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On the strongly damped wave equation with nonlinear damping and source terms[1]

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Abstract We consider a wave equation in a bounded domain with nonlinear dissipation and nonlinear source term. Characterizations with respect to qualitative properties of the solution: globality, boundedness, blow-up, convergence up to a subsequence towards the equilibria and exponential stability are given in this article.

Keywords: Wave equation; Nonlinear dissipation; Nonlinear source; Stable and unstable set; Global solution; Blow-up; Asymptotic behavior.

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1 Introduction

Let $\Omega \subseteq \mathbb{R}^N (N \geq 1)$ be a bounded domain with smooth boundary $\partial\Omega$. We are concerned with the behavior of the following superlinear wave equation with dissipation

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \mu |u_t|^{m-1} u_t = |u|^{p-1} u, & x \in \Omega,\ t \geq 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\omega \geq 0,\ \mu \geq 0,\ m \geq 1,\ p > 1$, and

$$\begin{cases} 1 < p \leq \begin{cases} \frac{N+2}{N-2} & \text{for } \omega > 0 \\ \frac{N}{N-2} & \text{for } \omega = 0 \end{cases} & \text{if } \ N \geq 3, \\ 1 < p < \infty & \text{if } \ N = 1, 2. \end{cases} \tag{1.2}$$

$$u_0 \in H_0^1(\Omega),\ u_1 \in L^2(\Omega). \tag{1.3}$$

We introduce some related works first and then explain in detail which are our main results. For the well posedness of problem (1.1) and why the natural regularity for the initial data is precisely that of (1.3), we refer to [8]. Equations with damping terms have been considered by many authors. For equations with linear weak damping, we refer to [7, 10, 14]. For equations with possibly nonlinear weak damping, we refer to [9, 12, 16, 20, 23]. Much less work is known for equations with strong damping, see the seminal paper by Levine [15] and also [18, 19], but still many problems unsolved. Gazzola et.al. [8] discussed the case when the weak damping term and the strong damping term are both linear ($m = 1$ in (1.1)). It is our purpose to shed some further light on damped wave equations of the kind in the problem (1.1) in both presence of nonlinear weak damping and linear strong damping.

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Cazenave [5] proved the boundedness of global solutions to (1.1) for $\omega = \mu = 0$, while Esquivel-Avila [7] recovered the same result for $\omega = 0$ and $\mu > 0$ and showed that this property may fail in presence of nonlinear disspation, however, by exploiting the same technique in [7], we proved, under the restrictions $E(t) \geq d, \forall\, t \geq 0$ (the energy goes beyond the mountain pass level all the times) and $m < p$, the global solutions can still be bounded even in presence of nonlinear weak damping.

From a different angle of consideration, it is interested to find out for which initial data (1.3) problem (1.1) does have a global solution. For the weakly damped case($\omega = 0$, $\mu > 0$), Iketa [12] proved that the solution is global and converges to equilibria $\phi \equiv 0$ as $t \to \infty$ if and only if $E(0) < d$ and $u_0 \in \mathcal{N}_+$. In Theorem 4.2 we extend this result to the case $\omega > 0$. For related asymptotic stability results the reader is referred to [2, 3], where the authors investigate qualitative aspects of global solutions of hyperbolic Kirchhoff systems, both in the classical framework and in a more general setting given by anisotropic Lebesgue and Sobolev spaces. In particular it is shown that a global solution u converges to an equilibrium state in the sense of the energy decay, provided that the initial data are sufficiently small.

Not all local solutions of (1.1) are global in time. For the weakly damped case($\omega = 0$, $\mu > 0$, $m = 1$), Pucci and Serrin [21] proved nonexistence of global solutions when $E(0) < d$ and $u_0 \in \mathcal{N}_-$. In the case when $\omega > 0$ and $\mu = 0$. Ono [19] showed that the solution of (1.1) blow up in finite time if $E(0) < 0$, which automatically implies $u_0 \in \mathcal{N}_+$. Ohta [18] improves this result by allowing $E(0) < d$ and $u_0 \in \mathcal{N}_+$. Gazzola and Squassina [8] extended this result to the case when $\mu \neq 0$ and $E(0) \leq d$. All those works mentioned above dealt with the linear damping case ($m = 1$) or when the weak damping is absent($\mu = 0$). In the case of (1.1) with $m > 1$ however, the most frequently used technique in the proof of blow up named "concavity argument" no longer apply, so it is necessary to use another approach, namely the blow up theorem 2.3 in [17] for all negative initial energies. In the recent paper [4], thanks to a new combination of the potential well and concavity methods, the global nonexistence of solutions has been proved for Kirchhoff systems when $\omega = 0$ and the initial energy is possibly above the critical level d.

The paper is organized as follows. In section 2, we state the local existence result and recall some notations and useful lemmas. In section 3, we present the boundedness result of global solutions under the assumptions $E(t) \geq d$ and $m < p$. In section 4, we state a sufficient and necessary condition on which the solution of (1.1) is global. In section 5, blow up behavior of (1.1) is investigated. In section 6, we present a exponential decay result.

2 Preliminaries

We specify some notations first. In this context, we denote $\|\cdot\|_q$ by the L^q norm for $1 \leq q \leq \infty$, and $\|\nabla u\|_2$ the Dirichlet norm of u in $H_0^1(\Omega)$. We define the C^1 functionals I, J, E: $H_0^1(\Omega) \to \mathbb{R}$

by:

$$I(u) = \|\nabla u\|_2^2 - \|u\|_{p+1}^{p+1}, \quad J(u) = \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1},$$
$$E(t) = E(u(t)) = \frac{1}{2}\|u_t\|_2^2 + J(u).$$

Note that $E(t)$ satisfies the energy identity

$$E(t) + \omega\int_s^t \|\nabla u_t(\tau)\|_2^2 \mathrm{d}\tau + \mu\int_s^t \|u_t(\tau)\|_{m+1}^{m+1}\mathrm{d}\tau = E(s),\ \forall\, 0 \le s \le t \le T_{max}, \tag{2.1}$$

where T_{max} is the maximal existence time of $u(t)$. The mountain pass level of J is defined as

$$d = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \max_{\lambda\ge 0} J(\lambda u). \tag{2.2}$$

Denote the best sobolev constant for the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ as C_{p+1}

$$C_{p+1} = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\|\nabla u\|_2}{\|u\|_{p+1}}. \tag{2.3}$$

We introduce the sets

$$S = \{\phi \in H_0^1(\Omega) : \phi \text{ is a stationary solution of (1.1)}\},$$
$$S_l = \{\phi \in S : J(\phi) = l\} \quad (l \in \mathbb{R}^+).$$

And the Nehari manifold $\mathcal{N}$ is defined by

$$\mathcal{N} = \{u \in H_0^1(\Omega) \setminus \{0\} : I(u) = 0\},$$

which intersects $H_0^1(\Omega)$ into two unbounded sets

$$\mathcal{N}_+ = \{u \in H_0^1(\Omega) \setminus \{0\} : I(u) > 0\} \cup \{0\},\ \mathcal{N}_- = \{u \in H_0^1(\Omega) \setminus \{0\} : I(u) < 0\}.$$

We also consider the sublevels of J

$$J^a = \{u \in H_0^1(\Omega) : J(u) < a\} \quad (a \in \mathbb{R}),$$

and we introduce the stable set $\mathcal{S}$ and the unstable set $\mathcal{U}$ defined by

$$\mathcal{S} = J^d \cap \mathcal{N}_+ \quad \textit{and} \quad \mathcal{U} = J^d \cap \mathcal{N}_-.$$

Denote $\beta = dist(0, \mathcal{N}) = \inf_{u\in\mathcal{N}} \|\nabla u\|_2$, the following lemma is a direct consequence of (2.2) and (2.3).

