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Multiple global bifurcation branches for nonlinear Picard problems

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Abstract

In this paper we prove the global bifurcation theorem for the nonlinear Picard problem. The right-hand side function φ is a Caratheodory map, not differentiable at zero, but behaving in the neighbourhood of zero as specified in details below. We prove that in some interval $[a, b] \subset \mathbb{R}$ the Leray-Schauder degree changes, hence there exists the global bifurcation branch. Later, by means of some approximation techniques, we prove that there exist at least two such branches.

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MSC: 34C23,34B24.

1 Main theorems

Let us consider the problem

$$\begin{cases} u''(t) + \varphi(t, u(t), u'(t), \lambda) = 0 \quad \text{a.e. in} \ \ (0, \pi) \\ u(0) = u(\pi) = 0, \end{cases} \tag{1}$$

where $\varphi : [0, \pi] \times \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is a Caratheodory map i.e. $\varphi(t, \cdot, \cdot, \cdot) : \mathbb{R} \times \mathbb{R} \times (0, +\infty) \to \mathbb{R}$ is continuous for $t \in [0, \pi]$, $\varphi(\cdot, x, y, \lambda) : [0, \pi] \to \mathbb{R}$ is measurable for $(x, y, \lambda) \in \mathbb{R} \times \mathbb{R} \times (0, +\infty)$ and for any $R > 0$ there exists an integrable function $m_R \in L^1(0, \pi)$, such that

$$\forall_{(x,y,\lambda)\in\mathbb{R}\times\mathbb{R}\times(0,+\infty)} \forall_{t\in[0,\pi]} |\lambda| + |x| + |y| \le R \Rightarrow |\varphi(t, x, y, \lambda)| \le m_R(t);$$

We will later assume that for each compact $\mathcal{K} \subset (0, +\infty)$ the function φ satisfies the condition

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{\lambda\in\mathcal{K}} \forall_{(x,y)\in\mathbb{R}^2} \forall_{t\in[0,\pi]} |x| + |y| \le \delta \Rightarrow$$

$$\Rightarrow |\varphi(t, x, y, \lambda) - m\lambda q_k(t, x)| \le \varepsilon(|x| + |y|),$$

where $q_k : [0,\pi] \times \mathbb{R} \to \mathbb{R}$ is given by $q_k(t,u) = \mathrm{sgn}(\sin(kt))|u|$, where

$$\mathrm{sgn}(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0. \end{cases}$$

for a fixed $k \in \{2,3,4,...\}$.

As a special case we are going to refer to the problem

$$\begin{cases} u''(t) + \lambda q_k(t,u(t)) = 0 \quad \text{a.e. in } (0,\pi) \\ u(0) = u(\pi) = 0. \end{cases} \tag{2}$$

Let us first observe that

Proposition 1 *The pair $(k^2, \sin kt)$ is the solution of (2).* □

Let $\Lambda(q_k)$ denote the set of all $\lambda \in [0,+\infty)$, such that there exists a solution (λ, u) of (2), such that $u \neq 0$. As we can see the set $\Lambda(q_k)$ is not empty. Let us also observe that $0 \notin \Lambda(q_k)$.

Let us further assume that the space $C^1[0,\pi]$ is equipped with the norm $\|u\|_1 = \|u\|_0 + \|u'\|_0$, where $\|u\|_0 = \sup_{t\in[0,\pi]} |u(t)|$.

In case φ satisfies $\varphi(t,0,0,\lambda) = 0$ for almost all $t \in [0,\pi]$ and all $\lambda \in (0,+\infty)$ each pair $(\lambda, 0) \in (0,+\infty) \times C^1[0,\pi]$ is the solution of (1).

We call all these pairs *trivial solutions* of (1). Let $\mathcal{R}_{(1)}$ denote the closure, in $(0,+\infty) \times C^1[0,\pi]$, of the set of nontrivial solutions of the problem (1).

Let $\mathcal{B}_{(1)}$ denote the set of all *bifurcation points* of the problem (1), i.e. $\mathcal{B}_{(1)} = \mathcal{R}_{(1)} \cap ((0,+\infty) \times \{0\})$.

The existence of bifurcation points and noncompact components of the set of solutions for boundary value problems (1) have been studied by many authors. The main ideas come from Krasnoselskii (see [10]) and Rabinowitz (see [12]). They studied the general nonlinear spectral problems in Banach spaces. Additionally Rabinowitz has studied the Sturm-Liouville problems (1) with φ linearizable at the origin. The problems with φ not differentiable at $(0,0)$ have also been studied (see e.g. [1],[2],[7],[13],[14]). In the mentioned papers the authors were mainly concentrated on the asymptotics such that $\varphi(t,u,u',\lambda) \geq 0$ for $u \geq 0$ and $|u| + |u'|$ small, which is not the case considered here.

The problems of the form

$$\begin{cases} u''(t) + \lambda a(t)u(t) + o(|u(t)| + |u'(t)|) = 0 \quad \text{a.e. in } (0,\pi) \\ l(u) = 0, \end{cases}$$

for l representing Sturm-Liouville boundary conditions, where a is not necessarily of constant sign, were studied e.g. in [7] and [9]. In [9] authors proved the important result for the linear case, which we are going to refer later.

By means of the topological degree methods we may prove the following theorem:

Theorem 1 *Let $m > 0$ and $\varphi : [0,\pi] \times \mathbb{R} \times \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ be a Caratheodory map, such that for each compact $\mathcal{K} \subset (0,+\infty)$*

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{\lambda\in\mathcal{K}} \forall_{(x,y)\in\mathbb{R}^2} \forall_{t\in[0,\pi]} |x| + |y| \leq \delta \Rightarrow \tag{3}$$

$$\Rightarrow |\varphi(t,x,y,\lambda) - m\lambda q_k(t,x)| \leq \varepsilon(|x|+|y|).$$

Then there exists the noncompact component $\mathcal{C}$ of $\mathcal{R}_{(1)}$ such, that $(\frac{\mu}{m},0) \in \mathcal{C}$ where $\mu \in \Lambda(q_k)$.

We can tell more about the structure of the solution set of the problem (1) when we study the linear eigenvalue problems

$$\begin{cases} u''(t) + \lambda a^+(t)u(t) = 0 \quad \text{a.e. in } (0,\pi) \\ u(0) = u(\pi) = 0 \end{cases} \tag{4}$$

and

$$\begin{cases} u''(t) + \lambda a^-(t)u(t) = 0 \quad \text{a.e. in } (0,\pi) \\ u(0) = u(\pi) = 0, \end{cases} \tag{5}$$

where $a^+ \in L^1(0,\pi)$ is the function given by $a^+(t) = \operatorname{sgn}(\sin(kt))$ and $a^- = -a^+$.

