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Multiple Positive Solutions for the System of Higher Order Two-Point Boundary Value Problems on Time Scales

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Abstract

In this paper, we establish the existence of at least three positive solutions for the system of higher order boundary value problems on time scales by using the well-known Leggett-Williams fixed point theorem. And then, we prove the existence of at least 2k-1 positive solutions for arbitrary positive integer k.

1 Introduction

The boundary value problems (BVPs) play a major role in many fields of engineering design and manufacturing. Major established industries such as the automobile, aerospace, chemical, pharmaceutical, petroleum, electronics and communications, as well as emerging technologies such as nanotechnology and biotechnology rely on the BVPs to simulate complex phenomena at different scales for design and manufactures of high-technology products. In these applied settings, positive solutions are meaningful. Due to their important role in both theory and applications, the BVPs have generated a great deal of interest over the recent years.

The development of the theory has gained attention by many researchers. To mention a few, we list some papers Erbe and Wang [7], Eloe and Henderson [5, 6], Hopkins and Kosmatov [9], Li [10], Atici and Guseinov [11], Anderson and Avery [2], Avery and Peterson [3] and Peterson, Raffoul and Tisdell [12]. For the time scale calculus and notation for delta differentiation, as well as

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concepts for dynamic equations on time scales, we refer to the introductory book on time scales by Bohner and Peterson [4]. By an interval we mean the intersection of real interval with a given time scale.

In this paper, we address the question of the existence of multiple positive solutions for the nonlinear system of boundary value problems on time scales,

$$\begin{cases} y_1^{\Delta^{(m)}} + f_1(t, y_1, y_2) = 0, & t \in [a, b] \\ y_2^{\Delta^{(n)}} + f_2(t, y_1, y_2) = 0, & t \in [a, b] \end{cases} \tag{1}$$

subject to the two-point boundary conditions

$$\begin{cases} y_1^{\Delta^{(i)}}(a) = 0, & 0 \leq i \leq m-2, \\ y_1(\sigma^q(b)) = 0, & \\ & \\ y_2^{\Delta^{(j)}}(a) = 0, & 0 \leq j \leq n-2, \\ y_2(\sigma^q(b)) = 0, & \end{cases} \tag{2}$$

where $f_i : [a, \sigma^q(b)] \times \mathbb{R}^2 \to \mathbb{R}$, $i = 1, 2$ are continuous, $m, n \geq 2$, $q = \min\{m, n\}$, and $\sigma^q(b)$ is right dense so that $\sigma^q(b) = \sigma^r(b)$ for $r \geq q$.

This paper is organized as follows. In Section 2, we prove some lemmas and inequalities which are needed later. In Section 3, we obtain existence and uniqueness of a solution for the BVP (1)-(2), due to Schauder fixed point theorem. In Section 4, by using the cone theory techniques, we establish sufficient conditions for the existence of at least three positive solutions to the BVP (1)-(2). The main tool in this paper is an applications of the Leggett-Williams fixed point theorem for operator leaving a Banach space cone invariant, and then, we prove the existence of at least $2k-1$ positive solutions for arbitrary positive integer k.

2 Green's function and bounds

In this section, we construct the Green's function for the homogeneous BVP corresponding to the BVP (1)-(2). And then we prove some inequalities which are needed later.

To obtain a solution $(y_1(t), y_2(t))$ of the BVP (1)-(2) we need the $G_n(t, s)$, $(n \geq 2)$ which is the Green's function of the BVP,

$$-y^{\Delta^{(n)}} = 0, \quad t \in [a, b] \tag{3}$$

$$y^{\Delta^{(i)}}(a) = 0, \quad 0 \leq i \leq n-2, \tag{4}$$

$$y(\sigma^n(b)) = 0. \tag{5}$$

Theorem 2.1 *The Green's function for the BVP* (3)-(5) *is given by*

$$G_n(t,s) = \frac{1}{(n-1)!}\begin{cases} \prod_{i=1}^{n-1} \frac{(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma^n(b)-\sigma^{i-1}(a))}, & \text{t} \leq \text{s}, \\ \prod_{i=1}^{n-1} \frac{(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma^n(b)-\sigma^{i-1}(a))} - \prod_{i=1}^{n-1}(t-\sigma^i(s)), & \sigma(\text{s}) \leq \text{t}. \end{cases}$$