Lemma 2.1 *d has the following characterizations*

$$d = \frac{p-1}{2(p+1)} C_{p+1}^{\frac{p+1}{p-1}} = \frac{p-1}{2(p+1)}\beta^2.$$

Now, we state the local existence theorem for the nonlinear wave equation (1.1).

Theorem 2.2 *Suppose that (1.2) holds, then for every initial data* (u_0, u_1) *satisfying (1.3), there exists a unique (local) weak solution* $(u(t), u_t(t)) = S(t)(u_0, u_1)$ *of problem* (1.1), *that is*

$$\frac{d}{dt}(u_t, w) + \omega(\nabla u_t, \nabla w) + \mu(|u_t|^{m-1}u_t, w)_2 = (|u|^{p-1}u, w)_2 \text{ a.e. in } (0,T), \forall\, w \in H_0^1(\Omega) \cap L^{m+1}(\Omega), \tag{2.4}$$

such that

$$u \in C([0,T]; H_0^1(\Omega)) \cap C^1(0,T; L^2(\Omega)), u_t \in L^{m+1}(\Omega \times (0,T)),$$

where $S(t)$ *denotes the corresponding semigroup on* $H_0^1(\Omega) \times L^2(\Omega)$, *generated by problem* (1.1). *Moreover, if*

$$T_{max} = \sup\{T > 0 :\ u = u(t) \text{ exists } \text{ on } [0,T]\} < \infty,$$

then $\lim\limits_{t \to T_{max}} \|u\|_q = \infty$ *for all* $q \geq 1$ *such that* $q > \dfrac{N(p-1)}{2}$.

We restricted ourselves to the case $\omega > 0$, $\mu \neq 0$ and $N \geq 3$, the other cases being similar. For a given $T > 0$, we choose the work space $\mathcal{H} = C([0,T]; H_0^1(\Omega)) \cap C^1(0,T; L^2(\Omega))$ endowed with the norm $\|u\|_{\mathcal{H}}^2 = \max\limits_{t \in [0,T]}(\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2)$. We divide the proof the local existence theorem into two lemmas.

Lemma 2.3 *For every* $T > 0$, *every* $w \in \mathcal{H}$ *and every initial data* (u_0, u_1) *satisfies (1.3), there exists a unique* $u \in \mathcal{H}$ *such that* $u_t \in L^2([0,T]; H_0^1(\Omega))$ *which satisfies the following problem*

$$\begin{cases} u_{tt} - \Delta u - \omega \Delta u_t + \mu |u_t|^{m-1} u_t = |w|^{p-1} w, & x \in \Omega,\ t \geq 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{2.5}$$

Proof. Existence. We consider a standard Galerkin approximation scheme for the solution of (2.5) based on the eigenfunction $\{e_k\}_{k=1}^{\infty}$ of the operator $-\Delta$ with null boundary condition on $\partial\Omega$. That is, we let $u_n(t) = \Sigma_{k=1}^n u_{n,k}(t) e_k$, where $u_n(t)$ satisfies

$$\begin{gathered} (u_{nt}, v) + (\nabla u_n, \nabla v) + \omega(\nabla u_{nt}, \nabla v) + \mu(|u_{nt}|^{m-1} u_{nt}, v) = (|w|^{p-1} w, v) \\ (u_n(0), v) = (u_0, v), \quad (u_{nt}(0), v) = (u_1, v) \end{gathered} \tag{2.6}$$

for all $v \in V_n :=$ the linear span of $\{e_1, e_2, \ldots, e_n\}$, $(\cdot,\cdot)$ denotes the standard $L^2(\Omega)$ inner product. By standard nonlinear ODE theory one obtains the existence of a global solution to (2.6) with the following a priori bounds uniformly in n

$$\begin{aligned} &\frac{1}{2}(\|\nabla u_n(t)\|_2^2 + \|u_{nt}(t)\|_2^2) + \mu \int_0^t \|u_{nt}(\tau)\|_{m+1}^{m+1} \mathrm{d}\tau + \omega \int_s^t \|\nabla u_{nt}(\tau)\|_2^2 \mathrm{d}\tau \\ = &\frac{1}{2}(\|\nabla u_n(0)\|_2^2 + \|u_{nt}(0)\|_2^2) + \int_0^t \int_\Omega |w(\tau)|^{p-1} w(\tau) u_{nt}(\tau) \mathrm{d}x \mathrm{d}\tau,\ \forall\, t \in (0,T]. \end{aligned} \tag{2.7}$$

We estimate the last term on the right-hand side

$$\begin{aligned} &\int_0^t \int_\Omega |w(\tau)|^{p-1} w(\tau) u_{nt}(\tau) \mathrm{d}x \mathrm{d}\tau \\ \leq &\int_0^t \|w(\tau)\|_{p+1}^p \|u_{nt}(\tau)\|_{p+1} \leq \frac{1}{2\omega} \int_0^t \|\nabla w(\tau)\|^{2p} \mathrm{d}\tau + \frac{\omega}{2} \int_0^t \|\nabla u_{nt}(\tau)\|_2^2 \mathrm{d}\tau \\ \leq &C(T) + \frac{\omega}{2} \int_0^t \|\nabla u_{nt}(\tau)\|_2^2 \mathrm{d}\tau,\ \forall\, t \in (0,T]. \end{aligned} \tag{2.8}$$

It follows from (2.7) and (2.8) that

$$(\|\nabla u_n(t)\|_2^2 + \|u_{nt}(t)\|_2^2) + \mu \int_0^t \|u_{nt}(\tau)\|_{m+1}^{m+1} \mathrm{d}\tau + \omega \int_s^t \|\nabla u_{nt}(\tau)\|_2^2 \mathrm{d}\tau \le C_T.$$

Hence, there exists a subsequence of u_n, which we still denoted by u_n, such that

$$\begin{aligned}
&u_n \to u \quad \text{weakly} * \quad \text{in } L^\infty([0,T]; H_0^1(\Omega)),\\
&u_{nt} \to u_t \quad \text{weakly} * \quad \text{in } L^\infty([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega)) \cap L^{m+1}(\Omega \times [0,T]),\\
&u_{ntt} \to u_{tt} \quad \text{weakly} * \quad \text{in } L^2([0,T]; H^{-1}(\Omega)).
\end{aligned}$$

Since $u \in H^1([0,T]; H_0^1(\Omega))$, we get $u \in C([0,T]; H_0^1(\Omega))$. Moreover, since $u_t \in L^2([0,T]; H_0^1(\Omega))$ and $u_{tt} \in L^2([0,T]; H^{-1}(\Omega))$, it follows from the Aubin compactness argument that $u_t \in C([0,T]; L^2(\Omega))$. The existence of u solving (2.5) is proved.

Uniqueness. If u_1, u_2 are two solutions of (2.5) with the same initial data, set $u = u_1 - u_2$, substracting the equations and test with u_t, we obtain

$$\frac{1}{2}(\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2) + \mu \int_0^t (g(u_{1t}) - g(u_{2t}))(u_{1t} - u_{2t}))\mathrm{d}\tau + \omega \int_0^t \|\nabla u_t(\tau)\|_2^2 \mathrm{d}\tau = 0.$$

Observe that $g(u) = |u|^{m-1}u$ is increasing, we immediately get $u_1 = u_2$. The proof of the lemma is complete.

