Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **35**, 1-14; **http://www.math.u-szeged.hu/ejqtde/**

Triple positive solutions for second-order four-point boundary value problem with sign changing nonlinearities

Dapeng Xie[a,*], Yang Liu[a], Chuanzhi Bai[b]
[a] *Department of Mathematics, Hefei Teachers College, Hefei, Anhui 230061, P R China*
[b] *Department of Mathematics, Huaiyin Teachers College, Huaian, Jiangsu 223300, P R China*

Abstract

In this paper, we study the existence of triple positive solutions for second-order four-point boundary value problem with sign changing nonlinearities. We first study the associated Green's function and obtain some useful properties. Our main tool is the fixed point theorem due to Avery and Peterson. The results of this paper are new and extent previously known results.

2000 Mathematics Subject Classication : **34B10, 34B18, 34B27**.

Keywords: Four-point boundary value problem; Triple positive solutions; Cone; Sign changing nonlinearities.

1. Introduction

Boundary value problems (BVP) with sign changing nonlinearities have received special attention from many authors in recent years. Recently, existence results for positive solutions of second-order three-point boundary value problems with sign changing nonlinearities have been studied by some authors (see [3-6]). In [4], by applying the Krasnoseiskii fixed-point theorem in a cone, Liu have proved the existence of at least one positive solution for the following second-order three-point BVP

$$\begin{cases} u''(t) + \lambda a(t) f(u(t)) = 0, \quad t \in (0,1), \\ u(0) = 0, \quad u(1) = \beta u(\eta). \end{cases} \tag{1.1}$$

where λ is a positive parameter, $0 < \beta < 1$, $0 < \eta < 1$, $f : [0,+\infty) \to (0,+\infty)$ is continuous and nondecreasing, $a : [0,1] \to (-\infty,+\infty)$ is continuous and such that $a(t) \geq 0$, $t \in [0,\eta]$; $a(t) \leq 0$, $t \in [\eta,1]$. Moreover, $a(t)$ does not vanish identically on any subinterval of $[0,1]$.

Very recently, in [2], by applying the Krasnoseiskii fixed-point theorem in a cone, we improved the results obtained in [4] and established the existence of at least one or two positive solutions for second-order four-point BVP as follows

E-mail address: xiedapeng9@yahoo.com.cn

$$\begin{cases} u''(t) + \lambda a(t) f(u(t)) = 0, & t \in (0,1), \\ u(0) = \alpha u(\xi), \quad u(1) = \beta u(\eta). \end{cases} \tag{1.2}$$

where λ is a positive parameter, $0 \le \alpha, \beta < 1$, $0 < \xi < \eta < 1$, $f : [0, +\infty) \to (0, +\infty)$ is continuous and nondecreasing, $a : [0, 1] \to (-\infty, +\infty)$ is continuous and such that $a(t) \le 0$, $t \in [0, \xi]$; $a(t) \ge 0$, $t \in [\xi, \eta]$; $a(t) \le 0$, $t \in [\eta, 1]$. Moreover, $a(t)$ does not vanish identically on any subinterval of $[0, 1]$.

Inspired by [2], in this paper, by using the fixed point theorem due to Avery and Peterson, we consider the following second-order four-point boundary value problem with sign changing nonlinearities

$$\begin{cases} u''(t) + a(t) f(u(t)) = 0, & t \in (0,1), \\ u(0) = \alpha u'(1), \quad u(\xi) = \beta u(\eta). \end{cases} \tag{1.3}$$

We make the following assumptions:

(H_0) $0 < \max\{\eta, 1-\eta\} < \xi \le 1$, $-\min\{\eta, 1-\eta\} \le \alpha \le -\frac{(1-\xi)(\xi-\beta\eta)}{\beta(1-\eta)-(1-\xi)}$, $\frac{(1-\xi)(\xi+\min\{\eta,1-\eta\})}{\min\{\eta,1-\eta\}(1-\eta)-\eta(1-\xi)} < \beta < 1$;

(H_1) $f : [0, +\infty) \to (0, +\infty)$ is continuous and nondecreasing;

(H_2) $a : [0, 1] \to (-\infty, +\infty)$ is continuous and such that $a(t) \ge 0$, $t \in [0, \eta]$; $a(t) \le 0$, $t \in [\eta, \xi]$; $a(t) \ge 0$, $t \in [\xi, 1]$. Moreover, $a(t)$ does not vanish identically on any subinterval of $[0, 1]$;

(H_3) There exists constant $\tau_1 \in (\beta\eta, \eta)$, such that

$$g(t) = \delta a^+(\eta - \delta t) - \frac{1}{\Lambda} a^-(\eta + t) \ge 0, \ t \in [0, \xi - \eta],$$

where $a^+(t) = \max\{a(t), 0\}$, $a^-(t) = -\min\{a(t), 0\}$ and

$$\delta = \frac{\eta - \tau_1}{\xi - \eta}, \quad \Lambda = \frac{\min\{-\alpha(1-\beta), \beta(1-\eta) - (1-\xi)\}\tau_1}{(\xi - \eta)[\xi + \eta - \beta(\alpha + \eta)]}.$$

The aim of this paper is to improve the previous existence results in [4]. By using the fixed point theorem due to Avery and Peterson, we will study the existence of triple positive solutions for the BVP (1.3) under some conditions concerning the function a that is sign-changing on $[0, 1]$. To the best of our knowledge, to date no paper has appeared in the literature which discusses the existence of positive solutions for the BVP (1.3). This paper attempts to fill this gap in the literature.