Proof: It is easy to check that the BVP (3)-(5) has only trivial solution. Let $y(t,s)$ be the Cauchy function for $-y^{\Delta^{(n)}} = 0$, and be given by

$$y(t,s) = \frac{-1}{(n-1)!} \underbrace{\int_{\sigma(s)}^{t} \int_{\sigma^2(s)}^{t} ... \int_{\sigma^{n-1}(s)}^{t}}_{(n-1)\ \text{times}} \Delta\tau\Delta\tau...\Delta\tau = \frac{-1}{(n-1)!}\prod_{i=1}^{n-1}(t-\sigma^i(s)).$$

For each fixed $s \in [a,b]$, let $u(.,s)$ be the unique solution of the BVP

$$-u^{\Delta^{(n)}}(.,s) = 0,$$

$$u^{\Delta^{(i)}}(a,s) = 0, \quad 0 \leq i \leq n-2 \quad \text{and} \quad u(\sigma^n(b),s) = -y(\sigma^n(b),s).$$

$$y(t,s)\mid_{t=\sigma^n(b)} = \frac{-1}{(n-1)!}\prod_{i=1}^{n-1}(\sigma^n(b)-\sigma^i(s)).$$

Since

$$u_1(t) = 1, u_2(t) = \int_a^t \Delta\tau, ..., \ u_n(t) = \underbrace{\int_a^t \int_{\sigma(a)}^t ... \int_{\sigma^{n-2}(a)}^t}_{(n-1)\ \text{times}} \Delta\tau\Delta\tau...\Delta\tau$$

are the solutions of $-u^{\Delta^{(n)}} = 0$,

$$u(t,s) = \alpha_1(s).1 + \alpha_2(s).\int_a^t \Delta\tau + ... + \alpha_n(s).\underbrace{\int_a^t \int_{\sigma(a)}^t ... \int_{\sigma^{n-2}(a)}^t}_{(n-1)\ \text{times}} \Delta\tau\Delta\tau...\Delta\tau$$

By using boundary conditions, $u^{\Delta^{(i)}}(a) = 0, \quad 0 \leq i \leq n-2$, we have $\alpha_1 = \alpha_2 = ... = \alpha_{n-1} = 0$. Therefore, we have

$$u(t,s) = \alpha_n \underbrace{\int_a^t \int_{\sigma(a)}^t ... \int_{\sigma^{n-2}(a)}^t}_{(n-1)\ \text{times}} \Delta\tau\Delta\tau...\Delta\tau = \alpha_n\prod_{i=1}^{n-1}(t-\sigma^{i-1}(a)).$$

Since,

$$u(\sigma^n(b), s) = -y(\sigma^n(b), s),$$

it follows that

$$\alpha_n \prod_{i=1}^{n-1} (\sigma^n(b) - \sigma^{i-1}(a)) = \frac{1}{(n-1)!} \prod_{i=1}^{n-1} (\sigma^n(b) - \sigma^i(s)).$$

From which implies

$$\alpha_n = \frac{1}{(n-1)!} \prod_{i=1}^{n-1} \frac{(\sigma^n(b) - \sigma^i(s))}{(\sigma^n(b) - \sigma^{i-1}(a))}.$$

Hence $G_n(t, s)$ has the form for $t \leq s$,

$$G_n(t, s) = \frac{1}{(n-1)!} \prod_{i=1}^{n-1} \frac{(t - \sigma^{i-1}(a))(\sigma^n(b) - \sigma^i(s))}{(\sigma^n(b) - \sigma^{i-1}(a))}.$$

