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Solvability for second order nonlinear impulsive boundary value problems

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Abstract. In this paper, we are concerned with the solvability for a class of second order nonlinear impulsive boundary value problem. New criteria are established based on Schaefer's fixed-point theorem. An example is presented to illustrate our main result. Our results essentially extend and complement some previous known results.

Key words: Impulsive boundary value problems; Solvability; Schaefer's fixed-point theorem; Periodic and anti-periodic boundary conditions.

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1 Introduction

Impulsive differential equations play a very important role in understanding mathematical models of real processes and phenomena studied in physics, chemical technology, population dynamics, biotechnology, economics and so on, see [1,2,8,10,17]. About wide applications of the theory of impulsive differential equations to different areas, we refer the readers to monographs [5,7,18,19] and the references therein. Some resent works on periodic and anti-periodic nonlinear impulsive boundary value problems can be found in [6,12,20,21]. Recently, J. Chen, C. Tisdell, and R. Yuan in [4] studied the following first order impulsive nonlinear periodic boundary value problem

$$\begin{cases} -u'(t) = f(t,u), \quad t \in [0,T], \quad t \neq t_1, \\ u(t_1^+) - u(t_1^-) = I(u(t_1)), \\ u(0) = u(T), \end{cases} \tag{1.1}$$

where $T > 0$ and $f : [0,T] \times R^n \to R^n$ is continuous on $(t,u) \in [0,T] \setminus \{t_1\} \times R^n$. The authors studied the existence of solutions to the problem (1.1) in view of differential inequalities and Schaefer's fixed-point theorem. Their results extend those of [9,14] in the sense that they allow superlinear growth in nonlinearity $\|f(t,p)\|$ in $\|p\|$.

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About further investigation, in 2007, Bai and Yang in [3] presented the existence results for the following second-order impulsive periodic boundary value problems

$$\begin{cases} u''(t) = f(t, u(t), u'(t)), \quad t \in [0, T] \setminus \{t_1\}, \\ u(t_1^+) = u(t_1^-) + I(u(t_1))), \\ u'(t_1^+) = u'(t_1^-) + J(u(t_1)), \\ u(0) = u(T), u'(0) = u'(T). \end{cases} \tag{1.2}$$

Inspired by [3,4], in this paper, we investigate the following second order impulsive nonlinear boundary value problems

$$\begin{cases} -u''(t) + p(t)u'(t) + q(t)u(t) = f(t, u(t), u'(t)), \quad t \in [0, T] \setminus \{t_1, t_2, ..., t_k\}, \\ \triangle u(t_i) = a_i, \quad i = 1, 2, ..., k, \\ \triangle u'(t_i) = b_i, \quad i = 1, 2, ..., k, \\ u(0) = \beta u(T), u'(0) = \gamma u'(T), \end{cases} \tag{1.3}$$

where $f : [0, T] \times R^n \times R^n \to R^n$ is continuous on $(t, u, v) \in [0, T] \setminus \{t_i\} \times R^n \times R^n$, $i = 1, 2, ..., k$, $p, q \in C([0, T])$, a_i, b_i are constants for $i = 1, 2, ..., k$, β, γ are constants satisfying $|\beta| \geq 1$, $|\gamma| \geq 1$. Notice that our results not only extend some known results from the nonimpulsive case [16] to the impulsive case, or from single impulse [3] to multiple impulses, but also extend those of [11] in the sense that we allow superlinear growth of $\|f(t, u, v)\|$ in $\|u\|$ and $\|v\|$. Furthermore, the impulsive boundary-value problem reduces to a periodic boundary value problem [15,22] for $\beta = \gamma = 1$, $p = q \equiv 0$, and anti-periodic boundary value problem [21] for $\beta = \gamma = -1$, $p = q \equiv 0$. Hence, the problem (1.3) can be considered as a generalization of periodic and anti-periodic boundary value problems.