Remark 1.1. We point out that condition (H_3) is reasonable. For example, we take $\eta = \frac{2}{5}$, $\xi = \beta = \frac{4}{5}$, $\alpha = -\frac{3}{10}$, $\tau_1 = \frac{9}{25}$, and

$$a(t) = \begin{cases} 40(\frac{2}{5} - t), & t \in [0, \frac{2}{5}], \\ \frac{560}{27}(t - \frac{2}{5})(t - \frac{4}{5}), & t \in [\frac{2}{5}, \frac{4}{5}], \\ (t - \frac{4}{5}), & t \in [\frac{4}{5}, 1], \end{cases}$$

Then, it is easy to check that (H_0) holds, and $\frac{8}{25} = \beta\eta < \tau_1 < \eta = \frac{2}{5}$, $\delta = \frac{1}{10}$, $\Lambda = \frac{27}{560}$. Moreover, for $t \in [0, \frac{2}{5}]$, we have

$$\begin{aligned} g(t) &= \delta a^+(\eta - \delta t) - \frac{1}{\Lambda} a^-(\eta + t) \\ &= \frac{1}{10} a^+(\frac{2}{5} - \frac{1}{10}t) - \frac{560}{27} a^-(\frac{2}{5} + t) \\ &= t^2 \geq 0. \end{aligned}$$

Thus, condition (H_3) holds, and condition (H_2) holds too.

2. Preliminary lemmas

Let ϑ and θ be nonnegative continuous convex functionals on K, κ be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K. Then for positive real numbers a, b, c and d, we define the following convex sets:

$$K(\vartheta, d) = \{u \in K : \vartheta(u) < d\};$$

$$K(\vartheta, \kappa, b, d) = \{u \in K : b \leq \kappa(u), \vartheta(u) \leq d\};$$

$$K(\vartheta, \theta, \kappa, b, c, d) = \{u \in K : b \leq \kappa(u), \theta(u) \leq c, \vartheta(u) \leq d\};$$

$$K(\vartheta, \psi, a, d) = \{u \in K : a \leq \psi(u), \vartheta(u) \leq d\}.$$

Lemma 2.1 [1]. *Let K be a cone in a Banach space E. Let ϑ and θ be nonnegative continuous convex functionals on K, κ be a nonnegative continuous concave functional on K, and ψ be a nonnegative continuous functional on K satisfying $\psi(\lambda u) \leq \lambda\psi(u)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d,*

$$\kappa(u) \leq \psi(u) \text{ and } \|u\| \leq M\vartheta(u) \tag{2.1}$$

for all $u \in \overline{K(\vartheta, d)}$. Suppose $T : \overline{K(\vartheta, d)} \to \overline{K(\vartheta, d)}$ is completely continuous and there exists positive numbers a, b and c with $a < b$ such that

(C_1) $\{u \in K(\vartheta, \theta, \kappa, b, c, d) : \kappa(u) > d\} \neq \emptyset$ and $\kappa(Tu) > b$ for $u \in K(\vartheta, \theta, \kappa, b, c, d)$;

(C_2) $\kappa(Tu) > b$, for $u \in K(\vartheta, \kappa, b, d)$ with $\theta(Tu) > c$;

(C_3) $0 \notin K(\vartheta, \psi, a, d)$ and $\psi(Tu) < a$ for $u \in K(\vartheta, \psi, a, d)$, with $\psi(u) = a$.

Then T has at least three fixed points u_1, u_2 and $u_3 \in \overline{K(\vartheta, d)}$ such that

$$\vartheta(u_i) \leq d \text{ for } i = 1, 2, 3,\ b < \kappa(u_1),\ a < \psi(u_2) \text{ with } \kappa(u_2) < b \text{ and } \psi(u_3) < a.$$

Lemma 2.2. *If $\Delta := \alpha + \xi - \beta(\alpha + \eta) \neq 0$, then for $y \in C[0, 1]$, the boundary value problem*

$$u''(t) + y(t) = 0, \quad t \in (0, 1), \tag{2.2}$$

$$u(0)=\alpha u'(1),\quad u(\xi)=\beta u(\eta), \tag{2.3}$$

has a unique solution

$$u(t)=\frac{\alpha}{\Delta}[(1-\beta)t-(\xi-\beta\eta)]\int_0^1 y(s)ds+\frac{\alpha+t}{\Delta}\int_0^\xi(\xi-s)y(s)ds$$
$$-\frac{\beta(\alpha+t)}{\Delta}\int_0^\eta(\eta-s)y(s)ds-\int_0^t(t-s)y(s)ds.$$

Proof. The proof follows by direct calculations.

Let $G(t,s)$ be the Green's function for the BVP (2.2)-(2.3). By direct calculation, we have

$$G(t,s)=\begin{cases} G_1(t,s), & s\in[0,\eta],\\ G_2(t,s), & s\in[\eta,\xi],\\ G_3(t,s), & s\in[\xi,1],\end{cases}$$

where

$$G_1(t,s)=\begin{cases} G_{11}(t,s)=\frac{s}{\Delta}[\xi-\beta\eta-(1-\beta)t], & s\le t,\\ G_{12}(t,s)=\frac{1}{\Delta}\{\alpha(1-\beta)(t-s)+[\xi-\beta\eta-(1-\beta)s]t\}, & t\le s,\end{cases}$$

$$G_2(t,s)=\begin{cases} G_{21}(t,s)=\frac{1}{\Delta}[\alpha\beta(\eta-s)+t(\beta\eta-s)+s(\xi-\beta\eta)], & s\le t,\\ G_{22}(t,s)=\frac{1}{\Delta}[\alpha(1-\beta)t+\alpha(\beta\eta-s)+t(\xi-s)], & t\le s,\end{cases}$$

$$G_3(t,s)=\begin{cases} G_{31}(t,s)=\frac{1}{\Delta}\{[\alpha(1-\beta)+\xi-\beta\eta]s-\alpha(\xi-\beta\eta)-(\xi-\beta\eta)t\}, & s\le t,\\ G_{32}(t,s)=\frac{\alpha}{\Delta}[(1-\beta)t-(\xi-\beta\eta)], & t\le s.\end{cases}$$

Lemma 2.3. *Suppose* (H_0) *holds. Then* $G(t,s)$ *has the following properties*

(i) $G(t,s)\ge 0$, *for all* $t,s\in[0,1]$;
(ii) $G(t,s_1)\ge\Lambda G(t,s_2)$, $t\in[0,1]$, $s_1\in[\tau_1,\eta]$, $s_2\in[\eta,\xi]$, *where* τ_1 *and* Λ *are given as in* (H_3).