And for $t \geq \sigma(s)$, $\quad G_n(t, s) = y(t, s) + u(t, s)$. It follows that

$$G_n(t, s) = \frac{1}{(n-1)!} \prod_{i=1}^{n-1} \frac{(t - \sigma^{i-1}(a))(\sigma^n(b) - \sigma^i(s))}{(\sigma^n(b) - \sigma^{i-1}(a))} - \frac{1}{(n-1)!} \prod_{i=1}^{n-1} (t - \sigma^i(s)).$$

□

Lemma 2.2 *For* $(t, s) \in [a, \sigma^n(b)] \times [a, b]$, *we have*

$$G_n(t, s) \leq G_n(\sigma(s), s). \tag{6}$$

Proof: For $\quad a \leq t \leq s \leq \sigma^n(b)$, we have

$$\begin{aligned} G_n(t, s) &= \frac{1}{(n-1)!} \prod_{i=1}^{n-1} \frac{(t - \sigma^{i-1}(a))(\sigma^n(b) - \sigma^i(s))}{(\sigma^n(b) - \sigma^{i-1}(a))} \\ &\leq \frac{1}{(n-1)!} \prod_{i=1}^{n-1} \frac{(\sigma(s) - \sigma^{i-1}(a))(\sigma^n(b) - \sigma^i(s))}{(\sigma^n(b) - \sigma^{i-1}(a))} \\ &= G_n(\sigma(s), s). \end{aligned}$$

Similarly, for $\quad a \leq \sigma(s) \leq t \leq \sigma^n(b)$, we have $G_n(t, s) \leq G_n(\sigma(s), s)$. Thus, we have

$$G_n(t, s) \leq G_n(\sigma(s), s), \quad \text{for all} \quad (t, s) \in [a, \sigma^n(b)] \times [a, b].$$

□

Lemma 2.3 *Let $I = [\frac{\sigma^n(b)+3a}{4}, \frac{3\sigma^n(b)+a}{4}]$. For $(t,s) \in I \times [a,b]$, we have*

$$G_n(t,s) \geq \frac{1}{16^{n-1}} G_n(\sigma(s),s). \tag{7}$$

Proof: The Green's function for the BVP (3)-(5) is given in the Theorem 2.1, clearly shows that

$$G_n(t,s) > 0 \ \text{ on } \ (a,\sigma^n(b)) \times (a,b).$$

For $\ a \leq t \leq s < \sigma^n(b)$ and $t \in I$, we have

$$\begin{aligned}
\frac{G_n(t,s)}{G_n(\sigma(s),s)} &= \prod_{i=1}^{n-1} \frac{(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{(\sigma(s)-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))} \\
&\geq \prod_{i=1}^{n-1} \frac{(t-\sigma^{i-1}(a))}{(\sigma^n(b)-a)} \\
&\geq \frac{1}{4^{n-1}}.
\end{aligned}$$

And for $\ a \leq \sigma(s) \leq t < \sigma^n(b)$ and $t \in I$, we have

$$\begin{aligned}
&\frac{G_n(t,s)}{G_n(\sigma(s),s)} \\
&= \frac{\prod_{i=1}^{n-1}(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s)) - \prod_{i=1}^{n-1}(t-\sigma^i(s))(\sigma^n(b)-\sigma^i(a))}{\prod_{i=1}^{n-1}(\sigma(s)-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))} \\
&\geq \frac{\prod_{i=1}^{n-1}(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s)) - \prod_{i=1}^{n-1}(t-\sigma^i(s))(\sigma^n(b)-\sigma^i(a))}{\prod_{i=1}^{n-1}(\sigma^n(b)-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))} \\
&\geq \frac{[(\sigma(s)-a)(\sigma^2(b)-t)]\prod_{i=2}^{n-1}(t-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(s))}{\prod_{i=1}^{n-1}(\sigma^n(b)-\sigma^{i-1}(a))(\sigma^n(b)-\sigma^i(a))} \\
&\geq \frac{1}{16^{n-1}}.
\end{aligned}$$

□

Remark:

$$G_n(t,s) \geq \gamma G_n(\sigma(s),s) \ \text{ and } \ G_m(t,s) \geq \gamma G_m(\sigma(s),s),$$

for all $(t,s) \in I \times [a, \sigma^q(b)]$, where $\gamma = \min\left\{\frac{1}{16^{n-1}}, \frac{1}{16^{m-1}}\right\}$.