Denote F the mapping defined by the equation (2.5), i.e., $u = F(w)$. Let $R^2 = 2(\|\nabla u_0\|^2 + \|u_1\|^2)$. Consider

$$B_R = \{u \in \mathcal{H} : u(0) = u_0, u_t(0) = u_1 \text{ and } \|u\|_{\mathcal{H}} \le R\}.$$

Lemma 2.4 *$F(B_R) \subseteq B_R$ and $F : B_R \to B_R$ is compact.*

Proof. By Lemma 2.3, for any given $w \in B_R$, the corresponding solution satisfies the following energy equality

$$\begin{aligned}
&\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 + 2\mu \int_0^t \|u_t(\tau)\|_{m+1}^{m+1} \mathrm{d}\tau + 2\omega \int_0^t \|\nabla u_t(\tau)\|_2^2 \mathrm{d}\tau\\
&= (\|\nabla u(0)\|_2^2 + \|u_t(0)\|_2^2) + 2 \int_0^t \int_\Omega |w(\tau)|^{p-1} w(\tau) u_t(\tau) \mathrm{d}x\mathrm{d}\tau. \qquad (2.9)
\end{aligned}$$

We estimate the last term on the right-hand side by using Hölder, Young's inequality and Sobolev embedding theorem

$$\begin{aligned}
&2 \int_0^t \int_\Omega |w(\tau)|^{p-1} w(\tau) u_t(\tau) \mathrm{d}x\mathrm{d}\tau\\
&\le |\Omega|^\alpha \int_0^T \|w(\tau)\|_{2^*}^p \|u_t(\tau)\|_{2^*} \mathrm{d}\tau \le C \int_0^T \|\nabla w(\tau)\|_2^{2p} \mathrm{d}\tau + 2\omega \int_0^T \|\nabla u_t(\tau)\|_2^2 \mathrm{d}\tau\\
&\le CTR^{2p} + 2\mu \int_0^t \|u_t(\tau)\|_{m+1}^{m+1} \mathrm{d}\tau + 2\omega \int_0^t \|\nabla u_t(\tau)\|_2^2 \mathrm{d}\tau, \qquad (2.10)
\end{aligned}$$

where $2^* = 2N/(N-2), \alpha = 1 - (p+1)/2^*, C = C(\Omega, \omega, p)$, but C is independent of T. Combining (2.9) with (2.10), by choosing T sufficiently small, we get $\|u\|_{\mathcal{H}} \le R$, which indicates that $F(B_R) \subseteq B_R$.

Observe that for any given ball $K \subseteq \mathcal{H}$, any solution to (2.5) with $w \in K$ with finite initial energy must satisfy

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \leq C(\|w\|_{\mathcal{H}}, \|\nabla u_0\|_2, \|u_1\|_2).$$

The above inequality and Simon's compactness lemma imply the compactness of $F(K)$. We need only to prove that $F : B_R \to B_R$ is continuous.

For this purpose, take $w_1, w_2 \in B_R$, substracting the two equations (2.5) for $u_1 = F(w_1)$ and $u_2 = F(w_2)$, set $u = u_1 - u_2$ and then we obtain for all $\eta \in H_0^1(\Omega)$,

$$\begin{aligned} &\langle u_{tt}, \eta\rangle + \int_\Omega \nabla u \nabla \eta + \omega \int_\Omega \nabla u_t \nabla \eta + \mu \int_\Omega (|u_{1t}|^{m-1} u_{1t} - |u_{2t}|^{m-1} u_{2t})\eta \\ = &\int_\Omega (|w_1|^{p-1} w_1 - |w_2|^{p-1} w_2)\eta, \end{aligned}$$

take $\eta = u_t$, integrate the above equality over $(0, t]$, notice that the last term on the left-hand side of the equality is nonnegative, we obtain

$$\begin{aligned} &\frac{1}{2}(\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2) + \omega \int_0^t \|\nabla u_t(\tau)\|_2^2 \mathrm{d}\tau \\ \leq &\int_0^t \int_\Omega p(|w_1(\tau)| + |w_2(\tau)|)^{p-1}(w_1 - w_2)u_t(\tau)\mathrm{d}x\mathrm{d}\tau \\ \leq &\, CR^{2(p-1)} T \|w_1 - w_2\|_{\mathcal{H}}^2 + \omega \int_0^t \|\nabla u_t(\tau)\|_2^2 \mathrm{d}\tau, \end{aligned}$$

which implies that

$$\|F(w_1) - F(w_2)\|_{\mathcal{H}}^2 \leq C\|w_1 - w_2\|_{\mathcal{H}}^2.$$

The proof of the lemma is complete.

Now we are ready to prove Theorem 2.2.

Proof of Theorem 2.2. Combining Lemma 2.3 and Lemma 2.4, the main statement of the theorem is a direct consequence of Schauder's fixed point theorem.

It follows from the above proof that, the local existence time of u merely depends on the norms of the initial data, therefore, if $T_{max} < \infty$, we obtain

$$\lim_{t \to T_{max}} \|u(t)\|_{\mathcal{H}}^2 = \infty \tag{2.11}$$

As a consequence of the energy identity (2.1), $E(t)$ is nonincreasing and the following inequality holds

$$\frac{1}{2}(\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2) \leq \frac{1}{p+1}\|u\|_{p+1}^{p+1} + E(0), \quad \forall\, t \in [0, T_{max}), \tag{2.12}$$

which, together with (2.11) yields

$$\lim_{t \to T_{max}} \|u\|_{p+1} = \infty \tag{2.13}$$

The Sobolev embedding theorem implies

$$\lim_{t \to T_{max}} \|\nabla u\|_2 = \infty \tag{2.14}$$

Moreover, by (2.12) we obtain

$$\|\nabla u\|_2^2 \leq 2E(0) + \frac{2}{p+1}\|u(t)\|_{p+1}^{p+1}, \quad t \in [0, T_{max}),$$

combining with the Gagliardo-Nirenberg inequality, it follows

$$C\|\nabla u\|_2^2 - C \leq \|u(t)\|_{p+1}^{p+1} \leq C\|u(t)\|_q^{(p+1)(1-\theta)}\|\nabla u(t)\|_2^{(p+1)\theta},$$

where $\theta = 2N(p+1-q)/((p+1)(2N+2q-Nq))$.
Since $N(p-1)/2 < q < p+1$ implies $\theta \in (0,1)$ and $(p+1)\theta < 2$, the above inequality combined with (2.14) immediately yields the last assertion of the theorem. This completes the proof of Theorem 2.2.

Lemma 2.5 ([7]) *For every solution of* (1.1)*, given by Theorem 2.2, only one of the following holds,*
(i) *there exists a $t_0 \geq 0$, such that $E(t_0) \leq d$, $u(t_0, \cdot) \in \mathcal{S}$, and remains there for all $t \in [t_0, T_{max})$,*
(ii) *there exists a $t_0 \geq 0$, such that $E(t_0) \leq d$, $u(t_0, \cdot) \in \mathcal{U}$, and remains there for all $t \in [t_0, T_{max})$,*
(iii) *$u(t, \cdot) \in \{u | E(u) \geq d\}$ for all $t \geq 0$.*

Lemma 2.6 *Under the assumptions of Lemma 2.5, the following inequalities hold*

$$J(u) > \frac{p-1}{2(p+1)}\|\nabla u\|_2^2 \quad if \quad 0 \neq \mu \in \mathcal{S}, \tag{2.15}$$

$$d < \frac{p-1}{2(p+1)}\|\nabla u\|_2^2 \quad if \quad 0 \neq \mu \in \mathcal{U}. \tag{2.16}$$

The proof of the above two lemmas are elemental, so we omit it.