Proof. (i) By calculating, we obtain

$$\beta>\frac{(1-\xi)(\xi+\min\{\eta,1-\eta\})}{\min\{\eta,1-\eta\}(1-\eta)+\eta(1-\xi)}>\frac{1-\xi}{1-\eta},$$

and

$$\alpha\le-\frac{(1-\xi)(\xi-\beta\eta)}{\beta(1-\eta)-(1-\xi)}\le-\frac{(1-\xi)(\xi-\beta\eta)}{\beta(\xi-\eta)}.$$

From this and (H_0), we have

$$G_{11}(t,s)=\frac{s}{\Delta}[\xi-\beta\eta-(1-\beta)t]\ge\frac{s}{\Delta}[\beta(1-\eta)-(1-\xi)]\ge 0,$$

$$G_{12}(t,s)=\frac{1}{\Delta}\{\alpha(1-\beta)(t-s)+[\xi-\beta\eta-(1-\beta)s]t\}$$

$$
\begin{aligned}
&\geq \frac{1}{\Delta}\{-\alpha(1-\beta)(s-t)+[\xi-\beta\eta-(1-\beta)\eta]t\} \\
&= \frac{1}{\Delta}[-\alpha(1-\beta)(s-t)+(\xi-\eta)t] \geq 0, \\
G_{21}(t,s) &= \frac{1}{\Delta}[\alpha\beta(\eta-s)+t(\beta\eta-s)+s(\xi-\beta\eta)] \\
&\geq \frac{1}{\Delta}[\alpha\beta(\eta-s)+(\beta\eta-s)+s(\xi-\beta\eta)] \\
&= \frac{1}{\Delta}[\beta\eta(1+\alpha)-s[\beta(\alpha+\eta)+(1-\xi)]\} \\
&\geq \frac{1}{\Delta}\{\beta\eta(1+\alpha)-\xi[\beta(\alpha+\eta)+(1-\xi)]\} \\
&= \frac{1}{\Delta}[-\alpha\beta(\xi-\eta)-(1-\xi)(\xi-\beta\eta)] \geq 0, \\
G_{22}(t,s) &= \frac{1}{\Delta}[\alpha(1-\beta)t+\alpha(\beta\eta-s)+t(\xi-s)] \\
&\geq \frac{1}{\Delta}[\alpha(1-\beta)s+\alpha(\beta\eta-s)+t(\xi-s)] \\
&= \frac{1}{\Delta}[-\alpha\beta(s-\eta)+t(\xi-s)] \geq 0, \\
G_{31}(t,s) &= \frac{1}{\Delta}\{[\alpha(1-\beta)+\xi-\beta\eta]s-\alpha(\xi-\beta\eta)-(\xi-\beta\eta)t\} \\
&\geq \frac{1}{\Delta}[\alpha(1-\beta)+(\xi-\beta\eta)\xi-\alpha(\xi-\beta\eta)-(\xi-\beta\eta)] \\
&= \frac{1}{\Delta}\{\alpha[(1-\beta)-(\xi-\beta\eta)]-(\xi-\beta\eta)(1-\xi)\} \\
&= \frac{1}{\Delta}\{\alpha[(1-\xi)-\beta(1-\eta)]-(\xi-\beta\eta)(1-\xi)\} \geq 0, \\
G_{32}(t,s) &= \frac{\alpha}{\Delta}[(1-\beta)t-(\xi-\beta\eta)] \geq \frac{\alpha}{\Delta}[(1-\beta)-(\xi-\beta\eta)] = \frac{-\alpha}{\Delta}[\beta(1-\eta)-(1-\xi)] \geq 0.
\end{aligned}
$$

Thus, (i) holds.

(ii) By $\beta > \frac{1-\xi}{1-\eta}$, we get

$$
\alpha(1-\beta)+\xi-\eta \geq \alpha(1-\frac{1-\xi}{1-\eta})+\xi-\eta = \frac{\alpha+1-\eta}{1-\eta} \geq 0. \tag{2.4}
$$

If $t \leq s_1$, in view of (2.4) and (H_0), we have

$$
\begin{aligned}
G_{12}(t,s_1) &= \frac{1}{\Delta}\{\alpha(1-\beta)(t-s_1)+[\xi-\beta\eta-(1-\beta)s_1]t\} \\
&= \frac{1}{\Delta}\{-\alpha(1-\beta)s_1+[\alpha(1-\beta)+\xi-\beta\eta-(1-\beta)s_1]t\}
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{1}{\Delta}\{-\alpha(1-\beta)s_1+[\alpha(1-\beta)+\xi-\beta\eta-(1-\beta)\eta]t\}\\
&> \frac{1}{\Delta}\{-\alpha(1-\beta)\tau_1+[\alpha(1-\beta)+\xi-\eta]t\}\\
&\geq \frac{-\alpha(1-\beta)\tau_1}{\Delta},
\end{aligned}
$$

$$
\begin{aligned}
G_{22}(t,s_2) &= \frac{1}{\Delta}[\alpha(1-\beta)t+\alpha(\beta\eta-s_2)+t(\xi-s_2)]\\
&\leq \frac{1}{\Delta}\{-\alpha(s_2-\beta\eta)+t[\alpha(1-\beta)+\xi-\eta]\}\\
&\leq \frac{1}{\Delta}\{-\alpha(\xi-\beta\eta)+\eta[\alpha(1-\beta)+\xi-\eta]\}\\
&= \frac{1}{\Delta}(\xi-\eta)(\eta-\alpha)\\
&\leq \frac{1}{\Delta}(\xi-\eta)[(\xi+\eta)-\beta(\alpha+\eta)].
\end{aligned}
$$

Thus,

$$
\frac{G(t,s_1)}{G(t,s_2)} = \frac{G_{12}(t,s_1)}{G_{22}(t,s_2)} \geq \Lambda.
$$