We shall establish the existence of solutions for impulsive BVP (1.3) by means of well-known Schaefer's fixed-point theorem. The rest of paper is organized as follows. In section 2, we present some definitions and lemmas, and the fixed point theorem which is key to our proof. In section 3, the new existence theorem of (1.3) is stated. An example is given in the last section to demonstrate the application of our main result.

2 Preliminaries

First, we introduce and denote the Banach space $PC([0, T], R^n)$ by

$$PC([0, T], R^n) = \{u : [0, T] \to R^n | u \in C([0, T] \setminus \{t_i\}, R^n),$$
$$u \text{ is left continuous at } t = t_i, \text{ the right} - \text{hand limit } u(t_i^+) \text{ exists}\}$$

with the norm

$$\|u\|_{PC} = \sup_{t \in [0,T]} \|u(t)\|,$$

where $\| \cdot \|$ is the usual Euclidean norm.

We denote the Banach space $PC^1([0, T]; R^n)$ by

$$PC^1([0, T], R^n) = \{u \in C^1([0, T] \setminus \{t_i\}, R^n),$$
$$u \text{ is left continuous at } t \neq t_i, u'(t_i^+), u'(t_i^-) \text{ exist}\}$$

with the norm

$$\|u\|_{PC^1} = \max\{\|u\|_{PC}, \|u'\|_{PC}\}.$$

The following fixed-point theorem due to Schaefer, is essential in the proof of our main result.

Lemma 2.1. Let E be a normed linear space and $\Phi : E \to E$ be a compact operator. Suppose that the set

$$S = \{x \in E | x = \lambda \Phi(x), \quad \text{for some} \quad \lambda \in (0,1)\}$$

is bounded. Then Φ has a fixed point in E.

Lemma 2.2. Assume $p \in C[0,T], q(t) \in C([0,T],(-\infty,0])$. Let ϕ_1, ϕ_2 be the solutions of

$$\begin{cases} \phi_1''(t) + p(t)\phi_1'(t) + q(t)\phi_1(t) = 0, \\ \phi_1(0) = 0, \quad \phi_1(T) = T, \end{cases} \tag{2.1}$$

and

$$\begin{cases} \phi_2''(t) + p(t)\phi_2'(t) + q(t)\phi_2(t) = 0, \\ \phi_2(0) = T, \quad \phi_2(T) = 0. \end{cases} \tag{2.2}$$

Then

(i) ϕ_1 is strictly increasing on [0,T];

(ii) ϕ_2 is strictly decreasing on [0,T].

Proof. The proof is similar to that of Lemma 2.1 in [13], so we omit it here.

Remark 2.3. It follows from Lemma 2.2 that

$$\phi_1(t)\phi_2(s) \le \phi_1(s)\phi_2(s) \le \phi_1(s)\phi_2(t), \quad 0 \le t \le s \le T. \tag{2.3}$$

In order to prove our main results, we present a useful lemma in this section. Consider the following impulsive boundary value problem

$$\begin{cases} -u''(t) + p(t)u'(t) + q(t)u(t) = h(t), \quad t \neq \{t_1, t_2, ..., t_k\}, \\ \triangle u(t_i) = a_i, \quad i = 1, 2, ..., k, \\ \triangle u'(t_i) = b_i, \quad i = 1, 2, ..., k, \\ u(0) = \beta u(T), u'(0) = \gamma u'(T), \end{cases} \tag{2.4}$$

where a_i, b_i are constants for $i = 1, 2, ..., k$, $h \in PC[0,T]$.