3 Boundedness of Global Solutions

Lemma 3.1 *Assume that $E(t) \geq d$ for all $t \geq 0$, then for every $t \geq 0$, there exists a positive constant C, such that*

$$\|\nabla u(t) - \nabla u(t+1)\| \leq C \quad for\ \omega > 0,$$
$$\|u(t) - u(t+1)\|_{m+1} \leq C \qquad for\ \omega = 0.$$

Theorem 3.2 *Assume that $\omega > 0$, let $m < p$ and $E(t) \geq d$ for all $t \geq 0$, then every global solution to* (1.1) *is bounded. Moreover, if $n = 1, 2$ or if $n \geq 3$ and $1 < p < \frac{n+2}{n-2}$, then there exists a positive constant l such that $\mathcal{S}_l \neq \emptyset$,*

$$\lim_{t\to\infty} E(t) = l, \quad \lim_{t\to\infty} \mathrm{dist}_{H_0^1}(u(t), \mathcal{S}_l) = 0, \quad \lim_{t\to\infty} \|u_t\|_2^2 = 0. \tag{3.1}$$

Proof. According to [8], the difficult part is to prove the boundedness of global solution. Once the boundedness result is established, the convergence up to a sequence of solutions of (1.1) towards a steady-state result of the theorem can be arrived by following the same arguments as in [8] step by step.

Taking into account that $u_t(\tau) \in H_0^1(\Omega)$ for a.e. $\tau \geq 0$, combine Poincaré inequality with the energy equality (2.1), we have for every $t > 0$,

$$\int_0^t \|u_t(\tau)\|_2^2 \mathrm{d}\tau \leq \frac{1}{\lambda_1} \int_0^t \|\nabla u_t(\tau)\|_2^2 \mathrm{d}\tau \leq C(E(0) - d).$$

Letting $t \to \infty$, we can conclude

$$\int_0^\infty \|u_t(\tau)\|_2^2 \mathrm{d}\tau < \infty \ \text{ and } \ \int_0^\infty \|\nabla u_t(\tau)\|_2^2 \mathrm{d}\tau < \infty. \tag{3.2}$$

It is easy to observe from the above inequality that, for every $t \geq 0$, there exists a positive constant C, such that

$$\|u_t(t)\|_2^2 \leq C. \tag{3.3}$$

Furthermore, by the definition of $E(t)$, we can obtain

$$\|u(t)\|_{p+1}^{p+1} \geq \frac{p+1}{2} \|\nabla u(t)\|_2^2 - (p+1)E(0). \tag{3.4}$$

Set $\tilde{E}(t) = \|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2$, inspired by [7], we shall prove

$$\int_t^{t+1} \tilde{E}(\tau) \mathrm{d}\tau \leq C, \tag{3.5}$$

where $C > 0$ is a constant.

For this purpose we introduce the function

$$H(t) \triangleq (u(t), u_t(t)) - ME(t),$$

where $M > 0$ to be specified later. Hence and from the energy equality, by applying Hölder and Young's inequality, in view of the convex property of the norm $\|u\|_{m+1}^{m+1}$, we have

$$\begin{aligned}
\dot{H}(t) &= \|u_t(t)\|_2^2 - \|\nabla u(t)\|_2^2 - \omega(\nabla u(t), \nabla u_t(t)) - \mu \int_\Omega |u_t(t)|^{m-1} u_t(t) u(t) \mathrm{d}x \\
&\quad + \|u(t)\|_{p+1}^{p+1} + M\|u_t(t)\|_{m+1}^{m+1} + M\|\nabla u_t(t)\|_2^2 \\
&\geq \|u_t(t)\|_2^2 - \|\nabla u(t)\|_2^2 - \frac{\varepsilon}{2}\|\nabla u(t)\|_2^2 - \frac{1}{2\varepsilon}\|\nabla u_t(t)\|_2^2 - \frac{\varepsilon}{m+1}\|u_t(t)\|_{m+1}^{m+1} \\
&\quad - \frac{m}{m+1}\varepsilon^{-\frac{1}{m}}\|u_t(t)\|_{m+1}^{m+1} + \|u(t)\|_{p+1}^{p+1} + M\|u_t(t)\|_{m+1}^{m+1} + M\|\nabla u_t(t)\|_2^2 \\
&\geq \|u_t(t)\|_2^2 - \|\nabla u(t)\|_2^2 - \frac{\varepsilon}{2}\|\nabla u(t)\|_2^2 - \frac{\varepsilon}{p+1}\|u(t)\|_{p+1}^{p+1} - \frac{\varepsilon}{2}\|u(t)\|_2^2 + \|u(t)\|_{p+1}^{p+1} \\
&\geq \|u_t(t)\|_2^2 + (1-\varepsilon)\|u(t)\|_{p+1}^{p+1} - \left(1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2\lambda_1}\right)\|\nabla u(t)\|_2^2 \\
&\geq \|u_t(t)\|_2^2 + \frac{(p+1)(1-\varepsilon)}{2}\|\nabla u(t)\|_2^2 - (1-\varepsilon)(p+1)E(0) - \left(1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2\lambda_1}\right)\|\nabla u(t)\|_2^2 \\
&\geq \|u_t(t)\|_2^2 + \left(\frac{(p+1)(1-\varepsilon)}{2} - \left(1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2\lambda_1}\right)\right)\|\nabla u(t)\|_2^2 - (p+1)E(0) \\
&\triangleq \delta\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 - (p+1)E(0),
\end{aligned} \tag{3.6}$$

we take $\varepsilon = (p+1)/(4+2p+2/\lambda_1)$, then $\delta = \delta(\varepsilon) \triangleq (p+1)(1-\varepsilon)/2 - (1+\varepsilon/2+\varepsilon/2\lambda_1) > 0$. For this chosen ε, take $M = \max\{\frac{1}{2\varepsilon}, \frac{m}{m+1}\varepsilon^{-\frac{1}{m}}\}$, then all the above inequalities hold.
Take $\eta = \min\{\delta, 1\} > 0$, we get from (3.6) that

$$\dot{H}(t) \geq \eta \tilde{E}(t) - (p+1)E(0). \tag{3.7}$$

Integrate the above inequality over $(t, t+1)$ and then estimate the integral on the left-hand side, from Hölder inequality and (3.3),

$$\begin{aligned}
\int_t^{t+1} \dot{H}(s)\mathrm{d}s &= (u(t+1), u_t(t+1)) - (u(t), u_t(t)) - ME(t+1) + ME(t) \\
&\leq \|u(t+1) - u(t)\| \|u_t(t+1) - u_t(t)\| + M(E(0) - d) \\
&\leq C\|\nabla u(t+1) - \nabla u(t)\| + M(E(0) - d),
\end{aligned}$$

combining (3.7) with Lemma 3.1, the above inequality yields (3.5).
Following the proof of Theorem 2.8[7], we can prove there exists a positive constant κ, such that

$$\tilde{E}(t) \leq \kappa(\tilde{E}(s) + 1) \tag{3.8}$$

for any $0 \leq s \leq t \leq s+1$.
Consequently, (3.5) and (3.8) imply

$$\|u_t(t)\|_2^2 + \|\nabla u(t)\|_2^2 = \int_{t-1}^t \tilde{E}(t)\mathrm{d}s \leq \kappa \int_{t-1}^t (\tilde{E}(s) + 1)\mathrm{d}s \leq \kappa(C+1).$$

The proof is complete.