Both of the above problems are left-definite and right-indefinite (see [9]). The Dirichlet boundary conditions are self-adjoint and separated, so we may apply theorem 3.1 of [9]. That is why there exists exactly one positive eigenvalue $\lambda^+ > 0$ of the problem (4) having the corresponding eigenvector with the constant sign. This eigenvalue is simple. Similarly there exists exactly one eigenvalue $\lambda^- > 0$ of the problem (5) having the corresponding eigenvector with the constant sign. This eigenvalue is simple as well.

Let us observe that for both problems (4) and (5) there exists also the negative eigenvalue with the above properties, but we are interested only in the eigenvalues belonging to the interval $(0,+\infty)$.

With the more detailed analysis we may prove the following fact:

Theorem 2 *Let $m > 0$ be fixed and $\varphi : [0,\pi] \times \mathbb{R} \times \mathbb{R} \times (0,+\infty) \to \mathbb{R}$ be the Caratheodory map such that for any compact $\mathcal{K} \subset (0,+\infty)$*

$$\forall_{\varepsilon>0} \exists_{\delta>0} \forall_{(x,y)\in\mathbb{R}^2} \forall_{t\in[0,\pi]} \forall_{\lambda\in\mathcal{K}} |x|+|y| \leq \delta \Rightarrow \tag{6}$$

$$\Rightarrow |\varphi(t,x,y,\lambda) - \lambda m q_k(t,x)| \leq \varepsilon|x|.$$

Then $\{(\frac{\lambda^+}{m},0), (\frac{\lambda^-}{m},0), (\frac{k^2}{m},0)\} \subset \mathcal{B}_{(1)}$.

Moreover, there exist noncompact, closed, in $(0,+\infty) \times C^1[0,\pi]$, connected sets $C_1^+, C_1^-, C_k \subset \mathcal{R}_{(1)}$, such that $(\frac{\lambda^+}{m},0) \in C_1^+$, $(\frac{\lambda^-}{m},0) \in C_1^-$, $(\frac{k^2}{m},0) \in C_k$, and

$$C_k \cap (C_1^+ \cup C_1^-) = \emptyset; \tag{7}$$

$$C_1^+ \subset (0,+\infty) \times \{u \in C^1[0,\pi] | u \geq 0\}; \tag{8}$$

$$C_1^- \subset (0,+\infty) \times \{u \in C^1[0,\pi] | u \leq 0\}; \tag{9}$$

$$\begin{aligned} &\textit{for } (\lambda,u) \in C_k \textit{ the function } u \textit{ has exactly } k-1 \textit{ zeroes} \\ &\textit{in } (0,\pi), \textit{ all zeroes of } u \textit{ are simple, and function } u \\ &\textit{is positive in a neighborhood } (0,\delta) \textit{ of } 0. \end{aligned} \tag{10}$$

Remark 1 *For the problems (4) and (5) it may happen that* $\lambda^+ = \lambda^-$. *For example in case of* $k = 2$ *we can prove the equality* $\mathcal{B}_{(1)} = \{(\frac{\lambda_0}{m}, 0), (\frac{4}{m}, 0)\}$ *where* λ_0 *is the minimal solution of the equation* $\tan(\sqrt{\lambda}\pi) = -\tanh(\sqrt{\lambda}\pi)$ *(see [5]).* $\square$

The example given below shows the application of Theorem 2 to the simple situation where $\varphi = \lambda q_3$ in the neighbourhood of 0.

Example 1 *Let* $\varphi : [0, \pi] \times \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}$ *be given by*

$$\varphi(t, x, y, \lambda) = \lambda(p(x)q_3(t, x) + (1 - p(x))x),$$

where $p : \mathbb{R} \to [0, 1]$ *is given by*

$$p(x) = \begin{cases} 1 & \textit{for } x \leq 1 \\ 2 - x & \textit{for } 1 \leq x \leq 2 \\ 0 & \textit{for } x \geq 2. \end{cases}$$

We will investigate the set of nontrivial solutions of (1) with φ *given as above.*

We can easily observe that $\varphi(t, \sin 3t, y, \lambda) = \lambda \sin(3t)$. *Moreover* $\varphi(t, x, y, \lambda) = \lambda q_3(t, x)$ *for* $x \leq 0$. *This means that we have two half-lines of nontrivial solutions of (1) given by* $(9, A\sin(3t))$ *and* (λ^-, Au^-) *for positive* $A > 0$. *Here* (λ^-, u^-) *is a nontrivial solution of (5) where* $u^-(t) < 0$ *for* $t \in (0, \pi)$. *The closures of these half-lines are the components* C_3 *and* C_1^- *given in Theorem 2.*

By Theorem 2 there exists one more component C_1^+ *of nontrivial solutions of (1) bifurcationg from* $(\lambda^+, 0)$. *As we can see the set* $(\lambda^+, Au^+) \subset C_1^+$ *for* $A \in [0, 1]$, *where* (λ^+, u^+) *is the nontrivial solution of (4) such that* $u^+(t) > 0$ *for* $t \in (0, \pi)$ *and* $\|u^+\| = 1$. *But the component* C_1^+ *is noncompact, so it must conatin more solutions then the interval* $\{(\lambda^+, Au^+) | A \in [0, 1]\}$. *Especially there exist positive solutions* (λ, u) *of (1) with* $\|u\|_0 > 1$.

As additional observation we can state that for $(\lambda, u) \in C_1^+$ *parameter* λ *must be bounded. This is because each positive solution of (1) satisfies*

$$\begin{cases} u''(t) + \lambda u(t) = 0 & \textit{for } t \in [0, \frac{\pi}{3}] \\ u(0) = 0, \end{cases}$$

hence $u(t) = \sin(\sqrt{\lambda}t)$ *for* $t \in [0, \frac{\pi}{3}]$. *For* $\lambda > 0$ *this means that there exist zero of* u *in the interval* $(0, \frac{\pi}{3})$ *which is not possible for positive* u.

So, in this case, we can describe two components as half-lines, while the third may be explicitely described in the neighbourhood of zero.

2 Auxiliary lemmas

In this section we are going to show some facts that will be used in the proofs of Theorems 1 and 2, but first we are going to specify the basic assumptions and notations.