If $s_1 \leq t \leq s_2$, we have by (2.4) and (H_0) that

$$
\begin{aligned}
G_{11}(t,s_1) &= \frac{1}{\Delta}\{s_1[\xi-\beta\eta-(1-\beta)t]\} \geq \frac{1}{\Delta}\{\tau_1[\xi-\beta\eta-(1-\beta)\xi]\}\\
&= \frac{1}{\Delta}[\beta\tau_1(\xi-\eta)] \geq \frac{\tau_1}{\Delta}[\beta(1-\eta)-(1-\xi)],
\end{aligned}
$$

$$
\begin{aligned}
G_{22}(t,s_2) &= \frac{1}{\Delta}[\alpha(t-s_2)+\alpha\beta(\eta-t)+t(\xi-s_2)]\\
&\leq \frac{1}{\Delta}\{-\alpha(s_2-\beta\eta)+t[\alpha(1-\beta)+\xi-\eta]\}\\
&\leq \frac{1}{\Delta}\{-\alpha(\xi-\beta\eta)+\xi[\alpha(1-\beta)+\xi-\eta]\}\\
&= \frac{1}{\Delta}(\xi-\eta)(\xi-\alpha\beta)\\
&\leq \frac{1}{\Delta}(\xi-\eta)[(\xi+\eta)-\beta(\alpha+\eta)]\}.
\end{aligned}
$$

So,

$$
\frac{G(t,s_1)}{G(t,s_2)} = \frac{G_{11}(t,s_1)}{G_{22}(t,s_2)} \geq \Lambda.
$$

If $t \geq s_2$, it follows from (H_0) that

$$G_{11}(t,s_1) \geq \frac{\tau_1}{\Delta}[\beta(1-\eta)-(1-\xi)],$$

$$\begin{aligned}
G_{21}(t,s_2) &= \frac{1}{\Delta}[\alpha\beta(\eta-s_2)+\beta\eta(t-s_2)+s_2(\xi-t)] \\
&= \frac{1}{\Delta}[-\alpha\beta(s_2-\eta)+(\beta\eta-s_2)t+s_2(\xi-\beta\eta)] \\
&\leq \frac{1}{\Delta}[-\alpha\beta(\xi-\eta)+(\beta\eta-\eta)\eta+\xi(\xi-\beta\eta)] \\
&= \frac{1}{\Delta}(\xi-\eta)[(\xi+\eta)-\beta(\alpha+\eta)].
\end{aligned}$$

Then,

$$\frac{G(t,s_1)}{G(t,s_2)} = \frac{G_{11}(t,s_1)}{G_{21}(t,s_2)} \geq \Lambda.$$

This completes the proof.

Let $C[0,1]$ be the Banach space with norm $\|u\| = \max\limits_{0\leq t\leq 1}|u(t)|$. Denote

$$X=\{\ u\in C[0,1]:\ \min_{0\leq t\leq 1} u(t)\geq 0 \text{ and } u(0)=\alpha u'(1),\ u(\xi)=\beta u(\eta)\ \},$$

and define the cone $K\subset X$ and the operator $T: C[0,1]\to C[0,1]$ by

$$K=\{\ u\in X\colon\ u(t) \text{ is concave on } [0,\eta]\cup[\xi,1],\ u(t) \text{ is convex on } [\eta,\xi]\ \},$$

and

$$(Tu)(t) = \int_0^1 G(t,s)a(s)f(u(s))ds.$$

Lemma 2.4. *Assume that $u\in K$, then*

$$u(t)\geq \phi(t)u(\eta),\ t\in[0,\eta], \quad u(t)\leq \phi(t)u(\eta),\ t\in[\eta,\xi], \tag{2.5}$$

where

$$\phi(t) = \begin{cases} \dfrac{t}{\eta}, & t\in[0,\eta], \\ \dfrac{\xi-\beta\eta+(\beta-1)t}{\xi-\eta}, & t\in[\eta,\xi]. \end{cases}$$

Proof. Since $u\in K$, then $u(t)$ is concave on $[0,\eta]$, $u(t)$ is convex on $[\eta,\xi]$. Thus, we have

$$u(t)\geq u(0)+\frac{u(\eta)-u(0)}{\eta}t \geq \frac{t}{\eta}u(\eta), \text{ for all } t\in[0,\eta],$$

and

$$u(t)\leq u(\xi)+\frac{u(\xi)-u(\eta)}{\xi-\eta}(t-\xi) = \frac{\xi-\beta\eta+(\beta-1)t}{\xi-\eta}u(\eta), \text{ for each } t\in[\eta,\xi].$$

So, (2.5) holds.

Lemma 2.5. *Assume that* $u \in K$.

(i) *If* $\max_{0\le t\le\eta} u(t) \ge \max_{\xi<t\le 1} u(t)$, *then* $u(t) \ge \gamma\|u\|$, *for* $t \in [\beta\eta, \tau_1]$, (2.6)

(ii) *If* $\max_{\xi<t\le 1} u(t) \ge \max_{0\le t\le\eta} u(t)$, *then* $u(t) \ge \gamma\|u\|$, *for* $t \in [\xi+\beta(1-\xi), \tau_2]$, (2.7)

where $\tau_1 \in (\beta\eta, \eta)$, $\tau_2 \in (\xi+\beta(1-\xi), 1)$, $\gamma = \min\left\{\beta, 1-\dfrac{\tau_1}{\eta}, 1-\dfrac{\tau_2-\xi}{1-\xi}\right\}$.

