(t)f(t,u(t)), \ 0 < t \le \sigma, \\ -(q-1)(\int_\sigma^t q(\tau)f(\tau,u(\tau))d\tau)^{q-2}q(t)f(t,u(t)), \ \sigma \le t < 1. \end{cases} \tag{3.17}$$

In view of (3.16) and (3.17), note that $(Au)''(t) \le 0$ for any $t \in (0,1)$, and $Au(0) \ge 0$, $Au(1) \ge 0$. Therefore, $Au(t)$ is concave and $Au(t) \ge 0$ on $[0,1]$. By Lemma 2.2, we have $Au(t) \ge \omega(t)||Au||$. Consequently, $Au \in K$, i.e., $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$.

By Lemma 2.3, we have that $A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ is completely continuous.

Now we shall show that

$$||Au|| < ||u|| \ \text{ for } \ u \in K \cap \partial\Omega_1. \tag{3.18}$$

Let $u \in K \cap \partial\Omega_1$, then $||u|| = r$. As in the proof of (3,7), we have

$$\begin{aligned} ||Au|| &= (Au)(\sigma) \\ &\le \phi_q(I^{-1}(G(r)))[\beta\phi_q(I^{-1}(q(\eta))) + \int_0^1 \phi_q(I^{-1}(q(t)))dt] \\ &= M_{\beta,\eta}(r) \\ &< r = ||u||. \end{aligned}$$

Consequently, $||Au|| < ||u||$. So (3.18) holds.

Furthermore, we give that

$$||Au|| \ge ||u|| \ \text{ for } \ u \in K \cap \partial\Omega_2. \tag{3.19}$$

Let $u \in K \cap \partial\Omega_2$, so $||u|| = R$ and $u(t) \geq \omega(t)R$ for $t \in [0,1]$. In particular, $u(t) \in [\omega_1(\delta)R, R]$ for $t \in [\delta, 1-\delta]$, where $\omega_1(\delta) = \min\{\frac{\delta}{\eta}, \frac{\delta}{1-\xi}\}$ (δ is the same as in (H_9)). By (H_3), (H_7) and (H_8), we obtain

$$
\begin{aligned}
||Au|| =&(Au)(\sigma)\\
=&\frac{1}{2}[\alpha\phi_q(\int_\xi^\sigma q(s)f(s,u(s))ds) + \int_0^\sigma \phi_q(\int_s^\sigma q(\tau)f(\tau,u(\tau))d\tau)ds\\
&+\beta\phi_q(\int_\sigma^\eta q(s)f(s,u(s))ds) + \int_\sigma^1 \phi_q(\int_\sigma^s q(\tau)f(\tau,u(\tau))d\tau)ds]\\
\geq&\frac{1}{2}[\alpha\phi_q(\int_\xi^\sigma q(s)f(s,u(s))ds) + \int_\delta^\sigma \phi_q(\int_s^\sigma q(\tau)f(\tau,u(\tau))d\tau)ds\\
&+\beta\phi_q(\int_\sigma^\eta q(s)f(s,u(s))ds) + \int_\sigma^{1-\delta} \phi_q(\int_\sigma^s q(\tau)f(\tau,u(\tau))d\tau)ds]\\
\geq&\frac{1}{2}\phi_q(f_1(R))\phi_q[1+\frac{f_2(\omega_1(\delta)R)}{f_1(\omega_1(\delta)R)}][\alpha\phi_q(\int_\xi^\sigma \mu(s)ds) + \int_\delta^\sigma \phi_q(\int_s^\sigma \mu(\tau)d\tau)ds\\
&+\beta\phi_q(\int_\sigma^\eta \mu(s)ds) + \int_\sigma^{1-\delta} \phi_q(\int_\sigma^s \mu(\tau)d\tau)ds]\\
\geq&\frac{1}{2}\phi_q(f_1(R))\phi_q[1+\frac{f_2(\omega_1(\delta)R)}{f_1(\omega_1(\delta)R)}]\min\{\alpha,\beta\}\min\{1,2^{2-q}\}\phi_q(\int_\xi^\eta \mu(s)ds)\\
\geq&R = ||u||.
\end{aligned}
$$

Consequently, $||Au|| \geq ||u||$. So (3.19) holds. By Theorem 2.5, we can obtain that A has a fixed point $u_1 \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ with $r < ||u_1|| \leq R$ and $u_1(t) \geq \omega(t)r$ for $t \in [0,1]$. Thus $u_1(t) > 0$ for $t \in (0,1)$. The proof is completed. □

Theorem 3.4. *Suppose* (H_{1a})-(H_7), (H_9) *and* (H_{10}) *hold. Then BVP* (1.1) *has two positive solutions* $u(t)$, $u_1(t)$ *with* $||u|| \leq r < ||u_1|| \leq R$.