Let $T : L^1(0, \pi) \to C^1[0, \pi]$ be the continuous linear map given by

$$(Th)(t) = -\int_0^t \int_0^s h(\tau) d\tau ds + \frac{t}{\pi} \int_0^\pi \int_0^s h(\tau) d\tau ds \tag{11}$$

Then we can see that $u = Th$ iff u is the solution of the boundary value problem

$$\begin{cases} u''(t) + h(t) = 0 & \text{a.e. on } (a,b) \\ u(0) = u(\pi) = 0 \end{cases} \tag{12}$$

for $h \in L^1(0,\pi)$.

For the problem (1) we may define the map $f : (0,+\infty) \times C^1[0,\pi] \to C^1[0,\pi]$ by

$$f(\lambda, u) = u - T\Phi(\lambda, u). \tag{13}$$

where $\Phi : (0,+\infty) \times C^1[0,\pi] \to C^1[0,\pi]$ is the Nemytskii map for a function φ. For each $\lambda \in (0,+\infty)$ the map $f(\lambda,\cdot)$ is completely continuous vector field and (λ, u) is zero of the map f iff it is a solution of (1).

Similarly, let us observe that $(\lambda, u) \in (0,+\infty) \times C^1[0,\pi]$ is the solution of (2) iff

$$f_0(\lambda, u) = 0,$$

where $f_0 : \mathbb{R} \times C^1[0,\pi] \to C^1[0,\pi]$ is given by

$$f_0(\lambda, u) = u - \lambda T Q_k(u). \tag{14}$$

and $Q_k : C^1[0,\pi] \to L^1(0,\pi)$ is the Nemytskii map for q_k, given by $Q_k(u)(t) = q_k(t, u(t))$.

Once we have this association we may define the closure $\mathcal{R}_f$ (in $(0,+\infty)\times C^1[0,\pi]$) of the set of all nontrivial zeroes of the map f and the set of bifurcation points $\mathcal{B}_f$ of the map f, and observe that $\mathcal{R}_f = \mathcal{R}_{(1)}$ and $\mathcal{B}_f = \mathcal{B}_{(1)}$.

Let $\alpha, \beta \in (0,+\infty)$ and $\alpha < \beta$ be such that $(\alpha, 0), (\beta, 0) \notin \mathcal{B}_f$. Then let us define *the bifurcation index of the map f on the interval* (α, β) by

$$s[f,\alpha,\beta] = \deg(f(\beta,\cdot), B(0,r), 0) - \deg(f(\alpha,\cdot), B(0,r), 0)$$

for $r > 0$ small enough. In the above formula $\deg(\cdot)$ stands for the Leray--Schauder degree. We may extend this definition to the case of $(\alpha, 0)$, $(\beta, 0)$ satisfying

$$\Big(((\alpha-\delta,\alpha) \cup (\beta,\beta+\delta)) \times \{0\}\Big) \cap \mathcal{B}_f = \emptyset$$

for some $\delta > 0$. This may be done by

$$s[f,\alpha,\beta] = \lim_{\delta \to 0^+} s[f,\alpha-\delta,\beta+\delta].$$

The classical sufficient condition for the existence of bifurcation points and the theorem describing the structure of the set $\mathcal{R}_f$ is given in [12]. There exist numerous extensions and modifications of this theorem (for more detailed comments and the list of references see e.g. [3], [8], [11]). We will refer here to the theorem given in [4] for multivalued maps. The theorem given below is the slight modification of this theorem to the case of single valued maps:

Theorem A *Let E be a real Banach space, $A \subset \mathbb{R}$ be an open interval and $f : A \times E \to E$ be given by $f(\lambda, x) = x - F(\lambda, x)$, where $F : A \times E \to E$ is completely continuous. Assume that there exists the interval $[\alpha, \beta] \subset A$ such that $\mathcal{B}_f \subset [\alpha,\beta] \times \{0\}$ and $s[f,\alpha,\beta] \neq 0$. Then there exists the noncompact component $\mathcal{C} \subset \mathcal{R}_f$ satisfying $\mathcal{C} \cap \mathcal{B}_f \neq \emptyset$.*

Now let us make the general observation that all zeroes of solution u of (1) are simple.

Lemma 1 *If $(\lambda, u) \in (0, +\infty) \times C^1[0, \pi]$ is the nontrivial solution of* (1) *where φ is the Caratheodory map satisfying (3) and $u(t_0) = 0$ for $t_0 \in [0, \pi]$, then u changes sign in t_0.*

Proof. Let us observe that if

$$\begin{cases} u''(t) + \varphi(t, u(t), u'(t), \lambda) = 0 & \text{a.e. on } \ t \in (a, b) \\ u(t_0) = u'(t_0) = 0, \end{cases} \tag{15}$$

then $u = 0$. This is because by (3), in some neighborhood of t_0 the following estimation holds

$$|\varphi(t, u(t), u'(t), \lambda)| \leq \lambda m |q(t, u(t))| + |u(t)|,$$

and for t close to t_0 there must be $u(t) = 0$. Hence we may conclude that each zero of u must be isolated.

□

From now on let $\langle \cdot, \cdot \rangle$ stands for the standard $L^2(0, \pi)$ inner product. It may be easily checked that for each $u \in C^1[0, \pi]$, such that $u' \in L^1(0, \pi)$ and $u(0) = u(\pi) = 0$ the relation holds $\langle u'', u_k \rangle = -k^2 \langle u, u_k \rangle$, where $u_k(t) = \sin kt$.

Lemma 2 *If $(\lambda, u) \in (0, +\infty) \times C^1[0, \pi]$ is the solution of (2) and $\lambda > k^2$, then $u = 0$.*

Proof. Let us take the solution $(\lambda, u) \in (0, +\infty) \times C^1[0, \pi]$ of the problem (2) such that $\lambda > k^2$. Then

$$0 = \langle u'', u_k \rangle + \lambda \langle Q_k(u), u_k \rangle = -k^2 \langle u, u_k \rangle + \lambda \langle Q_k(u), u_k \rangle$$

We can see that, $q_k(t, u(t)) \sin kt \geq 0$, so

$$q_k(t, u(t)) \sin kt = |q_k(t, u(t)) \sin kt| = |u(t)| |\sin kt|.$$

Hence

$$\lambda q_k(t, u(t)) \sin kt - k^2 u(t) q_k(t, \sin kt) \geq \lambda |u(t)| |\sin kt| - k^2 |u(t)| |\sin kt| \geq 0.$$

Assume now, that $u \neq 0$. Because all zeroes of u and u_k are isolated, then $\langle |u|, |u_k| \rangle > 0$, so for $\lambda > k^2$ we have

$$0 = -k^2 \langle u, u_k \rangle + \lambda \langle Q_k(u), u_k \rangle \geq (\lambda - k^2) \langle |u|, |u_k| \rangle > 0,$$

a contradiction.