Lemma 2.4. For $h(t) \in PC[0,T]$, the problem (2.4) has the unique solution

$$\begin{aligned} u(t) = M\phi_1(t) + N\phi_2(t) + \phi_1(t) \int_t^T \frac{1}{\rho} h(s)\phi_2(s)l(s)ds \\ + \phi_2(t) \int_0^t \frac{1}{\rho} h(s)\phi_1(s)l(s)ds + \sum_{t_i < t} b_i(t - t_i) + \sum_{t_i < t} a_i, \end{aligned} \tag{2.5}$$

where ϕ_1, ϕ_2 satisfies (2.1), (2.2) respectively, and

$$l(t) = \exp(\textstyle\int_0^t p(s)ds), \quad \rho := \phi_1'(0), \tag{2.6}$$

$$M=\frac{\gamma\int_0^T\frac{1}{\rho}h(s)\phi_1(s)l(s)ds\phi_2^{'}(T)-\int_0^T\frac{1}{\rho}h(s)\phi_2(s)l(s)ds\phi_1^{'}(0)+\gamma\sum\limits_{t_i<T}b_i}{\phi_1^{'}(0)-\gamma\phi_1^{'}(T)+\beta(\phi_2^{'}(0)-\gamma\phi_2^{'}(T))}$$

$$-\frac{\beta\left(\sum\limits_{t_i<T}b_i(T-t_i)+\sum\limits_{t_i<T}a_i\right)(\phi_2^{'}(0)-\gamma\phi_2^{'}(T))}{T[\phi_1^{'}(0)-\gamma\phi_1^{'}(T)+\beta(\phi_2^{'}(0)-\gamma\phi_2^{'}(T))]}. \tag{2.7}$$

$$N=\frac{\beta\gamma\int_0^T\frac{1}{\rho}h(s)\phi_1(s)l(s)ds\phi_2^{'}(T)-\beta\int_0^T\frac{1}{\rho}h(s)\phi_2(s)l(s)ds\phi_1^{'}(0)+\beta\gamma\sum\limits_{t_i<T}b_i}{\phi_1^{'}(0)-\gamma\phi_1^{'}(T)+\beta(\phi_2^{'}(0)-\gamma\phi_2^{'}(T))}$$

$$+\frac{\beta\left(\sum\limits_{t_i<T}b_i(T-t_i)+\sum\limits_{t_i<T}a_i\right)(\phi_1^{'}(0)-\gamma\phi_1^{'}(T))}{T[\phi_1^{'}(0)-\gamma\phi_1^{'}(T)+\beta(\phi_2^{'}(0)-\gamma\phi_2^{'}(T))]}. \tag{2.8}$$

Proof. Since ϕ_1,ϕ_2 are two linearly independent solutions of the equation

$$-u''(t)+p(t)u'(t)+q(t)u(t)=0,\quad t\in[0,T], \tag{2.9}$$

we know the solutions of (2.9) can be presented as

$$u(t)=c_1\phi_1(t)+c_2\phi_2(t),$$

where c_1,c_2 are any constants.

Let $u^*=c_1(t)\phi_1(t)+c_2(t)\phi_2(t)$ be a special solution of

$$-u''(t)+p(t)u'(t)+q(t)u(t)=h(t),\quad t\in[0,T]. \tag{2.10}$$

Employing the method of variation of parameter, by some calculation, we get

$$c_1(t)=\int_t^T\frac{1}{\rho}h(s)\phi_2(s)l(s)ds,\quad c_2(t)=\int_0^t\frac{1}{\rho}h(s)\phi_1(s)l(s)ds.$$

So the solution of (2.10) can be given as

$$u(t)=c_1\phi_1(t)+c_2\phi_2(t)+u^*.$$

Next, we consider

$$\begin{cases}-u''(t)+p(t)u'(t)+q(t)u(t)=h(t),\quad t\neq\{t_1,t_2,...,t_k\},\\ \triangle u(t_i)=a_i,\quad i=1,2,...,k,\\ \triangle u'(t_i)=b_i,\quad i=1,2,...,k.\end{cases} \tag{2.11}$$

It is easy to know the solution of (2.11) is as the following form

$$u(t)=c_1\phi_1(t)+c_2\phi_2(t)+u^*+\sum\limits_{t_i<t}b_i(t-t_i)+\sum\limits_{t_i<t}a_i. \tag{2.12}$$