Proof. Since $u \in K$, then $u(t) \ge 0$, for all $t \in [0,1]$, $u(t)$ is concave on $[0,\eta] \cup [\xi,1]$, $u(t)$ is convex on $[\eta,\xi]$, and $u(\xi) = \beta u(\eta) < u(\eta)$. Thus, we have

$$\|u\| = \max_{0\le t\le 1} |u(t)| = \max\{\max_{0\le t\le\eta} u(t), \max_{\xi<t\le 1} u(t)\}.$$

There are two cases to consider

(i) $\max_{0\le t\le\eta} u(t) \ge \max_{\xi<t\le 1} u(t)$, so $\max_{0\le t\le 1} |u(t)| = \max_{0\le t\le\eta} u(t)$. Let

$$\omega_1 = \inf\{\varepsilon \in [0,\eta] : \max_{0\le t\le\eta} u(t) = u(\varepsilon)\}.$$

If $t \in [0,\omega_1]$, then by the concavity of $u(t)$, we have

$$u(t) \ge u(0) + \frac{u(\omega_1)-u(0)}{\omega_1}t = \frac{\omega_1-t}{\omega_1}u(0) + \frac{t}{\omega_1}u(\omega_1) \ge \frac{t}{\omega_1}u(\omega_1) \ge \frac{t}{\eta}\|u\|. \tag{2.8}$$

If $t \in [\omega_1,\eta]$, similarly, we obtain

$$\begin{aligned} u(t) &\ge u(\omega_1) + \frac{u(\omega_1)-u(\eta)}{\omega_1-\eta}(t-\omega_1) = \frac{\eta-t}{\eta-\omega_1}u(\omega_1) + \frac{t-\omega_1}{\eta-\omega_1}u(\eta) \\ &\ge \frac{\eta-t}{\eta-\omega_1}u(\omega_1) \ge \frac{\eta-t}{\eta}\|u\|. \end{aligned} \tag{2.9}$$

In view of (2.8) and (2.9), we get

$$u(t) \ge \min\left\{\frac{t}{\eta}, 1-\frac{t}{\eta}\right\}\|u\|,\ t \in [0,\eta],$$

which yields

$$\min_{t\in[\beta\eta,\tau_1]} u(t) \ge \min\left\{\beta, 1-\frac{\tau_1}{\eta}\right\}\|u\| \ge \gamma\|u\|.$$

(ii) $\max_{\xi<t\le 1} u(t) \ge \max_{0\le t\le\eta} u(t)$, then $\max_{0\le t\le 1} |u(t)| = \max_{\xi\le t\le 1} u(t)$. Let

$$\omega_2 = \inf\{\epsilon \in (\xi,1] : \max_{\xi<t\le 1} u(t) = u(\epsilon)\}.$$

If $t \in (\xi,\omega_2]$, by the concavity of $u(t)$, we have

$$
\begin{aligned}
u(t) &\geq u(\xi) + \frac{u(\omega_2) - u(\xi)}{\omega_2 - \xi}(t - \xi) = \frac{\omega_2 - t}{\omega_2 - \xi} u(\xi) + \frac{t - \xi}{\omega_2 - \xi} u(\omega_2) \\
&\geq \frac{t - \xi}{\omega_2 - \xi} u(\omega_2) \geq \frac{t - \xi}{1 - \xi} \|u\|. \qquad (2.10)
\end{aligned}
$$

If $t \in [\omega_2, 1]$, similarly, we get

$$
\begin{aligned}
u(t) &\geq u(\omega_2) + \frac{u(\omega_2) - u(1)}{\omega_2 - 1}(t - \omega_2) = \frac{1 - t}{1 - \omega_2} u(\omega_2) + \frac{t - \omega_2}{1 - \omega_2} u(1) \\
&\geq \frac{1 - t}{1 - \omega_2} u(\omega_2) \geq \frac{1 - t}{1 - \xi} \|u\|. \qquad (2.11)
\end{aligned}
$$

It follows from (2.10) and (2.11) that

$$
u(t) \geq \min\left\{\frac{t - \xi}{1 - \xi}, 1 - \frac{t - \xi}{1 - \xi}\right\} \|u\|,\ t \in [\xi, 1].
$$

From this

$$
\min_{t \in [\xi + \beta(1 - \xi), \tau_2]} u(t) \geq \min\left\{\beta, 1 - \frac{\tau_2 - \xi}{1 - \xi}\right\} \|u\| \geq \gamma \|u\|.
$$

Lemma 2.6. *Assume that* (H_0)-(H_3) *hold. Then,*

$$
\int_{\tau_1}^{\eta} G(t, s) a^+(s) f(u(s)) ds \geq \int_{\eta}^{\xi} G(t, s) a^-(s) f(u(s)) ds. \qquad (2.12)
$$

Proof. If $x \in [0, \xi - \eta]$, then $\eta - \delta x \in [\tau_1, \eta]$, $\eta + x \in [\eta, \xi]$, where τ_1 and δ as in (H_3). By the definition of ϕ (see Lemma 2.4), we have

$$
\phi\left(\eta - \delta x\right) = 1 - \frac{\eta - \tau_1}{\eta(\xi - \eta)} x \text{ and } \phi(\eta + x) = 1 - \frac{1 - \beta}{\xi - \eta} x, \text{ for } x \in [0, \xi - \eta].
$$

By $\beta\eta < \tau_1 < \eta$, we get

$$
\phi(\eta - \delta x) \geq \phi(\eta + x), \text{ for all } x \in [0, \xi - \eta]. \qquad (2.13)
$$

Let $s = \eta - \delta x$, $\forall x \in [0, \xi - \eta]$. It follows from Lemmas 2.3 and 2.4, (2.13) and f is nondecreasing that

$$
\begin{aligned}
\int_{\tau_1}^{\eta} G(t, s) a^+(s) f(u(s)) ds &= \delta \int_0^{\xi - \eta} G(t, \eta - \delta x) a^+(\eta - \delta x) f(u(\eta - \delta x)) dx \\
&\geq \delta \Lambda \int_0^{\xi - \eta} G(t, \eta + x) a^+(\eta - \delta x) f(\phi(\eta - \delta x) u(\eta)) dx \\
&\geq \int_0^{\xi - \eta} G(t, \eta + x) a^+(\eta - \delta x) f(\phi(\eta + x) u(\eta)) dx. \qquad (2.14)
\end{aligned}
$$

On the other hand, let $s = \eta + x$, $x \in [0, \xi - \eta]$, by Lemma 2.4, one has

$$\int_{\eta}^{\xi} G(t,s)a^{-}(s)f(u(s))ds = \int_{0}^{\xi-\eta} G(t,\eta+x)a^{-}(\eta+x)f(u(\eta+x))dx$$
$$\leq \int_{0}^{\xi-\eta} G(t,\eta+x)a^{-}(\eta+x)f(\phi(\eta+x)u(\eta))dx. \tag{2.15}$$

Thus, in view of (2.14) and (2.15), we know that (2.12) holds.