□

Lemma 3 $s[f_0, \inf \Lambda(q_k), k^2] = -1.$

Proof. Let $\lambda \in (0, \inf \Lambda(q_k))$ and $r > 0$ be fixed. We can see, that the map $h_0 : [0, 1] \times \overline{B(0, r)} \to C^1[0, \pi]$ given by $h_0(\tau, u) = f_0(\lambda \tau, u)$ is the homotopy joining $f_0(\lambda, \cdot)$ with the identity map, so $\deg(f_0(\lambda, \cdot), B(0, r), 0) = 1$.

Now let us take $\lambda > k^2$. We are going to show that

$$\deg(f_0(\lambda, \cdot), B(0,r), 0) = 0.$$

Let us further denote $u_k(t) = \sin kt$ and let us define the homotopy $h : [0,1] \times \overline{B(0,r)} \to C^1[0,\pi]$ by

$$h(\tau, u) = f_0(\lambda, u) - \tau u_k.$$

We will show that for $\tau \in (0,1]$ there are no zeros of $h(\tau, \cdot)$. Assume, contrary to our claim, that $h(\tau, u) = 0$ for some $u \in C^1[0,\pi]$. Then we have

$$u - \lambda T(Q_k(u)) - \tau u_k = 0.$$

So

$$u''(t) + \lambda q_k(t, u(t)) - \tau u_k''(t) = 0,$$

and

$$\begin{aligned} 0 &= \langle u'', u_k\rangle + \lambda\langle Q_k(u), u_k\rangle + \tau k^2\langle u_k, u_k\rangle, \\ \lambda\langle Q_k(u), u_k\rangle - k^2\langle u, u_k\rangle &= -\tau k^2 \langle u_k, u_k\rangle < 0 \end{aligned} \tag{16}$$

Moreover $q_k(t, u(t))u_k(t) \geq 0$, so

$$q_k(t, u(t))u_k(t) = |q_k(t, u(t))u_k(t)| = |u(t)| \cdot |u_k(t)|.$$

Hence

$$\lambda q_k(t, u(t))u_k(t) - k^2 u(t)u_k(t) \geq \lambda|u(t)||u_k(t)| - k^2|u(t)||u_k(t)| \geq 0$$

for $\lambda > k^2$. This contradicts (16) and proves that $h(\tau, u) \neq 0$ for all $\tau \in (0,1]$ and $u \in C^1[0,\pi]$. That is why $\deg(f_0(\lambda, \cdot), B(0,r), 0) = 0$ what completes the proof.

□

Now let us make one more observation related to the problem (2): let us observe that the positive solution of (4) is also the solution of (2) and, similarly, the negative solution of (5) is the solution of (2). This is formulated in the next proposition.

Proposition 2 *There exist solutions $(\lambda^+, u^+), (\lambda^-, u^-)$ of the problem* (2)*, such that $u^+(t) > 0$ and $u^-(t) < 0$ for $t \in (0,\pi)$. Additionally for any positive $A > 0$ the pairs $(\lambda^+, Au^+), (\lambda^-, Au^-)$ are solutions of the problem* (2). □

Lemma 4 *If (λ_k, u_k) is a nontrivial solution of* (2)*, such that u_k has exactly $k-1$ zeroes in $(0,\pi)$, then $\lambda_k = k^2$.*

Proof. First let us assume that in the one of the intervals $[\frac{l\pi}{k}, \frac{(l+1)\pi}{k})$ where $l \in \{0, 1, ..., k-1\}$ there are two adjacent zeroes $t_1, t_2 \in [\frac{l\pi}{k}, \frac{(l+1)\pi}{k})$ of the function u_k. Then we have

$$\begin{cases} u_k''(t) + \lambda q_k(t, u_k(t)) = 0 & \text{a.e. on } (0,\pi) \\ u_k(t_1) = u_k(t_2) = 0, \end{cases}$$

for u with constant sign on (t_1, t_2). Hence there must be

$$\begin{cases} u_k''(t) + \lambda u_k(t) = 0 & \text{a.e. on } (0,\pi) \\ u_k(t_1) = u_k(t_2) = 0, \end{cases}$$

what implies that $\lambda = \frac{\pi^2}{(t_2-t_1)^2} > \frac{\pi^2}{(\frac{\pi}{k})^2} = k^2$, what is the contradiction with the lemma 2.

Similarly we can show that in each of the intervals $(\frac{l\pi}{k}, \frac{(l+1)\pi}{k}]$ where $l \in \{0, 1, ..., k-1\}$ there is at most one zero of the function u_k.

Assume now that in the interval $(0, \pi)$ there are exactly $k-1$ zeroes of u. Because there are $k-1$ intervals we may state that in each interval $(\frac{l\pi}{k}, \frac{(l+1)\pi}{k}]$ and $[\frac{l\pi}{k}, \frac{(l+1)\pi}{k})$ there is exactly one zero of function u.

From the above facts and $u(0) = 0$ we may conclude that there are no zero in the open interval $(0, \frac{\pi}{k})$, so there must be $u(\frac{\pi}{k}) = 0$. Hence $\lambda = k^2$, what completes the proof. $\square$

The lemma below is in fact a classical result (cf. example 3.2(a) in section XI of Hartman's book [6]) and will be given without a proof.

Lemma 5 *Let the function $p \in L^1(0, \pi)$ satisfy $0 < K \leq p(t) \leq L$, for positive constants $K, L \in (0, +\infty)$. Let u be the solution of the linear differential equation*

$$u''(t) + p(t)u(t) = 0$$

with two adjacent zeroes t_1, t_2, then the distance between the zeroes t_1 and t_2 may be estimated as follows:

$$\frac{\pi}{\sqrt{L}} \leq t_2 - t_1 \leq \frac{\pi}{\sqrt{K}}. \tag{17}$$

Within the proof of the theorem 2 we will refer to the sequence of functions $q_k^n : [0, \pi] \times \mathbb{R} \to \mathbb{R}$ given by

$$q_k^n(t, x) = \begin{cases} |x + \frac{1}{n}| - \frac{1}{n} & \text{for} \ \sin kt \geq 0 \\ -|x - \frac{1}{n}| + \frac{1}{n} & \text{for} \ \sin kt < 0. \end{cases}$$

Let $Q_k^n : C^1[0, \pi] \to L^1(0, \pi)$ denote the Nemytskii operator associated with q_k^n $(n = 1, 2, ...)$.