Finally, we consider the solution of (2.4). Substituting (2.12) into $u(0)=\beta u(T)$, $u'(0)=\gamma u'(T)$, we have

$$\begin{cases} c_2T - c_1\beta T - \beta \sum\limits_{t_i<T} b_i(T-t_i) - \beta \sum\limits_{t_i<T} a_i = 0, \\ \gamma \displaystyle\int_0^T \frac{1}{\rho} h(s)l(s)\phi_1(s)ds\phi_2'(T) + \gamma \sum_{t_i<T} b_i \\ -\displaystyle\int_0^T \frac{1}{\rho} h(s)l(s)\phi_2(s)ds\phi_1'(0) = c_1(\phi_1'(0) - \gamma\phi_1'(T)) + c_2(\phi_2'(0) - \gamma\phi_2'(T)). \end{cases} \tag{2.13}$$

By some calculations, we get

$$\begin{aligned} c_1 = {} & \frac{\gamma \int_0^T \frac{1}{\rho} h(s)\phi_1(s)l(s)ds\phi_2'(T) - \int_0^T \frac{1}{\rho} h(s)\phi_2(s)l(s)ds\phi_1'(0) + \gamma \sum\limits_{t_i<T} b_i}{\phi_1'(0) - \gamma\phi_1'(T) + \beta(\phi_2'(0) - \gamma\phi_2'(T))} \\ & - \frac{\beta\left(\sum\limits_{t_i<T} b_i(T-t_i) + \sum\limits_{t_i<T} a_i\right)(\phi_2'(0) - \gamma\phi_2'(T))}{T[\phi_1'(0) - \gamma\phi_1'(T) + \beta(\phi_2'(0) - \gamma\phi_2'(T))]} =: M, \\ c_2 = {} & \frac{\beta\gamma \int_0^T \frac{1}{\rho} h(s)\phi_1(s)l(s)ds\phi_2'(T) - \beta \int_0^T \frac{1}{\rho} h(s)\phi_2(s)l(s)ds\phi_1'(0) + \beta\gamma \sum\limits_{t_i<T} b_i}{\phi_1'(0) - \gamma\phi_1'(T) + \beta(\phi_2'(0) - \gamma\phi_2'(T))} \\ & + \frac{\beta\left(\sum\limits_{t_i<T} b_i(T-t_i) + \sum\limits_{t_i<T} a_i\right)(\phi_1'(0) - \gamma\phi_1'(T))}{T[\phi_1'(0) - \gamma\phi_1'(T) + \beta(\phi_2'(0) - \gamma\phi_2'(T))]} := N. \end{aligned}$$

Hence, the problem (2.4) has the unique solution

$$\begin{aligned} u(t) = {} & M\phi_1(t) + N\phi_2(t) + \phi_1(t) \int_t^T \frac{1}{\rho} h(s)\phi_2(s)l(s)ds \\ & + \phi_2(t) \int_0^t \frac{1}{\rho} h(s)\phi_1(s)l(s)ds + \sum_{t_i<t} b_i(t-t_i) + \sum_{t_i<t} a_i. \end{aligned}$$

Let $f : [0,T] \times R^n \times R^n \to R^n$ be continuous. We now introduce a mapping $A : PC^1([0,T], R^n) \to PC([0,T], R^n)$ defined by

$$\begin{aligned} Au(t) = {} & M\phi_1(t) + N\phi_2(t) + \phi_1(t) \int_t^T \frac{1}{\rho} f(s, u(s), u'(s))\phi_2(s)l(s)ds \\ & + \phi_2(t) \int_0^t \frac{1}{\rho} f(s, u(s), u'(s))\phi_1(s)l(s)ds + \sum_{t_i<t} b_i(t-t_i) + \sum_{t_i<t} a_i. \end{aligned} \tag{2.14}$$

In view of Lemma 2.4, we easily know that u is a fixed point of operator A iff u is a solution to the impulsive periodic boundary problem (1.3).