For the weakly damped case($\omega = 0$), we have the following

Theorem 3.3 *Assume that $\omega = 0$, let $m < p$ and $E(t) \geq d\,\forall\, t \geq 0$, suppose further that*

$$\begin{cases} 1 < p \leq \begin{cases} \frac{N}{N-2} & \text{for } N \geq 3, \\ 5 & \text{for } N = 2, \end{cases} \\ 1 < p < \infty & \text{for } N = 1. \end{cases}$$

Then every global solution to (1.1) *is bounded. Moreover, if $n = 1, 2$ or if $N \geq 3$ and $1 < p < \frac{N}{N-2}$, then there exists a positive constant l such that $\mathcal{S}_l \neq \emptyset$,*

$$\lim_{t\to\infty} E(t) = l, \quad \lim_{t\to\infty} \mathrm{dist}_{H_0^1}(u(t), \mathcal{S}_l) = 0, \quad \lim_{t\to\infty} \|u_t\|_2^2 = 0. \tag{3.9}$$

Similar proof can be done following the arguments of Theorem 3.2 by utilizing Lemma 3.1.

4 Global Existence

Theorem 4.1 *Assume that (1.2) and (1.3) being fulfilled, and let u be the unique local solution to* (1.1). *If $m \geq p$, then problem* (1.1) *admits a unique solution $u(t,x)$ such that for any $T > 0$,*

$$u(t,x) \in C([0,T]; H_0^1(\Omega)),\ u_t(t,x) \in C([0,T]; L^2(\Omega)) \cap L^{m+1}([0,T] \times \Omega).$$

Proof can be done by following the arguments in [9].

Now let us turn to the global existence of solutions starting with suitable initial data.

Theorem 4.2 *Assume that (1.2) and (1.3) being fulfilled, and let u be the unique local solution to (1.1) as in Theorem 2.2. Then there exists a $t_0 \in [0, T_{max})$, such that $u(t_0) \in \mathcal{S}$ and $E(u(t_0)) < d$ if and only if $T_{max} = \infty$ and $\lim_{t\to\infty} \|\nabla u(t)\|_2 = \lim_{t\to\infty} \|u_t(t)\|_2 = 0$.*

Proof. Necessity. Consider the case $\omega > 0, \mu > 0$. Since the energy function $E(t)$ is nonincreasing, by virtue of Lemma 2.5(i), we have $u(t) \in \mathcal{S}$ and $E(t) < d$, $\forall\, t \in [t_0, T_{max})$.

Combining (2.15) with the definition of $E(t)$, it yields, there exists a $M > 0$, such that

$$\|\nabla u(t)\|_2^2 + \|u_t(t)\|_2^2 \le M \quad \forall\, t \in [0, T_{max}) \tag{4.1}$$

which implies that $T_{max} = \infty$ by virtue of Theorem 2.2.

It follows again from the energy identity (2.1)

$$\int_{t_0}^{t} \|\nabla u_t(\tau)\|_2^2 \mathrm{d}\tau < \frac{d}{\omega}, \quad \int_{t_0}^{t} \|u_t(\tau)\|_{m+1}^{m+1} \mathrm{d}\tau < \frac{d}{\mu}, \quad \forall\, t \in [t_0, \infty). \tag{4.2}$$

By integrating over $[t_0, t]$ the trivial inequality

$$\frac{d}{dt}\left((1+t)E(t)\right) \le E(t)$$

we have

$$(1+t)E(t) \le (1+t_0)E(t_0) + \frac{1}{2}\int_{t_0}^{t} \|u_t(\tau)\|_2^2 \mathrm{d}\tau + \int_{t_0}^{t} J(u(\tau))\mathrm{d}\tau.$$

Since $J(u) \le CI(u)$(see [12] Lemma 2.5), the above inequality yields

$$(1+t)E(t) \le (1+t_0)E(t_0) + \frac{1}{2}\int_{t_0}^{t} \|u_t(\tau)\|_2^2 \mathrm{d}\tau + C\int_{t_0}^{t} I(u(\tau))\mathrm{d}\tau, \ \forall\, t \in [t_0, \infty). \tag{4.3}$$

Moreover, by testing the equation (1.1) with u, we have for all $t \in [t_0, \infty)$,

$$\langle u_{tt}(t), u(t)\rangle + \|\nabla u(t)\|_2^2 + \omega\int_\Omega \nabla u(t)\nabla u_t(t)\mathrm{d}x + \mu\int_\Omega |u_t(t)|^{m-1}u_t(t)u(t)\mathrm{d}x = \|u(t)\|_{p+1}^{p+1},$$

which implies

$$I(u(t)) = -\frac{d}{dt}\int_\Omega u_t(t)u(t)\mathrm{d}x + \|u_t(t)\|_2^2 - \mu\int_\Omega |u_t(t)|^{m-1}u_t(t)u(t)\mathrm{d}x - \omega\int_\Omega \nabla u(t)\nabla u_t(t)\mathrm{d}x.$$

By integrating the above equality over $[t_0, t]$, we have

$$\begin{aligned}\int_{t_0}^{t} I(u(\tau))\mathrm{d}\tau \le{}& \int_{t_0}^{t} \|u_t(\tau)\|_2^2 \mathrm{d}\tau + \|u(t)\|_2^2\|u_t(t)\|_2^2 + \|u_0\|_2^2\|u_1\|_2^2 \\ &+ \int_{t_0}^{t} \|\nabla u(\tau)\|_2^2\|\nabla u_t(\tau)\|_2^2\mathrm{d}\tau + \int_{t_0}^{t}\int_\Omega \left||u_t(\tau)|^{m-1}u_t(\tau)u(\tau)\right| \mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

In view of [12] Lemma 3.4, we get

$$\int_{t_0}^{t} \|u_t(\tau)\|_2^2 \mathrm{d}\tau \le C(t-t_0)^{\frac{m-1}{m+1}}\left(\int_{t_0}^{t} \|u_t(\tau)\|_{m+1}^{m+1}\mathrm{d}\tau\right)^{\frac{2}{m+1}},$$

$$\int_{t_0}^{t} \left||u_t(\tau)|^{m-1}u_t(\tau)u(\tau)\right| \mathrm{d}\tau \le C(t-t_0)^{\frac{1}{m+1}}\left(\int_{t_0}^{t} \|u_t(\tau)\|_{m+1}^{m+1}\mathrm{d}\tau\right)^{\frac{m}{m+1}}.$$

It follows from (4.1)–(4.3) that

$$(1+t)E(t) \le (1+t_0)E(0) + C_1(t-t_0)^{\frac{m-1}{m+1}} + C_2(t-t_0)^{\frac{1}{m+1}},$$

which indicates

$$\lim_{t\to\infty} E(t) = 0.$$

Since $u(t) \in \mathcal{S}$, $\forall\, t \in [t_0, \infty)$, it holds

$$\lim_{t\to\infty} \|u_t(t)\|_2^2 = \lim_{t\to\infty} J(u(t)) = 0.$$

Using (2.15) again and we obtain the final result.

Sufficiency. The Sobolev embedding theorem implies $\lim_{t\to\infty} \|u(t)\|_{p+1} = 0$, which indicates $\lim_{t\to\infty} J(u(t)) \le \lim_{t\to\infty} E(u(t,\cdot), u_t(t,\cdot)) = 0$ and $\lim_{t\to\infty} I(u(t)) = 0$. Note that $\mathcal{S}$ is a bounded neighborhood of 0 in $H_0^1(\Omega)$, we can conclude there exists a $t_0 \in [0,\infty)$, such that $u(t_0,\cdot) \in \mathcal{S}$ and $E(u(t_0,\cdot), u_t(t_0,\cdot)) < d$.
The proof of the theorem is then complete.