3 Existence and Uniqueness

In this section, we give the existence and local uniqueness of solution of the BVP (1)-(2). To prove this result, we define $B = E\times E$ and for $(y_1, y_2) \in B$, we denote the norm by $\|(y_1, y_2)\| = \|y_1\|_0+\|y_2\|_0$, where $E = \{y : y \in C[a, \sigma^q(b)]\}$ with the norm $\|y\|_0 = \max_{t\in[a,\sigma^q(b)]}\{|y(t)|\}$, obviously $(B, \| \,.\, \|)$ is a Banach space.

Theorem 3.1 *If M satisfies*

$$Q \leq M\epsilon,$$

where $\epsilon = \frac{1}{2\max\{\epsilon_m,\epsilon_n\}}$,

$$\epsilon_m = \max_{t\in[a,\sigma^q(b)]} \int_a^{\sigma(b)} G_m(t, s)\Delta s; \quad \text{and} \quad \epsilon_n = \max_{t\in[a,\sigma^q(b)]} \int_a^{\sigma(b)} G_n(t, s)\Delta s$$

and $Q > 0$ satisfies

$$Q \geq \max_{\|(y_1,y_2)\|\leq M} \{|f_1(t, y_1, y_2)|, |f_2(t, y_1, y_2)|\}, \quad for\ t \in [a, \sigma^q(b)],$$

then the BVP (1)-(2) has a solution in the cone P contained in B.

Proof: Set $P = \{(y_1, y_2) \in B :\| (y_1, y_2) \|\leq M\}$ the P is a cone in B, Note that P is closed, bounded and convex subset of B to which the Schauder fixed point theorem is applicable. Define $T : P \to B$ by

$$\begin{aligned} T(y_1, y_2)(t) &:= \left(\int_a^{\sigma(b)} G_m(t, s)f_1(s, y_1, y_2)\Delta s, \ \int_a^{\sigma(b)} G_n(t, s)f_2(s, y_1, y_2)\Delta s\right) \\ &:= (T_m(y_1, y_2)(t), T_n(y_1, y_2)(t)), \end{aligned}$$

for $t \in [a, \sigma^q(b)]$. Obviously the solution of the BVP (1)-(2) is the fixed point of operator T. It can be shown that $T : P \to B$ is continuous. Claim that $T : P \to P$. If $(y_1, y_2) \in P$, then

$$\begin{aligned} \| T(y_1, y_2) \| &=\| T_m(y_1, y_2) \|_0 + \| T_n(y_1, y_2) \|_0 \\ &= \max_{t\in[a,\sigma^q(b)]} |T_m(y_1, y_2)| + \max_{t\in[a,\sigma^q(b)]} |T_n(y_1, y_2)| \\ &\leq (\epsilon_m + \epsilon_n)Q \\ &\leq \frac{Q}{\epsilon}, \end{aligned}$$

where

$$Q \geq \max_{\|(y_1,y_2)\|\leq M} \{|f_1(t, y_1, y_2)|, |f_2(t, y_1, y_2)|\},$$

for $t \in [a, \sigma^q(b)]$. Thus we have

$$\| T(y_1, y_2) \| \leq M,$$

where M satisfies $Q \leq M\epsilon$. □

Corollary 3.2 *If the functions f_1, f_2, as defined in equation (1), are continuous and bounded. Then the BVP (1)-(2) has a solution.*

Proof: Choose $P > \sup\{|f_1(t, y_1, y_2)|, |f_2(t, y_1, y_2)|\}, \quad t \in [a, \sigma^q(b)]$. Pick M large enough so that $P < M\epsilon$, where $\epsilon = \frac{1}{2\max\{\epsilon_m, \epsilon_n\}}$. Then there is a number $Q > 0$ such that $P > Q$ where

$$Q \geq \max_{\|(y_1, y_2)\| \leq M} \{|f_1(t, y_1, y_2)|, |f_2(t, y_1, y_2)|\}, \quad t \in [a, \sigma^q(b)].$$