Theorem 3.5. *Suppose* (H_{1b})-(H_6) *and* (H_8)-(H_{10}) *hold. Then BVP* (1.1) *has two positive solutions* $u(t)$, $u_1(t)$ *with* $||u|| \leq r < ||u_1|| \leq R$.

4 Example

In this section, we give an explicit example to illustrate our main result.

Example 4.1. Consider four-point boundary value problem of second order differential equation

$$
\begin{cases}
u'' + \frac{a}{\sqrt{1-t}}(\frac{\sigma(t)}{u^{\frac{1}{4}}} + t) = 0, \ 0 < t < 1,\\
u(0) - u'(\frac{1}{4}) = 0, \ u(1) + u'(\frac{3}{4}) = 0,
\end{cases}
\tag{4.1}
$$

where $\sigma(t) = \max\{0, (t-\xi)(\eta-t)\}$, $a > 0$ is a constant.

Conclusion: BVP (4.1) has at least one positive solution.

Proof. Obviously, $p=2,\ \alpha=1,\ \beta=1,\ \xi=\frac{1}{4},\ \eta=\frac{3}{4}$ in BVP(4.1). Comparing to Theorem 3.1, we verify (H_{1a})-(H_7) as follows:

(H_{1a}) $q(t)=\frac{a}{\sqrt{1-t}}\in\mathbb{C}(0,1)\cap\mathbb{L}^1[0,1]$, $q(t)>0$ and nonincreasing on $(0,1)$;

(H_2) $f(t,u)=\frac{\sigma(t)}{u^{\frac{1}{4}}}+t$ is a continuous function on $[0,1]\times(0,+\infty)$;

(H_3) $0<f(t,u)=\frac{\sigma(t)}{u^{\frac{1}{4}}}+t\le\frac{1}{u^{\frac{1}{4}}}+1+u^4=f_1(u)+f_2(u)$, where $f_1(u)=\frac{1}{u^{\frac{1}{4}}}>0$ continuous, nonincreasing on $(0,+\infty)$ and $\int_0^L f_1(u)du<+\infty$ for any fixed $L>0$; $f_2(u)=u^4+1>0$ is continuous on $[0,+\infty)$; $\frac{f_2}{f_1}=u^{\frac{1}{4}}+u^{\frac{17}{4}}$ nondecreasing on $(0,+\infty)$;

(H_4) Let $R=1$, so $N(t)=\sup\limits_{u\in(0,1]}f(t,u)=t$ for $t\in[0,\frac{1}{4}]$, $n(t)=\inf\limits_{u\in(0,1]}f(t,u)=(t-\frac{1}{4})(\frac{3}{4}-t)+t$ for $t\in[\frac{1}{4},\frac{3}{4}]$, then

$$\int_0^{\frac{1}{4}}q(t)N(t)dt=a\int_0^{\frac{1}{4}}\frac{t}{\sqrt{1-t}}dt\doteq 0.034295227a,$$
$$\int_{\frac{1}{4}}^{\frac{3}{4}}q(t)n(t)dt=a\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{(t-\frac{1}{4})(\frac{3}{4}-t)+t}{\sqrt{1-t}}dt\doteq 0.4124355660a,$$
$$\Gamma=\frac{13}{3},\quad \Gamma a\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{(t-\frac{1}{4})(\frac{3}{4}-t)+t}{\sqrt{1-t}}dt\doteq 1.787220786a$$

so $a\int_0^{\frac{1}{4}}\frac{t}{\sqrt{1-t}}dt\le\Gamma a\int_{\frac{1}{4}}^{\frac{3}{4}}\frac{(t-\frac{1}{4})(\frac{3}{4}-t)+t}{\sqrt{1-t}}dt$;

(H_5) For each constant $H>0$, there exists a function $\psi_H(t)=t$ that is continuous on $[0,1]$ and positive on $(0,1)$ satisfying $f(t,u)\ge\psi_H(t)$ on $(0,1)\times(0,H]$;

(H_6) Clearly, $\int_0^{\frac{1}{4}}\frac{1}{\sqrt[4]{k_1 s}\sqrt{1-s}}ds<+\infty$, $\int_{\frac{3}{4}}^1\frac{1}{\sqrt[4]{k_2(1-s)}\sqrt{1-s}}ds<+\infty$, for any $k_1>0,\ k_2>0$;