5 Blow up

We come to a blow up result for solutions starting in the unstable set.

Theorem 5.1 *Suppose $m < p < \frac{2(m+1)}{N} + 1$, assume further that (1.2) and (1.3) hold and $(u(t), u_t(t)) = S(t)(u_0, u_1)$ be a local solution to problem* (1.1). *A necessary and sufficient condition for nonglobality, blow up by Theorem 2.2, is there exists a $t_0 \ge 0$, such that $u(t_0) \in \mathcal{U}$ and $E(u(t_0)) < d$.*

This theorem is an extension of Iketa's work [12], in which a necessary and sufficient condition of blowing up was given for the linear weakly damped case($\omega = 0, m = 1$). The concavity method no longer applies in this particular situation when nonlinear dissipation appears, we need the following blow up result here

Lemma 5.2 [17] *Let $m < p$, $\omega \ge 0$, and suppose the conditions (1.2) and (1.3) are fulfilled, then any weak solution to problem* (1.1) *blows up in finite time if the initial energy* $E(0)$ *is negative.*

Proof of Theorem 5.1. Sufficiency. Suppose on the contrary that for some initial data satisfies the condition of Theorem 5.1, the weak solution of problem (1.1) exists for all $t \ge 0$, then $E(t)$ has to be nonnegative for all $t \ge 0$. Since if there exists a t_1, such that $E(t_1) < 0$, by Lemma 5.2, the solution must blow up in finite time. Thus, we have $E(t) \ge 0$ for all $t \ge 0$, which leads to a constant control of the rate of energy decrease. That is, from the energy identity (2.1), we obtain

$$d \ge E(t_0) - E(t) = \omega \int_{t_0}^{t} \|\nabla u_t(\tau)\|_2^2 \mathrm{d}\tau + \mu \int_{t_0}^{t} \|u_t(\tau)\|_{m+1}^{m+1} \mathrm{d}\tau, \quad \forall\, t \ge t_0.$$

Denote $F(t) = \|u(t)\|_2^2$, it follows from equation of (1.1) that

$$F''(t) = 2\left(\|u_t(t)\|_2^2 - I(u(t)) - \omega(\nabla u(t), \nabla u_t(t)) - \mu \int_\Omega |u_t(t)|^{m-1} u_t(t) u(t) \mathrm{d}x \right). \tag{5.1}$$

To estimate the integral $\omega(\nabla u(t), \nabla u_t(t))$, we use Hölder and Young's inequality

$$|\omega(\nabla u(t), \nabla u_t(t))| \leq \varepsilon_1 \|\nabla u(t)\|_2^2 + C(\epsilon_1)\|\nabla u_t(t)\|_2^2. \tag{5.2}$$

To estimate the term $\mu \int_\Omega |u_t(t)|^{m-1} u_t(t) u(t) \mathrm{d}x$, we use Hölder inequality and interpolation inequality

$$\begin{aligned}
\mu \left| \int_\Omega |u_t(t)|^{m-1} u_t(t) u(t) \mathrm{d}x \right| &\leq \|u(t)\|_{m+1} \|u_t(t)\|_{m+1}^m \leq \|u(t)\|_2^\theta \|u(t)\|_{p+1}^{1-\theta} \|u_t(t)\|_{m+1}^m \\
&\leq C\|u_t(t)\|_{m+1}^m \|u(t)\|_{p+1}^{1-(p+1)/(m+1)-\theta+(p+1)\theta/2} \|u(t)\|_{p+1}^{(p+1)/(m+1)}
\end{aligned} \tag{5.3}$$

where $\theta = (\frac{1}{m+1} - \frac{1}{p+1})/(\frac{1}{2} - \frac{1}{p+1})$.

In the above estimates, we used the equality followed from Lemma 2.5(ii), i.e.,

$$\|u(t)\|_2^2 \leq \frac{1}{\lambda_1}\|\nabla u(t)\|_2^2 \leq \frac{1}{\lambda_1}\|u(t)\|_{p+1}^{p+1}, \ \forall\, t \geq t_0.$$

Since $1 - (p+1)/(m+1) - \theta + (p+1)\theta/2 = 0$, by using Young's inequality, we get from (5.3) that

$$\mu \left| \int_\Omega |u_t(t)|^{m-1} u_t(t) u(t) \mathrm{d}x \right| \leq \varepsilon_2 \|u(t)\|_{p+1}^{p+1} + C(\varepsilon_2)\|u_t(t)\|_{m+1}^{m+1}. \tag{5.4}$$

It follows from (5.1), (5.2) and (5.4) that

$$\begin{aligned}
&\frac{1}{2}F''(t) + C(\varepsilon_1)\|\nabla u_t(t)\|_2^2 + C(\varepsilon_2)\|u_t(t)\|_{m+1}^{m+1} \\
&\geq \|u_t(t)\|_2^2 - I(u(t)) - \varepsilon_1\|\nabla u(t)\|_2^2 - \varepsilon_2\|u(t)\|_{p+1}^{p+1}.
\end{aligned} \tag{5.5}$$

In view of the inequality

$$\begin{aligned}
-I(u(t)) &\geq -I(u(t)) + \sigma(E(t) - E(t_0)) \\
&\geq (1 - \sigma/(p+1))\|u(t)\|_{p+1}^{p+1} + \frac{\sigma}{2}\|u_t(t)\|_2^2 + (\frac{\sigma}{2} - 1)\|\nabla u(t)\|_2^2 - \sigma E(t_0),
\end{aligned}$$

where the constant $\sigma > 2$ will be chosen later.

We obtain from (5.5) the inequality

$$\begin{aligned}
&\frac{1}{2}F''(t) + C(\varepsilon_1)\|\nabla u_t(t)\|_2^2 + C(\varepsilon_2)\|u_t(t)\|_{m+1}^{m+1} \\
&\geq (1 + \sigma/2)\|u_t(t)\|_2^2 + (1 - \sigma/(p+1) - \varepsilon_2)\|u(t)\|_{p+1}^{p+1} \\
&\quad + (\sigma/2 - 1 - \varepsilon_1)\|\nabla u(t)\|_2^2 - \sigma E(t_0), \quad \forall\, t \geq t_0.
\end{aligned} \tag{5.6}$$

Choose the constant σ so that

$$\frac{2d(p+1)(1+\varepsilon_1)}{(p+1)d - (p-1)E(t_0)} \leq \sigma < p+1,$$

which is possible since $E(t_0) < d$, and this guarantees $\sigma > 2$.