Hence

$$\frac{1}{\epsilon} < \frac{M}{P} \leq \frac{M}{Q},$$

and then the BVP (1)-(2) has a solution by Theorem 3.1. □

4 Existence of Multiple Positive Solutions

In this section, we establish the existence of at least three positive solutions for the system of BVPs (1)-(2). And also we establish the $2k - 1$ positive solutions for arbitrary positive integer k.

Let B be a real Banach space with cone P. A map $S : P \to [0, \infty)$ is said to be a nonnegative continuous concave functional on P, if S is continuous and

$$S(\lambda x + (1 - \lambda)y) \geq \lambda S(x) + (1 - \lambda)S(y),$$

for all $x, y \in P$ and $\lambda \in [0, 1]$. Let a' and b' be two real numbers such that $0 < a' < b'$ and S be a nonnegative continuous concave functional on P. We define the following convex sets

$$P_{a'} = \{y \in P : \| y \| < a'\},$$

$$P(S, a', b') = \{y \in P : a' \leq S(y), \| y \| \leq b'\}.$$

We now state the famous Leggett-Williams fixed point theorem.

Theorem 4.1 *Let $T : \overline{P}_{c'} \to \overline{P}_{c'}$ be completely continuous and S be a non-negative continuous concave functional on P such that $S(y) \leq \| y \|$ for all $y \in \overline{P}_{c'}$. Suppose that there exist a', b', c', and d' with $0 < d' < a' < b' \leq c'$ such that*

(i)$\{y \in P(S, a', b') : S(y) > a'\} \neq \emptyset$ and $S(Ty) > a'$ for $y \in P(S, a', b')$,

(ii) $\| Ty \| < d'$ for $\| y \| \leq d'$,

(iii)$S(Ty) > a'$ for $y \in P(S, a', c')$ with $\| T(y) \| > b'$.

Then T has at least three fixed points y_1, y_2, y_3 in $\overline{P}_{c'}$ satisfying

$$\| y_1 \| < d', a' < S(y_2), \| y_3 \| > d', S(y_3) < a'. \qquad \square$$

For convenience, we let

$$C_m = \min_{t \in I} \int_{s \in I} G_m(t, s)\Delta s; \quad C_n = \min_{t \in I} \int_{s \in I} G_n(t, s)\Delta s.$$

Theorem 4.2 *Assume that there exist real numbers d_0, d_1, and c with $0 < d_0 < d_1 < \frac{d_1}{\gamma} < c$ such that*

$$f_1(t, y_1(t), y_2(t)) < \frac{d_0}{2\epsilon_m} \quad \textit{and} \quad f_2(t, y_1(t), y_2(t)) < \frac{d_0}{2\epsilon_n}, \tag{8}$$

for $t \in [a, \sigma^q(b)]$ and $(y_1, y_2) \in [0, d_0] \times [0, d_0]$,

$$f_1(t, y_1(t), y_2(t)) > \frac{d_1}{2C_m} \quad \textit{or} \quad f_2(t, y_1(t), y_2(t)) > \frac{d_1}{2C_n}, \tag{9}$$

for $t \in I$ and $(y_1, y_2) \in [d_1, \frac{d_1}{\gamma}] \times [d_1, \frac{d_1}{\gamma}]$,

$$f_1(t, y_1(t), y_2(t)) < \frac{c}{2\epsilon_m} \quad \textit{and} \quad f_2(t, y_1(t), y_2(t)) < \frac{c}{2\epsilon_n}, \tag{10}$$

for $t \in [a, \sigma^q(b)]$ and $(y_1, y_2) \in [0, c] \times [0, c]$.
Then the BVP (1)-(2) *has at least three positive solutions.*