Let us consider the family of boundary value problems

$$\begin{cases} u''(t) + \lambda m q_k^n(t, u(t)) + [\varphi(t, u(t), u'(t), \lambda) - \lambda m q_k(t, u(t))] = 0 \\ u(0) = u(\pi) = 0 \end{cases} \tag{18}$$

for $n \in \mathbb{N}$, and associated completely continuous vector fields $f_n : (0, +\infty) \times C^1[0, \pi] \to C^1[0, \pi]$ given by

$$f_n(\lambda, u) = u - \lambda m T Q_k^n(u) - T[\Phi(\lambda, u) - \lambda m Q_k(u)].$$

Lemma 6 *If $(\lambda_n, u_n) \in \mathcal{R}_{f_n}$ and the sequence $\{(\lambda_n, u_n)\}$ is bounded and $\{u_n\}$ is bounded away from zero, then it contains subsequence convergent (in $\mathbb{R} \times C^1[0, \pi]$) to $(\lambda_0, u_0) \in \mathcal{R}_f$.*

Proof. This is because

$$u_n = \lambda_n m T Q_k^n(u_n) + T[\Phi(\lambda_n, u_n) - \lambda_n m Q_k(u_n)]$$

and the sequence of maps $\{Q_k^n\}$ is uniformly bounded. Hence, we can select convergent subsequence of $\{u_n\}$. We can also select convergent subsequence of $\{\lambda_n\}$.

Let us now observe that for $u_n \to u_0$ the relation holds $Q_k^n(u_n) \to Q_k(u_0)$ in $L^1(0,\pi)$. This is because

$$|q_k^n(t,u_n(t)) - q_k(t,u_n(t))| \le \frac{2}{n}$$

and

$$|q_k(t,u_n(t)) - q_k(t,u_0(t))| \le \sup_{t\in[0,\pi]} |u_n(t) - u_0(t)|.$$

This completes the proof. $\square$

Lemma 7 *For any constant $\alpha \in (0, \min\{\frac{\min \Lambda(q_k)}{m}, \frac{2k^2}{3m}\})$ there exists $r_0 \in (0,+\infty)$, such that for $n \in \mathbb{N}$ each function $\|u_n\|_1 < r_0$ satisfying $f_n(\alpha, u_n) = 0$ and positive in some interval $(0,\delta)$, does not have exactly $k-1$ simple zeroes in $(0,\pi)$.*

Proof. Let us fix $\varepsilon \in (0, \frac{\alpha m}{2})$ and let $r_0 > 0$, be such that

$$\left| \frac{\varphi(t,x,y,\alpha) - \alpha m q_k(t,x)}{x} \right| \le \varepsilon$$

for $|x| + |y| \le r_0$.

Let us now denote by $t_n \in (0,\pi)$ the first zero of the function u_n. Then $u_n(t) > 0$ for $t \in (0, t_n)$ and

$$u_n''(t) + \alpha m q_k^n(t,u_n(t)) + \varphi(t,u_n(t),u_n'(t),\alpha) - \alpha m q_k(t,u_n(t)) = 0$$

$$u_n''(t) + \alpha m \frac{q_k^n(t,u_n(t))}{u_n(t)} \cdot u_n(t) + \qquad (19)$$

$$+ \frac{\varphi(t,u_n(t),u_n'(t),\alpha) - \alpha m q_k(t,u_n(t))}{u_n(t)} \cdot u_n(t) = 0$$

Assume that $t_n \in (0, \frac{\pi}{k})$. Then the equation (19) may be rewritten as

$$u_n''(t) + \alpha m u_n(t) + \frac{\varphi(t,u_n(t),u_n'(t),\alpha) - \alpha m q_k(t,u_n(t))}{u_n(t)} \cdot u_n(t) = 0$$

and $\alpha m + \frac{\varphi(t,u_n(t),u_n'(t),\alpha) - \alpha m q_k(t,u_n(t))}{u_n(t)} < \alpha m + \varepsilon < \frac{3\alpha m}{2}$. Then by lemma 5 we may estimate the distance between two adjacent zeroes of u_n by

$$\frac{\pi}{\sqrt{\frac{3\alpha m}{2}}} > \frac{\pi}{k}.$$

Hence u_n has no zero in the interval $(0, \frac{\pi}{k})$.

Similarly we may observe that for the interval $[\frac{l\pi}{k}, \frac{(l+1)\pi}{k}]$ for $l = 1,...,k-1$ the relations hold

(*) for l odd and $u(\frac{l\pi}{k}) \ge 0$ there is at most one zero in the interval $[\frac{l\pi}{k}, \frac{(l+1)\pi}{k}]$;

(**) for l even and $u(\frac{l\pi}{k}) \le 0$ there is at most one zero in the interval $[\frac{l\pi}{k}, \frac{(l+1)\pi}{k}]$;

(***) in any interval $[\frac{l\pi}{k}, \frac{(l+1)\pi}{k}]$ there exist at most 2 zeroes of the function u.

Let us now observe that if u changes sign in each interval $[\frac{i\pi}{k}, \frac{(i+1)\pi}{k}]$ (i.e. $u(\frac{i\pi}{k}) \cdot u(\frac{(i+1)\pi}{k}) < 0$) for $i = 1, ..., l$ then u has exactly l zeroes in the interval $[0, \frac{l\pi}{k}]$ (this is because $u(\frac{\pi}{k}) > 0$). Moreover, if in the interval $[\frac{l\pi}{k}, \frac{(l+1)\pi}{k}]$ function u has two zeroes, then there must exist the interval $[\frac{\bar{l}\pi}{k}, \frac{(\bar{l}+1)\pi}{k}]$ ($\bar{l} \in \{1, ..., l-1\}$) with no zeroes of u. Similarly, between two intervals containing two zeroes of u there must exist the interval with no zero of u.

This is why we may conclude that in the interval $[0, \frac{l\pi}{k}]$ there are at most l zeroes of u. So function u, satisfying $u(0) = u(\pi) = 0$, has at most $k-2$ zeroes in the open interval $(0, \pi)$ what contradicts our assumption.
□

Lemma 8 *For any constant $\beta > \frac{8k^2}{m}$ there exists $r_0 \in (0, +\infty)$, such that for $n \in \mathbb{N}$ each function $\|u_n\|_1 < r_0$ satisfying $f_n(\beta, u_n) = 0$ and positive in some interval $(0, \delta)$, does not have exactly $k-1$ simple zeroes in $(0, \pi)$.*

Proof. Let us fix $\varepsilon \in (0, \frac{\beta m}{2})$ and let $r_0 > 0$, be such that

$$\left| \frac{\varphi(t, x, y, \beta) - \beta m q_k(t, x)}{x} \right| \leq \varepsilon$$

for $|x| + |y| \leq r_0$.