Then, using this choice and (2.16) we obtain

$$(\sigma/2 - 1 - \varepsilon_1)\|\nabla u(t)\|_2^2 - \sigma E(t_0) \geq 0, \quad \forall\, t \geq t_0.$$

For this chosen σ, we choose ε_2 small enough so that

$$C_1 = 1 - \sigma/(p+1) - \varepsilon_2 > 0.$$

Finally, the inequality (5.6), Lemma 2.5 and Lemma 2.6 yield

$$\begin{aligned} & F''(t) + C(\varepsilon_1)\|\nabla u_t(t)\|_2^2 + C(\varepsilon_2)\|u_t(t)\|_{m+1}^{m+1} \\ \geq & \, C_1\|u(t)\|_{p+1}^{p+1} \geq C_1\|\nabla u(t)\|_2^2 \geq 2C_1 d(p+1)/(p-1), \quad \forall\, t \geq 0. \end{aligned} \tag{5.7}$$

Integrate two times the inequality (5.7) over $[t_0, t]$ and take into account

$$\int_0^t \|\nabla u_t(\tau)\|_2^2 + \|u_t(\tau)\|_{m+1}^{m+1} \mathrm{d}\tau \leq d,$$

we arrive at

$$F(t) \geq C_1\{d(p+1)/(p-1)\}t^2 + \{(-C(\varepsilon_1) - C(\varepsilon_2))d + F'(t_0)\}t + F(t_0), \tag{5.8}$$

thus, the norm $\|u(t)\|_2$ has at least linear growth for $t \geq t_0$. On the other hand, we estimate the norm $\|u(t)\|_2$ from above. For $t \geq t_0$, we have

$$\begin{aligned} \|u(t)\|_2 &\leq \|u(t_0)\|_2 + \int_{t_0}^t \|u_t(\tau)\|_2 \mathrm{d}\tau \\ &\leq \|u(t_0)\|_2 + C(t-t_0)^{\frac{m-1}{m+1}} \left(\int_{t_0}^t \|u_t(\tau)\|_{m+1}^{m+1} \mathrm{d}\tau \right)^{\frac{1}{m+1}} \\ &\leq \|u(t_0)\|_2 + C(t-t_0)^{\frac{m-1}{m+1}}, \end{aligned} \tag{5.9}$$

where in the above estimates we used the Hölder inequality with respect to t, the boundedness of the integral $\int_{t_0}^t \|u_t(\tau)\|_{m+1}^{m+1}\mathrm{d}\tau$. Obviously, the inequality (5.9) contradicts with the inequality (5.8).

Sufficiency. Suppose $T_{max} < \infty$, then it follows from the last assertion of Theorem 2.2 that

$$\lim_{t\to T_{max}} \|u(t)\|_{m+1} = \infty. \tag{5.10}$$

Observe the energy equality (2.1), we obtain

$$E(0) - E(t) \geq \mu \int_0^t \|u_t(\tau)\|_{m+1}^{m+1} \mathrm{d}\tau \geq \mu t^{-m} \left| \|u(t)\|_{m+1} - \|u_0\|_{m+1} \right|,$$

which combined with (5.10) imply

$$\lim_{t\to T_{max}} E(t) = -\infty.$$

On the other hand, since

$$\frac{p+1}{2}\|u_t(t)\|_2^2 + \frac{p-1}{2}\|\nabla u(t)\|_2^2 + I(u(t)) \leq (p+1)E(0),$$

we can conclude $\lim\limits_{t\to T_{max}} I(u(t)) = -\infty$, which implies there exists a $t_0 \in [0, T_{max})$, such that

$$J(u(t_0)) \leq E(u(t_0)) < d, \ I(u(t_0)) < 0.$$

The proof of Theorem 5.1 is then complete.

Remark 5.3 *As a byproduct of our proof, it is clear that under the restrictions on* m *and* p, $T_{max} < \infty$ *if and only if* $E(t) \to -\infty$ *as* $t \to T_{max}$. *In particular, the blow up has a full characterization in terms of negative energy blow up.*

Remark 5.4 *It can be observed from the proof that the condition* $m < p$ *was given for necessity, and* $p < 2(m+1)/N + 1$ *was given for sufficiency.*

6 Exponential Decay

In what follows, we shall assume, without loss of generality, that $\omega = \mu = 1$.

Theorem 6.1 *Suppose that* $\max\{m, p\} \leq \frac{N+2}{N-2}, u_0 \in \mathcal{N}_+$ *and* u_0, u_1 *satisfies*

$$\alpha \triangleq C_{p+1}^{p+1} \left(\frac{2(p+1)}{p-1} E(0) \right)^{\frac{p-1}{2}} < 1. \tag{6.1}$$

Then there exist positive constants C *and* β *such that the global solution to problem* (1.1) *satisfies*

$$E(t) \leq Ce^{-\beta t}.$$

Lemma 6.2 *Under the assumptions of Theorem 6.1, we have for all* $t \geq 0$, $u(t) \in \mathcal{N}_+$.

Proof . Since $I(u_0) > 0$, there exists a $T > 0$, such that $I(u(t)) \geq 0$ for all $t \in [0, T)$, which tells

$$\begin{aligned} J(u(t)) &= \frac{1}{2}\|\nabla u(t)\|_2^2 - \frac{1}{p+1}\|u(t)\|_{p+1}^{p+1} \\ &\geq \frac{p-1}{2(p+1)}\|\nabla u(t)\|_2^2 \quad \forall t \in [0, T). \end{aligned} \tag{6.2}$$

Therefore,

$$\|\nabla u(t)\|_2^2 \leq \frac{2(p+1)}{p-1}J(u(t)) \leq \frac{2(p+1)}{p-1}E(u(t)) \leq \frac{2(p+1)}{p-1}E(0) \quad \forall t \in [0, T). \tag{6.3}$$

The Sobolev embedding theorem entails

$$\begin{aligned} \|u(t)\|_{p+1}^{p+1} &\leq C_{p+1}^{p+1}\|\nabla u(t)\|_2^{p+1} = C_{p+1}^{p+1}\|\nabla u(t)\|_2^2 C_{p+1}^{p+1}\|\nabla u(t)\|_2^{p-1} \\ &\leq C_{p+1}^{p+1}\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{p-1}{2}} \|\nabla u(t)\|_2^2 \\ &= \alpha\|\nabla u(t)\|_2^2 < \|\nabla u(t)\|_2^2 \quad \forall t \in [0, T), \end{aligned} \tag{6.4}$$

which implies $u(t) \in \mathcal{N}_+$, $\forall t \in [0, T)$. Note that $\lim\limits_{t \to T} C_{p+1}^{p+1}(\frac{2(p+1)}{p-1}E(u(t), u_t(t)))^{\frac{p-1}{2}} \leq \alpha$, we can repeat the procedure and extend T to $2T$, by continuing the argument and the lemma is so proved.

Proof of Theorem 6.1. We modify the function defined in Section 3 as follows

$$G(t) \triangleq \varepsilon\left((u(t), u_t(t)) + \frac{1}{2}\|\nabla u(t)\|_2^2\right) + E(t).$$

We shall prove, for ε sufficiently small, there exist two positive constants c_1 and c_2 such that

$$c_1 E(t) \leq G(t) \leq c_2 E(t). \tag{6.5}$$

Actually,

$$
\begin{aligned}
G(t) &\leq E(t) + \frac{\varepsilon}{2}(\|u_t(t)\|_2^2 + \|u(t)\|_2^2 + \|\nabla u(t)\|_2^2) \\
&\leq (1+\varepsilon)E(t) + \frac{\varepsilon}{2}\left(1 + \frac{1}{\lambda_1}\right)\|\nabla u(t)\|_2^2 \\
&\leq (1+\varepsilon)E(t) + \frac{\varepsilon}{2}\left(1 + \frac{1}{\lambda_1}\right)\frac{2(p+1)}{p-1}E(t) \triangleq c_2 E(t),
\end{aligned}
$$

and

$$
\begin{aligned}
G(t) &\geq E(t) - \varepsilon\left\{\delta\|u(t)\|_2^2 + \frac{1}{4\delta}\|u_t(t)\|_2^2\right\} + \frac{\varepsilon}{2}\|\nabla u(t)\|_2^2 \\
&\geq E(t) - \frac{\varepsilon}{4\delta}\|u_t(t)\|_2^2 + \varepsilon\left(\frac{1}{2} - \frac{\delta}{\lambda_1}\right)\|\nabla u(t)\|_2^2.
\end{aligned}
$$