Lemma 2.5. Let $f : [0,T] \times R^n \times R^n \to R^n$ be continuous. Then $A : PC^1([0,T], R^n) \to PC([0,T], R^n)$ is a compact map.

Proof. This is similar to that of Lemma 3.2 in [4].

For convenience, let

$$\|\phi_1\|_0 = \max_{0\le t\le T} |\phi_1(t)|, \quad \|\phi_2\|_0 = \max_{0\le t\le T} |\phi_2(t)|, \quad G_1 = \max_{0\le t\le T} |\phi_1(t)\phi_2(t)|,$$

$$L=\max_{0\le t\le T}|l(t)|,\quad \|\phi_1'\|_0=\max_{0\le t\le T}|\phi_1'(t)|,\quad \|\phi_2'\|_0=\max_{0\le t\le T}|\phi_2'(t)|. \tag{2.15}$$

Now we are in the position to present our main results.

3 Main results

Theorem 3.1. Suppose that $f:[0,T]\times R^n\times R^n\to R^n$ is continuous and $p\in C([0,T],[0,+\infty)), q(t)\equiv q\le 0$, $|\beta|\ge 1$, $|\gamma|\ge 1$, a_i,b_i are constants for $i=1,2,...,k$. If there exist nonnegative constants α,Q such that

$$\|f(t,u,v)\|\le 2\alpha\langle v,pv+qu-f(t,u,v)\rangle+Q,\qquad (t,u,v)\in([0,T]\setminus\{t_1,t_2,...,t_k\})\times R^n\times R^n. \tag{3.1}$$

Then BVP (1.3) has at least one solution.

Proof. Let $u\in PC([0,T],R^n)$ be such that $u=\lambda Au$ for some $\lambda\in(0,1)$. That is,

$$\begin{cases} -u''(t)+p(t)u'(t)+qu(t)=\lambda f(t,u(t),u'(t)),\quad t\in[0,T]\setminus\{t_1,t_2,...,t_k\},\\ \triangle u(t_i)=\lambda a_i,\quad i=1,2,...,k,\\ \triangle u'(t_i)=\lambda b_i,\quad i=1,2,...,k,\\ u(0)=\beta u(T),u'(0)=\gamma u'(T). \end{cases} \tag{3.2}$$

By Lemma 2.5, A is a compact map. In order to utilize Lemma 2.1, next, we will show $S=\{u\in PC^1|u=\lambda Au,\ \ \lambda\in(0,1)\}$ is bounded. By(2.3), (2.14)-(2.15) together with (3.1)-(3.2), we obtain

$$\begin{aligned}
\|u(t)\| &= \lambda\|Au(t)\|\\
&=\lambda\|M\phi_1(t)+N\phi_2(t)+\frac{1}{\rho}\int_t^T\phi_1(t)\phi_2(s)l(s)f(s,u(s),u'(s))ds\\
&\quad+\frac{1}{\rho}\int_0^t\phi_1(s)l(s)\phi_2(t)f(s,u(s),u'(s))ds+\sum_{t_i<t}b_i(t-t_i)+\sum_{t_i<t}a_i\|\\
&\le |M|\|\phi_1(t)\|+|N|\|\phi_2(t)\|+|\frac{1}{\rho}\phi_1(t)\phi_2(t)L|\int_t^T\lambda\|f(s,u(s),u'(s))\|ds\\
&\quad+|\frac{1}{\rho}\phi_1(t)\phi_2(t)L|\int_0^t\lambda\|f(s,u(s),u'(s))\|ds+\sum_{t_i<T}|b_i|(T-t_i)+\sum_{t_i<T}|a_i|\\
&\le |M|\|\phi_1\|_0+|N|\|\phi_2\|_0+\frac{2}{|\rho|}G_1L\int_0^T(2\alpha\langle u'(s),\lambda p(s)u'(s)+\lambda qu(s)-\lambda f(s,u(s),u'(s))\rangle+Q)ds\\
&\quad+\sum_{t_i<T}|b_i|(T-t_i)+\sum_{t_i<T}|a_i|\\
&=\frac{2}{|\rho|}G_1L[\int_0^T2\alpha\langle u'(s),p(s)u'(s)+qu(s)-\lambda f(t,u,u')-(1-\lambda)p(s)u'(s)-(1-\lambda)qu(s)\rangle ds+QT]\\
&\quad+|M|\|\phi_1\|_0+|N|\|\phi_2\|_0+\sum_{t_i<T}|b_i|(T-t_i)+\sum_{t_i<T}|a_i|\\
&=\frac{2}{|\rho|}G_1L[\int_0^T2\alpha\langle u'(s),u''(s)\rangle ds-2\alpha(1-\lambda)\int_0^T\langle u'(s),p(s)u'(s)\rangle ds-2\alpha(1-\lambda)q\int_0^T\langle u'(s),u(s)\rangle ds
\end{aligned}$$