Let us now denote by $t_n \in (0, \pi)$ the first zero of the function u_n and assume that $u_n(t) > 0$ for $t \in (0, t_n)$.

Similarly as in the proof of the lemma 7 let us consider, in the interval $(0, t_n)$, the equation

$$u_n''(t) + \beta m \frac{q_k^n(t, u_n(t))}{u_n(t)} \cdot u_n(t) + \tag{20}$$

$$+ \frac{\varphi(t, u_n(t), u_n'(t), \beta) - \beta m q_k(t, u_n(t))}{u_n(t)} \cdot u_n(t) = 0.$$

Assume that $t_n > \frac{\pi}{k}$, then $q_k^n(t, u_n(t)) = u_n(t)$ for $t \in (0, \frac{\pi}{k})$ and the above equation (20) may be rewritten as

$$u_n''(t) + \beta m u_n(t) + \frac{\varphi(t, u_n(t), u_n'(t), \beta) - \beta m q_k(t, u_n(t))}{u_n(t)} \cdot u_n(t) = 0$$

and $\beta m + \frac{\varphi(t, u_n(t), u_n'(t), \beta) - \beta m q_k(t, u_n(t))}{u_n(t)} > \beta m - \varepsilon > \frac{\beta m}{2}$. Then by lemma 5 we may estimate the distance between two adjacent zeroes of u_n by

$$\frac{\pi}{\sqrt{\frac{\beta m}{2}}} \leq \frac{\pi}{2k}.$$

This means that in the interval $(0, \frac{\pi}{2k})$ there exists the zero of u_n.

Assume u_n has exactly $k-1$ zeroes in the open interval $(0, \pi)$. As we know from lemma 1 all zeroes of u_n are simple (so u_n changes sign exactly $k-1$ times), so in some neighborhood $(\pi - \delta, \pi)$ of the point π the relation holds $q_k^n(t, u_n(t)) = u_n(t)$. So, we may repeat the above arguments and state that in the interval $(\pi - \frac{\pi}{2k}, \pi)$ there exists the zero of u_n.

The similar reasoning may be applied for each interval $[\frac{l\pi}{k}, \frac{(l+1)\pi}{k}]$, $l = 1, 2, ..., k-1$, what means that we have the following facts:

(*) for l even and $u(\frac{l\pi}{k}) \geq 0$ there exists at least one zero in the interval $[\frac{l\pi}{k}, \frac{l\pi}{k} + \frac{\pi}{2k})$;

(**) for l odd and $u(\frac{l\pi}{k}) \leq 0$ there exists at least one zero in the interval $[\frac{l\pi}{k}, \frac{l\pi}{k} + \frac{\pi}{2k})$.

Let us now assume, that in the interval $[\frac{l_0\pi}{k}, \frac{(l_0+1)\pi}{k})$ there is no zero of u, and that l_0 is the minimal number with this property. In case each interval $[\frac{l\pi}{k}, \frac{(l+1)\pi}{k})$ for $l = 1, ..., l_0 - 1$ has exactly one zero and there are exactly two zeroes in the interval $[0, \frac{\pi}{k})$, then $u(\frac{l\pi}{k}) < 0$ for l odd and $u(\frac{l\pi}{k}) > 0$ for l even. This implies that also in the interval $[\frac{l_0\pi}{k}, \frac{(l_0+1)\pi}{k})$ there is at least one zero of u, a contradiction. So at least one interval $[\frac{l\pi}{k}, \frac{(l+1)\pi}{k})$ for $l = 1, ..., l_0$ must contain at least two zeroes of u. So, the interval $(0, \frac{(l_0+1)\pi}{k})$ contains at least l_0 zeroes.

Moreover, if in the interval $[\frac{l_0\pi}{k}, \frac{(l_0+\frac{1}{2})\pi}{k})$ there are no zeroes of u, then $u(\frac{l_0\pi}{k}) > 0$ for l_0 odd and $u(\frac{l_0\pi}{k}) < 0$ for l_0 even, and u does not change sign in the interval $[\frac{l_0\pi}{k}, \frac{(l_0+1)\pi}{k})$, so there exists at least one zero in the interval $[\frac{(l_0+1)\pi}{k}, \frac{(l_0+2)\pi}{k})$.

Similarly as above, between two intervals $[\frac{l\pi}{k}, \frac{(l+1)\pi}{k})$ with no zero of u there exists at least one interval with two zeroes. So, each interval $(0, \frac{l\pi}{k} + \frac{\pi}{2k})$ contains at least l zeroes of u. Because, as we have shown above, there exists zero of u in the interval $(\pi - \frac{\pi}{2k}, \pi)$ the function u has at least k zeroes in the interval $(0, \pi)$. A contradiction.

□

3 Proofs of the theorems

Proof.[Proof of theorem 1] Let us take the map $f_0 : (0, +\infty) \times C^1[0, \pi] \to C^1[0, \pi]$ given by $f_0(\lambda, u) = u - m\lambda T Q_k(u)$.

We are going to refer to the theorem A given in the section 2. First let us observe that by lemmas 2 and 3

$$\emptyset \neq \mathcal{B}_{f_0} \subset [\inf \Lambda(q_k), k^2] \times \{0\}.$$

and $\mathcal{B}_f \subset \mathcal{B}_{f_0}$.

Now let us observe that $s[f, \inf \Lambda(q_k), k^2] = s[f_0, \inf \Lambda(q_k), k^2] = -1$. This is because for any $\lambda \in (0, +\infty) \setminus [\inf \Lambda(q_k), k^2]$ there exists positive $r > 0$, such that the maps $f(\lambda, \cdot), \tilde{f}(\lambda, \cdot) : \overline{B(0, r)} \to C^1[0, \pi]$ may be joined by homotopy $h : [0, 1] \times \overline{B(0, r)} \to C^1[0, \pi]$ given by

$$h(\tau, u) = u - m\lambda T Q_k(u) + \tau T[m\lambda Q_k(u) - \Phi(\lambda, u)].$$

Hence all assumptions of theorem A are satisfied and there exists the noncompact component C of $\mathcal{R}_f$, such that $C \cap \mathcal{B}_f \neq \emptyset$. □

Proof.[Proof of theorem 2]

Step 1.