Proof: We consider the Banach space $B = E \times E$ where $E = \{y | y \in C[a, \sigma^q(b)]\}$ with the norm

$$\| y \|_0 = \max_{t \in [a, \sigma^q(b)]} | y(t) | .$$

And for $(y_1, y_2) \in B$, we denote the norm by $\| (y_1, y_2) \| = \| y_1 \|_0 + \| y_2 \|_0$. Then define a cone P in B by

$$P = \{(y_1, y_2) \in B : y_1(t) \geq 0 \ \text{ and } \ y_2(t) \geq 0, \quad t \in [a, \sigma^q(b)]\}.$$

For $(y_1, y_2) \in P$, we define

$$S(y_1, y_2) = \min_{t \in I} \{y_1(t)\} + \min_{t \in I} \{y_2(t)\}.$$

We denote

$$T_m(y_1, y_2)(t) := \int_a^{\sigma(b)} G_m(t, s) f_1(s, y_1(s), y_2(s)) \Delta s,$$

$$T_n(y_1, y_2)(t) := \int_a^{\sigma(b)} G_n(t, s) f_2(s, y_1(s), y_2(s)) \Delta s,$$

for $t \in [a, \sigma^q(b)]$ and the operator $T(y_1, y_2)(t) := (T_m(y_1, y_2)(t), T_n(y_1, y_2)(t))$.

It is easy to check that S is a nonnegative continuous concave functional on P with $S(y_1, y_2)(t) \leq \| (y_1, y_2) \|$ for $(y_1, y_2) \in P$ and that $T : P \to P$ is completely continuous and fixed points of T are solutions of the BVP (1)-(2). First, we prove that if there exists a positive number r such that $f_1(t, y_1(t), y_2(t)) < \frac{r}{2\epsilon_m}$ and $f_2(t, y_1(t), y_2(t)) < \frac{r}{2\epsilon_n}$ for $(y_1, y_2) \in [0, r] \times [0, r]$, then $T : \overline{P}_r \to P_r$. Indeed, if $(y_1, y_2) \in \overline{P}_r$, then for $t \in [a, \sigma^q(b)]$.

$$\begin{aligned} \| T(y_1, y_2) \| &= \max_{t \in [a, \sigma^q(b)]} \left| \int_a^{\sigma(b)} G_m(t, s) f_1(s, y_1(s), y_2(s)) \Delta s \right| \\ &\quad + \max_{t \in [a, \sigma^q(b)]} \left| \int_a^{\sigma(b)} G_n(t, s) f_2(s, y_1(s), y_2(s)) \Delta s \right| \\ &< \frac{r}{2\epsilon_m} \int_a^{\sigma(b)} G_m(t, s) \Delta s + \frac{r}{2\epsilon_n} \int_a^{\sigma(b)} G_n(t, s) \Delta s = r. \end{aligned}$$

Thus, $\| T(y_1, y_2) \| < r$, that is, $T(y_1, y_2) \in P_r$. Hence, we have shown that if (8) and (10) hold, then T maps $\overline{P}_{d_0}$ into P_{d_0} and $\overline{P}_c$ into P_c. Next, we show that $\{(y_1, y_2) \in P(S, d_1, \frac{d_1}{\gamma}) : S(y_1, y_2) > d_1\} \neq \emptyset$ and $S(T(y_1, y_2)) > d_1$ for all $(y_1, y_2) \in P(S, d_1, \frac{d_1}{\gamma})$. In fact, the constant function

$$\frac{d_1 + \frac{d_1}{\gamma}}{2} \in \left\{ (y_1, y_2) \in P(S, d_1, \frac{d_1}{\gamma}) : S(y_1, y_2) > d_1 \right\}.$$