$$+QT]+|M|\|\phi_1\|_0+|N|\|\phi_2\|_0+\sum_{t_i<T}|b_i|(T-t_i)+\sum_{t_i<T}|a_i|$$

$$=\frac{2}{|\rho|}G_1L[\alpha\int_0^T\frac{d}{ds}\|u'(s)\|^2ds-2\alpha(1-\lambda)\int_0^T\langle\sqrt{p(s)}u'(s),\sqrt{p(s)}u'(s)\rangle ds-\alpha(1-\lambda)q\int_0^T\frac{d}{ds}\|u(s)\|^2ds$$

$$+QT]+|M|\|\phi_1\|_0+|N|\|\phi_2\|_0+\sum_{t_i<T}|b_i|(T-t_i)+\sum_{t_i<T}|a_i|$$

$$=\frac{2}{|\rho|}G_1L[\alpha(\|u'(T)\|^2-\|u'(0)\|^2)-2\alpha(1-\lambda)\int_0^T\|\sqrt{p(s)}u'(s)\|^2ds-\alpha(1-\lambda)q(\|u(T)\|^2-\|u(0)\|^2)$$

$$+QT]+|M|\|\phi_1\|_0+|N|\|\phi_2\|_0+\sum_{t_i<T}|b_i|(T-t_i)+\sum_{t_i<T}|a_i|$$

$$\leq\frac{2}{|\rho|}G_1L[\alpha(1-\gamma^2)\|u'(T)\|^2-\alpha(1-\lambda)q(1-\beta^2)\|u(T)\|^2+QT]$$

$$+|M|\|\phi_1\|_0+|N|\|\phi_2\|_0+\sum_{t_i<T}|b_i|(T-t_i)+\sum_{t_i<T}|a_i|$$

$$\leq\frac{2}{|\rho|}G_1LQT+|M|\|\phi_1\|_0+|N|\|\phi_2\|_0+\sum_{t_i<T}|b_i|(T-t_i)+\sum_{t_i<T}|a_i|.$$

A similar calculation yields an estimate on u': differentiating both sides of the integration and taking norms yields, for each $t\in[0,T]$, we have

$$\|u'(t)\|=\lambda\|(Au)'(t)\|$$

$$=\lambda\|M\phi_1^{'}(t)+N\phi_2^{'}(t)+\int_t^T\frac{1}{\rho}f(s,u(s),u'(s))\phi_2(s)l(s)ds\phi_1^{'}(t)$$

$$+\int_0^t\frac{1}{\rho}f(s,u(s),u'(s))\phi_1(s)l(s)ds\phi_2^{'}(t)+\sum_{t_i<t}b_i\|$$

$$\leq|M|\|\phi_1^{'}\|_0+|N|\|\phi_2^{'}\|_0+\frac{1}{|\rho|}(\|\phi_1^{'}\|_0\|\phi_2\|_0+\|\phi_1\|_0\|\phi_2^{'}\|_0)L\int_0^T\lambda\|f(s,u(s),u'(s))\|ds+\sum_{t_i<T}|b_i|$$