(H_7) When $r>0$,

$$\phi_q(r)=r,\quad I^{-1}(r)=\sqrt{2r},$$
$$G(r)=\int_0^r[f_1(s)+f_2(s)]ds=\int_0^r[\frac{1}{s^{\frac{1}{4}}}+1+s^4]ds=\frac{4}{3}r^{\frac{3}{4}}+r+\frac{1}{5}r^5,$$
$$M_{\beta,\eta}(r)=\phi_q(I^{-1}(G(r)))[\beta\phi_q(I^{-1}(q(\eta)))+\int_0^1\phi_q(I^{-1}(q(t)))dt]$$
$$=\sqrt{\frac{8}{3}r^{\frac{3}{4}}+2r+\frac{2}{5}r^5}(2+\frac{4}{3}\sqrt{2})\sqrt{a},$$

let $r=0.1,\ a=10^{-4}$, so $\frac{r}{M_{\beta,\eta}(r)}\doteq 3.134308209>1$.

Therefore, By Theorem 3.1, we obtain that BVP (4.1) has at least one positive solution. □

Acknowledgement. The authors thank the referee for his/her careful reading of the manuscript and valuable suggestions.

References

[1] Lan K, Webb J L, Positive solutions of semilinear differential equations with singularities, *J. Differential Equations* **148** (1998) 407-421.

[2] Agarwal R P, O'Regan D, Nonlinear superlinear singular and nonsingular second order boundary value problems, *J. Differential Equations* **143** (1998) 60-95.

[3] Agarwal R P, O'Regan D, Existence theory for single and multiple solutions to singular positone boundary value problems, *J. Differential Equations* **175** (2001) 393-414.

[4] Agarwal R P, O'Regan D, Twin solutions to singular Dirichlet problems, *J. Math. Anal. Appl.* **240** (1999) 433-445.

[5] Bonheure D, Coster C De, Forced singular oscillators and the method of lower and upper solutions, *Topol. Methods Nonlinear Anal.* **22** (2003) 297-317.

[6] Jiang D, Chu J, Zhang M, Multiplicity of positive periodic solutions to superlinear repulsive singular equations, *J. Differential Equations* **211** (2005) 282-302.

[7] Ha K, Lee Y, Existence of multiple positive solutions of singular boundary value problems, *Nonlinear Anal.* **28** (1997) 1429-1438.

[8] Liu Y, Qi A, Positive solutions of nonlinear singular boundary value problem in abstract space, *Comput. Appl. Math.* **47** (2004) 683-688.

[9] Khan R A, Positive solutions of four-point singular boundary value problems, *Appl. Math. Comput.* **201** (2008) 762-773.

[10] Liu B, Liu L, Wu Y, Positive solutions for singular second order three-point boundary value problems, *Nonlinear Anal.* **66** (2007) 2756-2766.

[11] Ma D, Han J, Chen X, Positive solution of three-point boundary value problem for the one-dimensional p-Laplacian with singularities, *J. Math. Anal. Appl.* **324** (2006) 118-133.

[12] Ma D, Ge W, Positive solution of multi-point boundary value problem for the one-dimensional p-Laplacian with singularities, *Rocky Mountain J. Math.* **137(4)** (2007) 1229-1250.

[13] Ma D, Ge W, The existence of positive solution of multi-point boundary value problem for the one-dimensional p-Laplacian with singularities, *Acta Mech. Sinica* (Beijing) **48(6)** (2005) 1079-1088.

[14] Wei Zhong-Li, Pang Chang-Ci, Positive solutions of some singular m-point boundary value problems at non-resonance, *Appl. Math. Comput.* **171** (2005) 433-449.

[15] Xu X, Positive solutions for singular m-point boundary value problems with positive parameter, *J. Math. Anal. Appl.* **291** (2004) 352-367.

[16] Zhang X, Liu L, Positive solutions of fourth-order multi-point boundary value problems with bending term, *Appl. Math. Comput.* **194** (2007) 321-332.

[17] Zhang X, Liu L, Positive solutions of fourth-order four-point boundary value problems with p-Laplacian operator, *J. Math. Anal. Appl.* **336** (2007) 1414-1423.

[18] Lian H, Ge W, Positive solutions for a four-point boundary value problem with the p-Laplacian, *Nonlinear Anal.* **68** (2008) 3493-3503.

[19] Krasnosel'skii M A, Positive solutions of operator equations, Noordhoff, Groningen, 1964.

(Received March 3, 2009)