Moreover, for $(y_1, y_2) \in P(S, d_1, \frac{d_1}{\gamma})$, we have

$$\frac{d_1}{\gamma} \geq \| (y_1, y_2) \| \geq y_1(t) + y_2(t) \geq \min_{t\in I} \{y_1(t)\} + \min_{t\in I} \{y_2(t)\} = S(y_1, y_2) \geq d_1,$$

for all $t \in I$. Thus, in view of (9) we see that

$$\begin{aligned}
&S(T(y_1, y_2)) \\
&= \min_{t\in I} \left\{ \int_a^{\sigma(b)} G_m(t, s) f_1(s, y_1(s), y_2(s))\Delta s \right\} \\
&\quad + \min_{t\in I} \left\{ \int_a^{\sigma(b)} G_n(t, s) f_2(s, y_1(s), y_2(s))\Delta s \right\} \\
&\geq \min_{t\in I} \left\{ \int_{s\in I} G_m(t, s) f_1(s, y_1(s), y_2(s))\Delta s \right\} \\
&\quad + \min_{t\in I} \left\{ \int_{s\in I} G_n(t, s) f_2(s, y_1(s), y_2(s))\Delta s \right\} \\
&> \frac{d_1}{2C_m} \min_{t\in I} \left\{ \int_{s\in I} G_m(t, s)\Delta s \right\} + \frac{d_1}{2C_n} \min_{t\in I} \left\{ \int_{s\in I} G_n(t, s)\Delta s \right\} = d_1,
\end{aligned}$$

as required. Finally, we show that if $(y_1, y_2) \in P(S, d_1, c)$ and $\| T(y_1, y_2) \| > \frac{d_1}{\gamma}$, then $S(T(y_1, y_2)) > d_1$. To see this, we suppose that $(y_1, y_2) \in P(S, d_1, c)$ and $\| T(y_1, y_2) \| > \frac{d_1}{\gamma}$, then, by Lemma 2.3, we have

$$\begin{aligned}
S(T(y_1, y_2)) &= \min_{t\in I} \left\{ \int_a^{\sigma(b)} G_m(t, s) f_1(s, y_1(s), y_2(s))\Delta s \right\} \\
&\quad + \min_{t\in I} \left\{ \int_a^{\sigma(b)} G_n(t, s) f_2(s, y_1(s), y_2(s))\Delta s \right\} \\
&\geq \gamma \int_a^{\sigma(b)} G_m(\sigma(s), s) f_1(s, y_1(s), y_2(s))\Delta s \\
&\quad + \gamma \int_a^{\sigma(b)} G_n(\sigma(s), s) f_2(s, y_1(s), y_2(s))\Delta s \\
&\geq \gamma \max_{t\in[a,\sigma^q(b)]} \left\{ \int_a^{\sigma(b)} G_m(t, s) f_1(s, y_1(s), y_2(s))\Delta s \right\} \\
&\quad + \gamma \max_{t\in[a,\sigma^q(b)]} \left\{ \int_a^{\sigma(b)} G_m(t, s) f_1(s, y_1(s), y_2(s))\Delta s \right\},
\end{aligned}$$

for all $t \in [a, \sigma^q(b)]$. Thus

$$
\begin{aligned}
S(T(y_1, y_2)) \geq \gamma \max_{t \in [a, \sigma^q(b)]} &\left\{ \int_a^{\sigma(b)} G_m(t,s) f_1(s, y_1(s), y_2(s)) \Delta s \right\} \\
&+ \gamma \max_{t \in [a, \sigma^q(b)]} \left\{ \int_a^{\sigma(b)} G_m(t,s) f_1(s, y_1(s), y_2(s)) \Delta s \right\} \\
&= \gamma \parallel T(y_1, y_2) \parallel > \gamma \frac{d_1}{\gamma} = d_1.
\end{aligned}
$$

To sum up the above, all the hypotheses of Theorem 4.2 are satisfied. Hence T has at least three fixed points, that is, the BVP (1)-(2) has at least three positive solutions (y_1, y_2), (u_1, u_2), and (w_1, w_2) such that

$$
\parallel (y_1, y_2) \parallel < d_0, \; d_1 < \min_{t \in I}(u_1, u_2), \quad \parallel (w_1, w_2) \parallel > d_0, \; \min_{t \in I}(w_1, w_2) < d_1.
$$

□

Now, we establish the existence of at least $2k - 1$ positive solutions for the BVP (1)-(2), by using induction on k.