First we are going to show that $(\frac{\lambda^+}{m}, 0) \in \mathcal{B}_f$ and $(\frac{\lambda^-}{m}, 0) \in \mathcal{B}_f$.

Let us observe that if $(\lambda, u) \in (0, +\infty) \times C^1[0, \pi]$ is the solution of

$$\begin{cases} u''(t) + \lambda m a^+(t)u(t) + [\varphi(t, u(t), u'(t), \lambda) - \lambda m q_k(t, u(t))] = 0 \\ u(0) = u(\pi) = 0, \end{cases} \tag{21}$$

such that $u \geq 0$, then (λ, u) is the solution of (1). This is because for $u \geq 0$ the relation $q_k(t, u) = a^+(t)u(t)$ holds.

Let $f^+ : (0, +\infty) \times C^1[0, \pi] \to C^1[0, \pi]$ be the map associated with the problem (21). Because all eigenvalues of the linear problem (4) are simple (see [9]) we can observe that for the above problem (21) we may apply the Rabinowitz global bifurcation theorem (see [12], theorem 2.3). That is why there exists the noncompact, connected, closed subset $C_1^+ \subset \mathcal{R}_{f+}$ such that $u \geq 0$ for all $(\lambda, u) \in C_1^+$ and $(\lambda^+, 0) \in C_1^+$. The set C_1^+ is also the closed, connected and noncompact subset of $\mathcal{R}_f$.

Similar observation may be made for the problem

$$\begin{cases} u''(t) + \lambda m a^-(t)u(t) + [\varphi(t, u(t), u'(t), \lambda) - \lambda m q_k(t, u(t))] = 0 \\ u(0) = u(\pi) = 0. \end{cases} \tag{22}$$

Similarly as above for each solution $(\lambda, u) \in (0, +\infty) \times C^1[0, \pi]$ such that $u \leq 0$, the pair (λ, u) is the solution of (1). So, there exists the closed, connected and noncompact subset of $C_1^- \subset \mathcal{R}_f$.

Step 2.

Now we are going to prove the existence of the component C_k.

Let us consider the family of boundary value problems (18)

$$\begin{cases} u''(t) + \lambda m q_k^n(t, u(t)) + [\varphi(t, u(t), u'(t), \lambda) - \lambda m q_k(t, u(t))] = 0 \\ u(0) = u(\pi) = 0. \end{cases}$$

Let us observe that $Q_k^n(u) = u$ for $\|u\|_1 \leq \frac{1}{n}$. Moreover, by (6), for any positive $\varepsilon > 0$, there exists $\delta > 0$, such that for $\|u\|_1 \leq \delta$ the relation holds

$$|\varphi(t, u(t), u'(t), \lambda) - \lambda m q_k(t, u(t))| \leq \varepsilon \|u\|_1.$$

Hence all assumptions of the Rabinowitz global bifurcation theorem (see [12]) are satisfied for the map f_n. The theorem 2.3 of [12] implies that there exists the connected, noncompact and closed set $C_{k,+}^n \subset \mathcal{R}_{f_n}$, such that $(\frac{k^2}{m}, 0) \in C_{k,+}^n$ and for $(\lambda, u) \in C_{k,+}^n$ and $u \neq 0$, the function u has exactly $k - 1$ zeroes in the interval $(0, \pi)$, all zeroes of u are simple and $u(t) > 0$ in some neighbourhood of 0.

Let the constants $\alpha, \beta \in (0, +\infty)$ satisfy all the assumptions given in the lemmas 7 and 8, and $\frac{k^2}{m} \in (\alpha, \beta)$. Additionally let $r_0 > 0$ be such that

$$\left| \frac{\varphi(t, x, y, \lambda) - \lambda m q_k(t, x)}{x} \right| \leq \varepsilon < \frac{\alpha m}{2} < \frac{\beta m}{2}$$

for $|x| + |y| \leq r_0$ and $\lambda \in [\alpha, \beta]$.

Because $(\frac{k^2}{m}, 0) \in C_{k,+}^n$, then it is not possible that $\|u\|_1 > r_0$ for all $(\lambda, u) \in C_{k,+}^n$. Hence let us assume, that for $n \in \mathbb{N}$ large enough if $(\lambda_n, u_n) \in C_{k,+}^n$ and $\lambda_n \in [\alpha, \beta]$ then $\|u_n\|_1 < r_0$. Then, because the set $C_{k,+}^n$ is noncompact and connected, there must exist either $(\alpha, u_n) \in C_{k,+}^n$ or $(\beta, u_n) \in C_{k,+}^n$. As shown in lemmas 7 and 8 neither of these situations is possible. So, for almost all $n \in \mathbb{N}$ there exist pairs $(\lambda_n, u_n) \in C^n_{k,+}$, such that $\lambda_n \in [\alpha, \beta]$ and $\|u_n\|_1 = r_0$.

From the above observation we may conclude that there exists the bifurcation point $(\lambda^k, 0) \in \mathcal{B}_f$, such that for $n \in \mathbb{N}$ large enough, there exist points $(\lambda_n, u_n) \in C^n_{k,+}$ laying arbitrarily close to $(\lambda^k, 0)$. This is because the sets $C^n_{k,+}$ are connected, and for each $r \in (0, r_0]$ there exists the sequence $\{(\lambda^r_n, u^r_n)\}$, such that $(\lambda^r_n, u^r_n) \in C^n_{k,+}$ and $\|u^r_n\|_1 = r$. As stated in the lemma 6 this sequence contains subsequence convergent to $(\lambda^r, u^r) \in \mathcal{R}_f$. With any sequence $r_n \to 0$ we may take the subsequence of $\{\lambda^{r_n}\}$ convergent to some $\lambda^k \in [0, \pi]$.

Let C_k denote the component of $\mathcal{R}_f$ such that $(\lambda^k, 0) \in C_k$.

Step 3.

Now we are going to show that C_k is not compact.

Let us assume now that there exists $\varepsilon > 0$ such that $O_\varepsilon(C_k) \cap C^n_{k,+} = \emptyset$ for infinitely many $n \in \mathbb{N}$, where

$$O_\varepsilon(A) = \{(\lambda, u) \in (0, +\infty) \times C^1[0, \pi] | \exists_{(\mu,v)\in A} |\lambda - u| + \|u - v\|_1 < \varepsilon\}$$

for the set $A \subset (0, +\infty) \times C^1[0, \pi]$.