Lemma 2.7. *Assume that* (H_0)-(H_3) *hold, then* $T: K \to K$ *is completely continuous.*

Proof. Firstly, we claim $T(K) \subset K$. In fact, for all $u \in K$, we have by Lemma 2.6 that

$$\begin{aligned}(Tu)(t) &= \int_0^1 G(t,s)a(s)f(u(s))ds \\ &= \Big(\int_0^{\tau_1} G(t,s)a^{+}(s)f(u(s))ds + \int_{\tau_1}^{\eta} G(t,s)a^{+}(s)f(u(s))ds \\ &\quad - \int_{\eta}^{\xi} G(t,s)a^{-}(s)f(u(s))ds + \int_{\xi}^{1} G(t,s)a^{+}(s)f(u(s))ds\Big) \geq 0.\end{aligned}$$

Moreover, $(Tu)(0) = \alpha(Tu)'(1)$, $(Tu)(\xi) = \beta(Tu)(\eta)$, Then, $T: K \to X$. On the other hand,

$$(Tu)''(t) = -a^{+}(t)f(u(t)) \leq 0, \quad t \in [0,\eta],$$

$$(Tu)''(t) = a^{-}(t)f(u(t)) \geq 0, \quad t \in [\eta,\xi],$$

$$(Tu)''(t) = -a^{+}(t)f(u(t)) \leq 0, \quad t \in [\xi,1],$$

This show that $T: K \to K$. It can be shown that $T: K \to K$ is completely continuous by Arzela-Ascoli theorem.

3. Main result

Let the nonnegative continuous concave functional κ on K, the nonnegative continuous convex functionals θ, ϑ and the nonnegative continuous functional ψ be defined on the cone K by

$$\kappa(u) = \min\{\min_{\beta\eta \leq t \leq \tau_1} |u(t)|, \min_{\xi+\beta(1-\xi) \leq t \leq \tau_2} |u(t)|\},\ \vartheta(u) = \theta(u) = \psi(u) = \max_{0 \leq t \leq 1} |u(t)|, \text{ for } u \in K.$$

For the sake of brevity, we denote

$$N = \max_{0 \leq t \leq 1} \Big(\int_0^{\eta} G(t,s)a^{+}(s)ds + \int_{\xi}^{1} G(t,s)a^{+}(s)ds\Big),$$
$$M = \min\{M_1, M_2, M_3, M_4\},$$

where

$$M_1 = \int_{\beta\eta}^{\tau_1} G(\beta\eta,s)a^{+}(s)ds, \quad M_2 = \int_{\beta\eta}^{\tau_1} G(\tau_1,s)a^{+}(s)ds,$$

$$M_3 = \int_{\xi+\beta(1-\xi)}^{\tau_2} G(\xi+\beta(1-\xi), s)a^+(s)ds, \quad M_4 = \int_{\xi+\beta(1-\xi)}^{\tau_2} G(\tau_2, s)a^+(s)ds.$$

Therorem 3.1. *Suppose that* (H_0)-(H_3) *hold. Let* $0 < a < b < \gamma d$, *and the following conditions hold,*

(H_4) $f(u(t)) \leq \frac{d}{N}$, *for all* $t \in [0,1]$, $u \in [0,d]$,

(H_5) $f(u(t)) \geq \frac{b}{M}$, *for all* $t \in [\beta\eta, \tau_1] \cup [\xi+\beta(1-\xi), \tau_2]$, $u \in [b, \frac{b}{\gamma}]$,

(H_6) $f(u(t)) \leq \frac{a}{N}$, *for all* $t \in [0,1]$, $u \in [0,a]$.

Then the BVP (1.3) *have at least three positive solutions* u_1, u_2 *and* u_3 *such that*

$$\max_{0\leq t\leq 1}|u_i(t)| \leq d \text{ for } i=1,2,3,\ b < \min_{\beta\eta\leq t\leq\tau_1}|u_1(t)| \ (\text{ or } b < \min_{\xi+\beta(1-\xi)\leq t\leq\tau_2}|u_1(t)|\ ),\ \max_{0\leq t\leq 1}|u_1| \leq d,$$
$$a < \max_{0\leq t\leq 1}|u_2(t)| < \frac{b}{\gamma} \text{ with } \min_{\beta\eta\leq t\leq\tau_1}|u_2(t)| < b \ (\text{or } \min_{\xi+\beta(1-\xi)\leq t\leq\tau_2}|u_2(t)| < b\ ), \text{ and } \max_{0\leq t\leq 1}|u_3(t)| < a.$$

Proof. Firstly, we check $T : \overline{K(\vartheta,d)} \to \overline{K(\vartheta,d)}$ is completely continuous operator.