$$\leq\frac{1}{|\rho|}(\|\phi_1^{'}\|_0\|\phi_2\|_0+\|\phi_1\|_0\|\phi_2^{'}\|_0)L\int_0^T[2\alpha\langle u'(s),\lambda p(s)u'(s)+\lambda qu(s)$$

$$-\lambda f(s,u(s),u'(s))\rangle+Q]ds+|M|\|\phi_1^{'}\|_0+|N|\|\phi_2^{'}\|_0+\sum_{t_i<T}|b_i|$$

$$=\frac{1}{|\rho|}(\|\phi_1^{'}\|_0\|\phi_2\|_0+\|\phi_1\|_0\|\phi_2^{'}\|_0)L[\int_0^T2\alpha\langle u'(s),p(s)u'(s)+qu(s)$$

$$-\lambda f(t,u,u')-(1-\lambda)p(s)u'(s)-(1-\lambda)qu(s)\rangle ds+QT]+|M|\|\phi_1^{'}\|_0+|N|\|\phi_2^{'}\|_0+\sum_{t_i<T}|b_i|$$

$$=\frac{1}{|\rho|}(\|\phi_1^{'}\|_0\|\phi_2\|_0+\|\phi_1\|_0\|\phi_2^{'}\|_0)L[\int_0^T2\alpha\langle u'(s),u''(s)\rangle ds-2\alpha(1-\lambda)\int_0^T\langle u'(s),p(s)u'(s)\rangle ds$$

$$-2\alpha(1-\lambda)q\int_0^T\langle u'(s),u(s)\rangle ds+QT]+|M|\|\phi_1^{'}\|_0+|N|\|\phi_2^{'}\|_0+\sum_{t_i<T}|b_i|$$

$$=\frac{1}{|\rho|}(\|\phi_1^{'}\|_0\|\phi_2\|_0+\|\phi_1\|_0\|\phi_2^{'}\|_0)L[\alpha\int_0^T\frac{d}{ds}\|u'(s)\|^2ds-2\alpha(1-\lambda)\int_0^T\langle\sqrt{p(s)}u'(s),\sqrt{p(s)}u'(s)\rangle ds$$

$$-\alpha(1-\lambda)q\int_0^T\frac{d}{ds}\|u(s)\|^2ds+QT]+|M|\|\phi_1^{'}\|_0+|N|\|\phi_2^{'}\|_0+\sum_{t_i<T}|b_i|$$

$$=\frac{1}{|\rho|}(\|\phi_1^{'}\|_0\|\phi_2\|_0+\|\phi_1\|_0\|\phi_2^{'}\|_0)L[\alpha(\|u'(T)\|^2-\|u'(0)\|^2)-2\alpha(1-\lambda)\int_0^T\|\sqrt{p(s)}u'(s)\|^2ds$$

$$-\alpha(1-\lambda)q(\|u(T)\|^2-\|u(0)\|^2)+QT]+|M|\|\phi_1^{'}\|_0+|N|\|\phi_2^{'}\|_0+\sum_{t_i<T}|b_i|$$

$$\le\frac{1}{|\rho|}(\|\phi_1^{'}\|_0\|\phi_2\|_0+\|\phi_1\|_0\|\phi_2^{'}\|_0)L[\alpha(1-\gamma^2)\|u'(T)\|^2$$

$$-\alpha(1-\lambda)q(1-\beta^2)\|u(T)\|^2+QT]+|M|\|\phi_1^{'}\|_0+|N|\|\phi_2^{'}\|_0+\sum_{t_i<T}|b_i|.$$

$$\le\frac{1}{|\rho|}(\|\phi_1^{'}\|_0\|\phi_2\|_0+\|\phi_1\|_0\|\phi_2^{'}\|_0)LQT+|M|\|\phi_1^{'}\|_0+|N|\|\phi_2^{'}\|_0+\sum_{t_i<T}|b_i|.$$