Take $\delta = \varepsilon$, then choose ε small enough , we see there exists a $c_1 > 0$, such that $G(t) \geq c_1(t)$. By Poincaré inequality and keeping in mind the energy equality (2.1), one obtains

$$
\begin{aligned}
G'(t) &= -(\|u_t(t)\|_{m+1}^{m+1} + \|\nabla u(t)\|_2^2) \\
&\quad +\varepsilon\left(\|u_t(t)\|_2^2 - \|\nabla u(t)\|_2^2 - \int_\Omega |u_t(t)|^{m-1}u_t(t)u(t)\mathrm{d}x + \|u(t)\|_{p+1}^{p+1}\right) \\
&\leq -\|u_t(t)\|_{m+1}^{m+1} - \left(1 - \frac{\varepsilon}{\lambda_1}\right)\|\nabla u(t)\|_2^2 - \varepsilon\|\nabla u(t)\|_2^2 \\
&\quad -\varepsilon\int_\Omega |u_t(t)|^{m-1}u_t(t)u(t)\mathrm{d}x + \varepsilon\|u(t)\|_{p+1}^{p+1}
\end{aligned}
\tag{6.6}
$$

To estimate the integral $\int_\Omega |u_t(t)|^{m-1}u_t(t)u(t)\mathrm{d}x$, we use (6.3), Poincaré and Young's inequality

$$
\begin{aligned}
\int_\Omega |u_t(t)|^{m-1}u_t(t)u(t)\mathrm{d}x &\leq \delta\|u(t)\|_{m+1}^{m+1} + C(\delta)\|u_t(t)\|_{m+1}^{m+1} \\
&\leq \delta C_0\|\nabla u(t)\|_2^{m+1} + C(\delta)\|u_t(t)\|_{m+1}^{m+1} \\
&= \delta C_0\|\nabla u(t)\|_2^2\|\nabla u(t)\|_2^{m-1} + C(\delta)\|u_t(t)\|_{m+1}^{m+1} \\
&\leq \delta C_0\left(\frac{2(p+1)}{p-1}E(0)\right)^{\frac{m-1}{2}}\frac{2(p+1)}{p-1}E(t) + C(\delta)\|u_t(t)\|_{m+1}^{m+1} \\
&\leq \delta C E(t) + C(\delta)\{\|u_t(t)\|_{m+1}^{m+1} + \|\nabla u_t(t)\|_2^2\}.
\end{aligned}
$$

To estimate the norm $\|u(t)\|_{p+1}^{p+1}$ use Lemma 6.2 to obtain for some $0 < \lambda < 1$

$$
\begin{aligned}
\|u(t)\|_{p+1}^{p+1} &= \lambda\|u(t)\|_{p+1}^{p+1} + (1-\lambda)\|u(t)\|_{p+1}^{p+1} \\
&\leq \lambda\left(\frac{p+1}{2}\|u_t(t)\|_2^2 + \frac{p+1}{2}\|\nabla u(t)\|_2^2 - (p+1)E(t)\right) + (1-\lambda)\alpha\|\nabla u(t)\|_2^2.
\end{aligned}
$$

Then (6.6) turns into

$$\begin{aligned}
G'(t) &\le -\|u_t(t)\|_{m+1}^{m+1} - \left(1-\frac{\varepsilon}{\lambda_1}\right)\|\nabla u(t)\|_2^2 - \varepsilon\|\nabla u(t)\|_2^2 + \varepsilon\delta C E(t) \\
&\quad +\varepsilon\delta C\{\|u_t(t)\|_{m+1}^{m+1} + \|\nabla u_t(t)\|_2^2\} + (1-\lambda)\varepsilon\alpha\|\nabla u\|_2^2 \\
&\quad +\lambda\varepsilon\left(\frac{p+1}{2}\|u_t(t)\|_2^2 + \frac{p+1}{2}\|\nabla u(t)\|_2^2 - (p+1)E(t)\right) \\
&\le -(1-\varepsilon C(\delta))\|u_t(t)\|_{m+1}^{m+1} - \left[1-\varepsilon\left(\frac{1}{\lambda_1} + C(\delta) + \frac{\lambda}{\lambda_1}\frac{p+1}{2}\right)\right]\|\nabla u_t(t)\|_2^2 \\
&\quad -\varepsilon[\lambda(p+1) - \delta C]E(t) + \varepsilon\left[(1-\lambda)\alpha + \frac{\lambda(p+1)}{2} - 1\right]\|\nabla u\|_2^2,
\end{aligned}$$

take $\gamma = 1-\alpha$ in the above inequality, one obtains

$$\begin{aligned}
G'(t) &\le -(1-\varepsilon C(\delta))\|u_t(t)\|_{m+1}^{m+1} - \left[1-\varepsilon\left(\frac{1}{\lambda_1} + C(\delta) + \frac{\lambda}{\lambda_1}\frac{p+1}{2}\right)\right]\|\nabla u_t(t)\|_2^2 \\
&\quad -\varepsilon[\lambda(p+1) - \delta C]E(t) + \varepsilon\left[\frac{p-1}{2}\lambda - \gamma(1-\lambda)\right]\|\nabla u(t)\|_2^2.
\end{aligned}$$

By taking λ close to 1, so that $(p-1)\lambda/2 - \gamma(1-\lambda) > 0$, in view of (6.3), we obtain

$$\begin{aligned}
G'(t) &\le -(1-\varepsilon C(\delta))\|u_t(t)\|_{m+1}^{m+1} - \left[1-\varepsilon\left(\frac{1}{\lambda_1} + C(\delta) + \frac{\lambda}{\lambda_1}\frac{p+1}{2}\right)\right]\|\nabla u_t(t)\|_2^2 \\
&\quad -\varepsilon[\lambda(p+1) - \delta C]E(t) + \varepsilon\left[(p+1)\lambda - \frac{2(p+1)}{p-1}\gamma(1-\lambda)\right]E(t) \\
&= -(1-\varepsilon C(\delta))\|u_t(t)\|_{m+1}^{m+1} - \left[1-\varepsilon\left(\frac{1}{\lambda_1} + C(\delta) + \frac{\lambda}{\lambda_1}\frac{p+1}{2}\right)\right]\|\nabla u_t(t)\|_2^2 \\
&\quad -\varepsilon\left[\frac{2(p+1)}{p-1}\gamma(1-\lambda) - \delta C\right]E(t)
\end{aligned} \tag{6.7}$$

Then we choose δ small enough to guarantee

$$\frac{2(p+1)}{p-1}\gamma(1-\lambda) - \delta C > 0.$$

For this chosen δ, take ε sufficiently small to satisfy

$$1-\varepsilon C(\delta) \ge 0, 1-\varepsilon\left(\frac{1}{\lambda_1} + C(\delta) + \frac{\lambda}{\lambda_1}\frac{p+1}{2}\right) > 0,$$

we reach the following differential inequality

$$G'(t) \le -\frac{\varepsilon}{c_1}\left(\frac{2(p+1)}{p-1}\gamma(1-\lambda) - \delta C\right)G(t).$$

A simple integration yields

$$E(t) \le \frac{1}{c_1}G(t) \le \frac{1}{c_1}G(0)e^{-\beta t} \triangleq Ce^{-\beta t}$$

where $\beta = \frac{\varepsilon}{c_1}(\gamma\frac{2(p+1)}{p-1}(1-\lambda) - \delta C)$. The proof is complete.

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