If $u \in \overline{K(\vartheta,d)}$, then $\vartheta(u) = \max_{0\leq t\leq 1}|u(t)| \leq d$. It follows from (H_4) that

$$\begin{aligned}
\vartheta(Tu) &= \max_{0\leq t\leq 1}|Tu(t)| = \max_{0\leq t\leq 1}\int_0^1 G(t,s)a(s)f(u(s))ds \\
&= \max_{0\leq t\leq 1}\left(\int_0^\eta G(t,s)a^+(s)f(u(s))ds - \int_\eta^\xi G(t,s)a^-(s)f(u(s))ds\right. \\
&\quad \left. + \int_\xi^1 G(t,s)a^+(s)f(u(s))ds\right) \\
&\leq \max_{0\leq t\leq 1}\left(\int_0^\eta G(t,s)a^+(s)f(u(s))ds + \int_\xi^1 G(t,s)a^+(s)f(u(s))ds\right) \\
&\leq \frac{d}{N}\cdot\max_{0\leq t\leq 1}\left(\int_0^\eta G(t,s)a^+(s)ds + \int_\xi^1 G(t,s)a^+(s)ds\right) = d.
\end{aligned}$$

Therefore, $T : \overline{K(\vartheta,d)} \to \overline{K(\vartheta,d)}$. From this and Lemma 2.7, we know that $T : \overline{K(\vartheta,d)} \to \overline{K(\vartheta,d)}$ is completely continuous operator.

To check condition (C_1) of Lemma 2.1, we choose $u(t) = \frac{b}{\gamma}$, $0 \leq t \leq 1$. It is easy to check that $u(t) = \frac{b}{\gamma} \in K(\vartheta,\theta,\kappa,b,\frac{b}{\gamma},d)$ and $\kappa(u) = \kappa(\frac{b}{\gamma}) > b$, and so $\{u \in K(\vartheta,\theta,\kappa,b,\frac{b}{\gamma},d) : \kappa(u) > b\} \neq \emptyset$. Thus, for all $u \in K(\vartheta,\theta,\kappa,b,\frac{b}{\gamma},d)$, we have that $b \leq u(t) \leq \frac{b}{\gamma}$, for $t \in [\beta\eta,\tau_1] \cup [\xi+\beta(1-\xi),\tau_2]$. We consider the following two cases.

Case (i) Suppose that $\max_{0\leq t\leq\eta} u(t) \geq \max_{\xi<t\leq 1} u(t)$. For all $u \in K(\vartheta,\theta,\kappa,b,\frac{b}{\gamma},d)$, then we have $Tu \in K$, and so Tu is concave on $[0,\eta]$, we have to distinguish two cases: (1) $\min_{\beta\eta\leq t\leq\tau_1} Tu(t) = Tu(\beta\eta)$; (2) $\min_{\beta\eta\leq t\leq\tau_1} Tu(t) = Tu(\tau_1)$. For case (1), we have from Lemma 2.6 that

$$\begin{aligned}
\min_{\beta\eta\le t\le\tau_1} Tu(t) = Tu(\beta\eta) &= \int_0^1 G(\beta\eta,s)a(s)f(u(s))ds \\
&= \int_0^{\tau_1} G(\beta\eta,s)a^+(s)f(u(s))ds + \int_{\tau_1}^{\eta} G(\beta\eta,s)a^+(s)f(u(s))ds \\
&\quad - \int_{\eta}^{\xi} G(\beta\eta,s)a^-(s)f(u(s))ds + \int_{\xi}^{1} G(\beta\eta,s)a^+(s)f(u(s))ds \\
&\ge \int_0^{\tau_1} G(\beta\eta,s)a^+(s)f(u(s))ds + \int_{\xi}^{1} G(\beta\eta,s)a^+(s)f(u(s))ds \\
&\ge \int_{\beta\eta}^{\tau_1} G(\beta\eta,s)a^+(s)f(u(s))ds \\
&\ge \frac{b}{M}\cdot\int_{\beta\eta}^{\tau_1} G(\beta\eta,s)a^+(s)ds \ge b.
\end{aligned}$$

For case (2), similarly, we get

$$\min_{\beta\eta\le t\le\tau_1} Tu(t) = Tu(\tau_1) \ge \int_{\beta\eta}^{\tau_1} G(\tau_1,s)a^+(s)f(u(s))ds \ge \frac{b}{M}\cdot\int_{\beta\eta}^{\tau_1} G(\tau_1,s)a^+(s)ds \ge b.$$

Thus,

$$\min_{\beta\eta\le t\le\tau_1} Tu(t) \ge b, \text{ for all } u\in K(\vartheta,\theta,\kappa,b,\tfrac{b}{\gamma},d) \text{ with } b\le u(t)\le \tfrac{b}{\gamma},\ t\in[\beta\eta,\tau_1]. \tag{3.1}$$

Case (ii) Suppose that $\max_{\xi<t\le 1} u(t) \ge \max_{0\le t\le\eta} u(t)$. For all $u\in K(\vartheta,\theta,\kappa,b,\frac{b}{\gamma},d)$, then we have $Tu\in K$, and so Tu is concave on $[\xi,1]$, we have to distinguish two cases: (1) $\min_{\xi+\beta(1-\xi)\le t\le\tau_2} Tu(t) = Tu(\xi+\beta(1-\xi))$; (2) $\min_{\xi+\beta(1-\xi)\le t\le\tau_2} Tu(t) = Tu(\tau_2)$. For case (1), it follows from Lemma 2.6 that

$$\begin{aligned}
\min_{\xi+\beta(1-\xi)\le t\le\tau_2} Tu(t) = Tu(\xi+\beta(1-\xi)) &\ge \int_{\xi+\beta(1-\xi)}^{\tau_2} G(\xi+\beta(1-\xi),s)a^+(s)f(u(s))ds \\
&\ge \frac{b}{M}\cdot\int_{\xi+\beta(1-\xi)}^{\tau_2} G(\xi+\beta(1-\xi),s)a^+(s)ds \ge b.
\end{aligned}$$

For case (2), similarly, we obtain

$$\begin{aligned}
\min_{\xi+\beta(1-\xi)\le t\le\tau_2} Tu(t) = Tu(\tau_2) &\ge \int_{\xi+\beta(1-\xi)}^{\tau_2} G(\xi+\beta(1-\xi),s)a^+(s)f(u(s))ds \\
&\ge \frac{b}{M}\cdot\int_{\xi+\beta(1-\xi)}^{\tau_2} G(\xi+\beta(1-\xi),s)a^+(s)ds \ge b.
\end{aligned}$$

Hence,

$$\min_{\xi+\beta(1-\xi)\le t\le\tau_2} Tu(t) \ge b, \text{ for all } u\in K(\vartheta,\theta,\kappa,b,\tfrac{b}{\gamma},d) \text{ with } b\le u(t)\le \tfrac{b}{\gamma},\ t\in[\xi+\beta(1-\xi),\tau_2]. \tag{3.2}$$

In view of (3.1) and (3.2), we have

$$\kappa(Tu) = \min\{\min_{\beta\eta\leq t\leq\tau_1} Tu(t), \min_{\xi+\beta(1-\xi)\leq t\leq\tau_2} Tu(t)\} \geq b, \text{ for all } u \in K(\vartheta,\theta,\kappa,b,\tfrac{b}{\gamma},d).$$

This shows that condition (C_1) of Lemma 2.1 is sastisfied.