Thus, we conclude that

$$\|u\|_{PC^1}\le\max\{\frac{2}{|\rho|}G_1LQT+|M|\|\phi_1\|_0+|N|\|\phi_2\|_0+\sum_{t_i<T}|b_i|(T-t_i)+\sum_{t_i<T}|a_i|,$$

$$\frac{1}{|\rho|}(\|\phi_1^{'}\|_0\|\phi_2\|_0+\|\phi_1\|_0\|\phi_2^{'}\|_0)LQT+|M|\|\phi_1^{'}\|_0+|N|\|\phi_2^{'}\|_0+\sum_{t_i<T}|b_i|\}.$$

As a result, set S is bounded. Applying Scheafer's fixed-point theorem, the problem (3.2) has at least one fixed point, which means that (1.3) has at least one solution. We complete the proof.

A similar discuss as Theorem 3.1 leads to the following result.

Remark 3.2. If the condition (3.1) is replaced by

$$\|f(t,u,v)\|\le 2\alpha\langle v,pv-f(t,u,v)\rangle+Q,\qquad (t,u,v)\in([0,T]\setminus\{t_1,t_2,...,t_k\})\times R^n\times R^n,\tag{3.3}$$

and all the other assumptions are satisfied in Theorem 3.1, then the problem (1.3) has at least one solution.

4 An example

In this section, an example is given to highlight our main result. Consider the scalar impulsive periodic BVP given by

$$\begin{cases} -u''(t) + t^2u'(t) - 9u(t) = f(t, u(t), u'(t)), \quad t \in [0,1] \setminus \{t_1, t_2, ..., t_k\}, \\ \triangle u(t_i) = a_i, \quad i = 1, 2, ..., k, \\ \triangle u'(t_i) = b_i, \quad i = 1, 2, ..., k, \\ u(0) = u(1), u'(0) = u'(1), \end{cases} \tag{4.1}$$

where $0 < t_1 < \cdots < t_k < 1$, $f(t, u, v) = \dfrac{8}{\pi} \arctan u + (1 - t^2)v^2 - v^3$, $p(t) = t^2$, and $q = -9$. We claim that (4.1) has at least one solution.

Proof. Let $T = 1$, $\beta = \gamma = 1$, and $f(t, u, u') = (1 - t^2)u'^2 - u'^3 + \dfrac{8}{\pi} \arctan u$. It is easy to check that

$$x^4 - 2x^3 - x^2 - 4x + 12 \geq 0, \qquad x \geq 0. \tag{4.2}$$

And we see that

$$\begin{aligned} |f(t, u, v)| &= |\frac{8}{\pi} \arctan u + (1 - t^2)v^2 - v^3| \\ &\leq \frac{8}{\pi} \times \frac{\pi}{2} + |v|^2 + |v|^3 \\ &\leq 4 + |v|^2 + |v|^3, \qquad (t, u, v) \in [0,1] \times R^2. \end{aligned} \tag{4.3}$$

On the other hand, for $\alpha = \dfrac{1}{2}$, $Q = 16$, we have

$$\begin{aligned} &2\alpha\langle v, pv - f(t, u, v)\rangle + Q \\ &\quad = v(t^2v - (1 - t^2)v^2 + v^3 - \frac{8}{\pi} \arctan u) + 16 \\ &\quad = v^4 - (1 - t^2)v^3 + t^2v^2 - \frac{8}{\pi} \arctan uv + 16 \\ &\quad \geq |v|^4 - |v|^3 - 4|v| + 16, \quad (t, u, v) \in [0,1] \times R^2. \end{aligned} \tag{4.4}$$

In view of (4.2), we have

$$|v|^4 - |v|^3 - 4|v| + 16 \geq |v|^3 + |v|^2 + 4. \tag{4.5}$$

By (4.3)-(4.5), we obtain that

$$\|f(t, u, v)\| \leq 2\alpha\langle v, pv - f(t, u, v)\rangle + Q.$$

Thus, condition (3.3) holds. By Remark 3.2, we conclude that the solvability of (4.1) follows.

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