Theorem 4.3 *Let k be an arbitrary positive integer. Assume that there exist numbers $a_i (1 \leq i \leq k)$ and $b_j (1 \leq j \leq k-1)$ with $0 < a_1 < b_1 < \frac{b_1}{\gamma} < a_2 < b_2 < \frac{b_2}{\gamma} < \ldots < a_{k-1} < b_{k-1} < \frac{b_{k-1}}{\gamma} < a_k$ such that*

$$
f_1(t, y_1(t), y_2(t)) < \frac{a_i}{2\epsilon_m} \quad \textit{and} \quad f_2(t, y_1(t), y_2(t)) < \frac{a_i}{2\epsilon_n}, \tag{11}
$$

for $t \in [a, \sigma^q(b)]$ and $(y_1, y_2) \in [0, a_i] \times [0, a_i], 1 \leq i \leq k$

$$
f_1(t, y_1(t), y_2(t)) > \frac{b_j}{2C_m} \quad \textit{or} \quad f_2(t, y_1(t), y_2(t)) > \frac{b_j}{2C_n} \tag{12}
$$

for $t \in I$ and $(y_1, y_2) \in [b_j, \frac{b_j}{\gamma}] \times [b_j, \frac{b_j}{\gamma}], 1 \leq j \leq k-1$.
Then the BVP (1)-(2) has at least $2k - 1$ positive solutions in $\overline{P}_{a_k}$.

Proof: We use induction on k. First, for $k = 1$, we know from (11) that $T : \overline{P}_{a_1} \to P_{a_1}$, then, it follows from Schauder fixed point theorem that the BVP (1)-(2) has at least one positive solution in $\overline{P}_{a_1}$. Next, we assume that this conclusion holds for $k = r$. In order to prove that this conclusion holds for $k = r + 1$, we suppose that there exist numbers $a_i (1 \leq i \leq r + 1)$ and

$b_j (1 \le j \le r)$ with $0 < a_1 < b_1 < \frac{b_1}{\gamma} < a_2 < b_2 < \frac{b_2}{\gamma} < ... < a_r < b_r < \frac{b_r}{\gamma} < a_{r+1}$ such that

$$f_1(t, y_1(t), y_2(t)) < \frac{a_i}{2\epsilon_m} \text{ and } f_2(t, y_1(t), y_2(t)) < \frac{a_i}{2\epsilon_n}, \tag{13}$$

for $t \in [a, \sigma^q(b)]$ and $(y_1, y_2) \in [0, a_i] \times [0, a_i], 1 \le i \le r+1$

$$f_1(t, y_1(t), y_2(t)) > \frac{b_j}{2C_m} \text{ or } f_2(t, y_1(t), y_2(t)) > \frac{b_j}{2C_n} \tag{14}$$

for $t \in I$ and $(y_1, y_2) \in [b_j, \frac{b_j}{\gamma}] \times [b_j, \frac{b_j}{\gamma}], 1 \le j \le r$. By assumption, the BVP (1)-(2) has at least $2r-1$ positive solutions $(u_i, u_i')(i = 1, 2, ..., 2r-1)$ in $\overline{P}_{a_r}$. At the same time, it follows from Theorem 4.2, (13) and (14) that the BVP (1)-(2) has at least three positive solutions $(u_1, u_1'), (v_1, v_2)$ and (w_1, w_2) in $\overline{P}_{a_{r+1}}$ such that, $\| (u_1, u_1') \| < a_r, b_r < \min_{t \in I}(v_1(t), v_2(t)), \| (w_1, w_2) \| > a_r, \min_{t \in I}(w_1(t), w_2(t)) < b_r$. Obviously, (v_1, v_2) and (w_1, w_2) are different from $(u_i, u_i')(i = 1, 2, ..., 2r-1)$. Therefore, the BVP(1)-(2) has at least $2r+1$ positive solutions in $\overline{P}_{a_{r+1}}$ which shows that this conclusion also holds for $k = r+1$. □

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