This assumption leads to a contradiction, because, as shown above, for some $r > 0$ sufficiently small, from the sequence $(\lambda^r_n, u^r_n) \in C^n_{k,+}$ such that $\|u^r_n\|_1 = r$ and $\lambda^r_n \in [a, b]$ we may select subsequence converging to the point $(\lambda^r, u^r) \in \mathcal{R}_f$ being arbitrarily close to the bifurcation point $(\lambda_k, 0)$.

So for any positive $\varepsilon > 0$ the relation $O_\varepsilon(C_k) \cap C^n_{k,+} \neq \emptyset$ holds for almost all $n \in \mathbb{N}$.

Now let us assume, contrary to our claim, that the set C_k is compact. Then there exists the interval $[c, d] \subset [c - \delta, d + \delta] \subset (0, +\infty)$ and the constant $R > 0$, such that $C_k \subset (c, d) \times B(0, R)$. We may also assume that $\mathcal{B}_f \subset (c, d)$. So, the sequence of noncompact sets $C^n_{k,+}$ satisfies:

(a) $C^n_{k,+} \cap \partial([c, d] \times \overline{B(0, R)}) \neq \emptyset$;

(b) for any positive $\varepsilon > 0$, there exists the subsequence $C^{\gamma(n)}_{k,+}$ of $C^n_{k,+}$, such that for $n \in \mathbb{N}$ large enough $C^{\gamma(n)}_{k,+} \cap O_\varepsilon(C_k) \neq \emptyset$.

Let us denote

$$\mathcal{R}_0 = \mathcal{R}_f \cap \left([c, d] \times \overline{B(0, R)}\right).$$

We are going to show, that there exists the sequence $(\lambda_{\gamma(n)}, u_{\gamma(n)}) \in C^n_{k,+} \cap \partial([c, d] \times \overline{B(0, R)})$ convergent to $(\lambda_0, u_0) \in \mathcal{R}_0$. As we have shown above the limit point is the zero of f. We will observe that this zero is not trivial. Let us further denote $u_n = u_{\gamma(n)}$ and assume that $u_n \to 0$. Then

$$u_n = T\lambda_n m Q^n_k(u_n) + T[\Phi(\lambda_n, u_n) - \lambda_n m Q_k(u_n)]$$

and

$$v_n = T\lambda_n m Q^n_k(v_n) + T[\frac{\Phi(\lambda_n, u_n) - \lambda_n m Q_k(u_n)}{\|u_n\|_1}]$$

where $v_n = \frac{u_n}{\|u_n\|_1}$.

So, the sequence v_n contains subsequence convergent to a function v_0. Because neither $(c, 0)$ and $(d, 0)$ is the bifurcation point of f, then $u_n \not\to 0$ and this means that the limit point (λ_0, u_0) belongs to $\mathcal{R}_0$.

The set $\mathcal{R}_0$ is a compact metric space, and $X = C_k$ and $Y = \{(\lambda_0, u_0)\}$ its closed subsets, not belonging to the same component of $\mathcal{R}_0$. By the separation lemma (see [15]) there exists the separation $\mathcal{R}_0 = \mathcal{R}_x \cup \mathcal{R}_y$ of $\mathcal{R}_0$, where $\mathcal{R}_x$ and $\mathcal{R}_y$ are closed and disjoint, and such that $C_k \subset \mathcal{R}_x$ and $(\lambda_0, u_0) \in \mathcal{R}_y$. Moreover, the set $\mathcal{R}_y$ may be selected in such way, that it is bounded away from the line of trivial solutions.

This implies, that there exist open and disjoint subsets $U_x, U_y \subset (c - \delta, d + \delta) \times B(0, R + \delta)$, such that $(\lambda_0, u_0) \in U_y$ and $C_k \subset U_x$ and $\mathcal{R}_0 \subset U_x \cup U_y$. Additionally we may assume that $\overline{U_y}$ does not intersect the line of trivial solutions.

Because for $n \in \mathbb{N}$ large enough the components $C_{k,+}^{\gamma(n)}$ intersect both U_x and U_y, from its connectedness we may conclude that there exist the sequence $\{(\lambda_n, u_n)\} \subset \partial U_y$. This sequence contains subsequence convergent to $(\lambda_0, u_0) \in \mathcal{R}_0 \cap (\partial U_y) = \emptyset$, a contradiction.

Step 4.

Now we are going to prove (10). As we have shown above there exists pairs $(\lambda_n, u_n) \in C_{k,+}^n$, such that $|\lambda_n - \lambda| + \|u_n - u\|_1 \to 0$ for some $(\lambda, u) \in C_k$. So $(\lambda, u) \in \overline{\mathcal{C}_{k,+}}$, where $\mathcal{C}_{k,+}$ denotes the set of functions $u \in C^1[0, \pi]$ having exactly $k - 1$ zeroes in the interval $(0, \pi)$, with all zeroes simple, and positive in a small neighborhood $(0, \delta)$ of 0. We can observe that for $u \in \partial\mathcal{C}_{k,+}$ function u must have double zero. This is not possible (by lemma 1), so because $C_k \cap ((0, +\infty) \times \mathcal{C}_{k,+}) \neq \emptyset$ and $C_k \cap ((0, +\infty) \times \partial\mathcal{C}_{k,+}) = \emptyset$, there must be $C_k \subset (0, +\infty) \times \mathcal{C}_{k,+}$, what proves (10).

Step 5.

It remains to prove that $(\frac{k^2}{m}, 0) \in C_k$. Let us take the sequence $\{(\lambda_n, u_n)\} \subset C_k$ such that $0 \neq \|u_n\|_1 \to 0$ and $\lambda_n \to \lambda_k$. Then we have

$$u_n = \lambda_n m T Q_k^n(u_n) + T[\Phi(\lambda_n, u_n) - \lambda_n m Q_k(u_n)]$$
$$v_n = \lambda_n m T Q_k^n(v_n) + T\frac{\Phi(\lambda_n, u_n) - \lambda_n m Q_k(u_n)}{\|u_n\|_1}$$

where $v_n = \frac{u_n}{\|u_n\|_1}$. We may select subsequence of $\{v_n\}$ convergent to v_0, such that

$$v_0 = \lambda_k m T Q_k(v_0).$$

Because $\{v_n\} \subset \mathcal{C}_{k,+}$ and $v_n \notin \partial\mathcal{C}_{k,+}$ the function v_0 has exactly $k - 1$ zeroes in the interval $(0, \pi)$. By lemma 4 $\lambda_k = \frac{k^2}{m}$, what completes the proof. $\square$

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