Secondly, we show (C_2) of Lemma 2.1 is satisfied. If $\max_{0\leq t\leq\eta} u(t) \geq \max_{\xi<t\leq 1} u(t)$. By Lemma 2.5 (i) and Lemma 2.7, we have

$$\min_{\beta\eta\leq t\leq\tau_1} Tu(t) \geq \gamma\|Tu\| = \gamma\theta(Tu) > b, \text{ for all } u \in K(\vartheta,\kappa,b,d) \text{ with } \theta(Tu) > \tfrac{b}{\gamma}. \tag{3.3}$$

If $\max_{\xi<t\leq 1} u(t) \geq \max_{0\leq t\leq\eta} u(t)$. It follows from Lemma 2.5 (ii) and Lemma 2.7 that

$$\min_{\xi+\beta(1-\xi)\leq t\leq\tau_2} Tu(t) \geq \gamma\|Tu\| = \gamma\theta(Tu) > b, \text{ for all } u \in K(\vartheta,\kappa,b,d) \text{ with } \theta(Tu) > \tfrac{b}{\gamma}. \tag{3.4}$$

Therefore, we obtain by (3.3) and (3.4) that

$$\kappa(Tu) = \min\{\min_{\beta\eta\leq t\leq\tau_1} Tu(t), \min_{\xi+\beta(1-\xi)\leq t\leq\tau_2} Tu(t)\} \geq b, \text{ for all } u \in K(\vartheta,\kappa,b,d) \text{ with } \theta(Tu) > \tfrac{b}{\gamma}.$$

Finally, we show condition (C_3) of Lemma 2.1 is also satisfied. Obviously, as $\psi(0) = 0 < a$, there holds $0 \notin R(\vartheta,\psi,a,d)$. Suppose $u \in R(\vartheta,\psi,a,d)$ with $\psi(u) = a$. Then we have by condition (H_6) that

$$\begin{aligned}
\psi(Tu) &= \max_{0\leq t\leq 1} |Tu(t)| = \max_{0\leq t\leq 1} \int_0^1 G(t,s)a(s)f(u(s))ds \\
&= \max_{0\leq t\leq 1} \left(\int_0^\eta G(t,s)a^+(s)f(u(s))ds - \int_\eta^\xi G(t,s)a^-(s)f(u(s))ds \right. \\
&\quad \left. + \int_\xi^1 G(t,s)a^+(s)f(u(s))ds \right) \\
&\leq \max_{0\leq t\leq 1} \left(\int_0^\eta G(t,s)a^+(s)f(u(s))ds + \int_\xi^1 G(t,s)a^+(s)f(u(s))ds \right) \\
&\leq \frac{a}{N} \cdot \max_{0\leq t\leq 1} \left(\int_0^\eta G(t,s)a^+(s)f(u(s))ds + \int_\xi^1 G(t,s)a^+(s)f(u(s))ds \right) \\
&= a.
\end{aligned}$$

So, condition (C_3) of Lemma 2.1 is satisfied. Therefore, according to Lemma 2.1, there exist three positive solutions u_1, u_2 and u_3 for the BVP (1.3) such that

$$\max_{0\leq t\leq 1} |u_i(t)| \leq d \text{ for } i = 1,2,3,\ b < \min_{\beta\eta\leq t\leq\tau_1} |u_1(t)| \text{ (or } b < \min_{\xi+\beta(1-\xi)\leq t\leq\tau_2} |u_1(t)| \text{)}, \max_{0\leq t\leq 1} |u_1| \leq d,$$
$$a < \max_{0\leq t\leq 1} |u_2(t)| < \tfrac{b}{\gamma} \text{ with } \min_{\beta\eta\leq t\leq\tau_1} |u_2(t)| < b \text{ (or } \min_{\xi+\beta(1-\xi)\leq t\leq\tau_2} |u_2(t)| < b \text{), and } \max_{0\leq t\leq 1} |u_3(t)| < a.$$

Acknowledgements

Project supported by the National Natural Science Foundation of China (10771212) and the

Natural Science Foundation of Jiangsu Education Office (06KJB110010).

References

[1] R. Avery and A. Peterson, Three positive fixed points of nonlinear operators on ordered Banach spaces, Comput. Math. Appl., 42(2001) 313-322.

[2] C.Z. Bai, D.P. Xie, Y. Liu and C.L. Wang, Positive solutions for second-order four-point boundary value problems with alternating coefficient, Nonlinear Anal., 70(2009) 2014-2023.

[3] B. Liu, Positive solutions of second-order three-point boundary value problems with change of sign in Banach spaces, Nonlinear Anal., 64(2006) 1336-1355.

[4] B. Liu, Positive Solutions of Second-Order Three-Point Boundary Value Problems with Change of Sign, Comput. Math. Appl., 47(2004)1351-1361.

[5] I.Y. Karaca, Nonlinear triple-point problems with change of sign, Comput. Math. Appl., 55(2008) 691-703.

[6] I.Y. Karaca, On existence of positive solutions three-point boundary value problems with alternating coefficient, Math. Comput. Modelling, 47(2008) 1019-1034.

(Received February 25, 2009)