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Existence Results for Abstract Partial Neutral Integro-differential Equation with Unbounded Delay

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Abstract

In this paper we study the existence and regularity of mild solutions for a class of abstract partial neutral integro-differential equations with unbounded delay.

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1 Introduction

In this paper we study the existence and regularity of mild solutions for a class of abstract partial neutral integro-differential equations in the form

$$\frac{d}{dt}\left(x(t)+f(t,x_t)\right) = Ax(t)+\int_0^t B(t-s)x(s)ds+g(t,x_t), \quad t\in I=[0,a], \tag{1.1}$$

$$x_0 = \varphi\in\mathcal{B}, \tag{1.2}$$

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where $A : D(A) \subset X \to X$ and $B(t) : D(B(t)) \subset X \to X$, $t \geq 0$, are closed linear operators; $(X, \| \cdot \|)$ is a Banach space; the history $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$, belongs to some abstract phase space $\mathcal{B}$ defined axiomatically and $f, g : I \times \mathcal{B} \to X$ are appropriated functions.

Abstract partial neutral integro-differential equations with unbounded delay arises, for instance, in the theory development in Gurtin & Pipkin [13] and Nunziato [28] for the description of heat conduction in materials with fading memory. In the classic theory of heat conduction, it is assumed that the internal energy and the heat flux depends linearly on the temperature $u(\cdot)$ and on its gradient $\nabla u(\cdot)$. Under these conditions, the classic heat equation describes sufficiently well the evolution of the temperature in different types of materials. However, this description is not satisfactory in materials with fading memory. In the theory developed in [13, 28], the internal energy and the heat flux are described as functionals of u and u_x. The next system, see for instance [2, 4, 5, 24], has been frequently used to describe this phenomena,

$$\begin{aligned} \frac{d}{dt}\left[c_0 u(t,x) + \int_{-\infty}^{t} k_1(t-s)u(s,x)ds\right] &= c_1 \triangle u(t,x) + \int_{-\infty}^{t} k_2(t-s)\triangle u(s,x)ds, \\ u(t,x) &= 0, \qquad x \in \partial\Omega. \end{aligned}$$

In this system, $\Omega \subset \mathbb{R}^n$ is open, bounded and with smooth boundary; $(t,x) \in [0,\infty) \times \Omega$; $u(t,x)$ represents the temperature in x at the time t; c_1, c_2 are physical constants and $k_i : \mathbb{R} \to \mathbb{R}$, $i = 1, 2$, are the internal energy and the heat flux relaxation respectively. By assuming that the solution $u(\cdot)$ is known on $(-\infty, 0]$, we can transform this system into an abstract partial neutral integro-differential system with unbounded delay.

The literature relative to ordinary neutral differential equations is very extensive and we refer the reader to [14] related this matter. Concerning partial neutral functional differential equations ($B \equiv 0$), we cite the pioneer Hale paper [15] and Wu & Xia [29, 31] for equations with finite delay and Hernández et al. [16, 17, 18, 19] for systems with unbounded delay. To the best of our knowledge, the study of the existence and qualitative properties of solutions of neutral integro-differential equations with unbounded delay described in the abstract form (1.1) are untreated topics in the literature and this, is the main motivation of this article.

2 Preliminaries

Throughout this paper, $(X, \| \cdot \|)$ is a Banach space and $A : D(A) \subset X \to X$, $B(t) \subset D(B(t)) \subset X \to X$, $t \geq 0$, are closed linear operators defined on a common domain $D(A)$

which is dense in X. To obtain our results, we assume that the abstract Cauchy problem

$$\left.\begin{aligned} x'(t) &= Ax(t) + \int_0^t B(t-s)x(s)ds, \quad t \geq 0, \\ x(0) &= z \in X, \end{aligned}\right\} \tag{2.1}$$

has an associated resolvent operator of bounded linear operators $(R(t))_{t\geq 0}$ on X.

Definition 2.1 *A family of bounded linear operators $(R(t))_{t\geq 0}$ from X into X is a resolvent operator family for system (2.1) if the following conditions are verified.*

(i) $R(0) = I_d$ *and* $R(\cdot)x \in C([0,\infty) : X)$ *for every* $x \in X$*;*

(ii) $AR(\cdot)x \in C([0,\infty) : X)$ *and* $R(\cdot)x \in C^1([0,\infty) : X)$ *for every* $x \in D(A)$*;*

(iii) *For every* $x \in D(A)$ *and* $t \geq 0$*,*

$$R'(t)x = AR(t)x + \int_0^t B(t-s)R(s)xds, \tag{2.2}$$

$$R'(t)x = R(t)Ax + \int_0^t R(t-s)B(s)xds. \tag{2.3}$$

Let $h \in C([0,b]; X)$ and consider the integro-differential abstract Cauchy problem

$$\left\{\begin{aligned} x'(t) &= Ax(t) + \int_0^t B(t-s)x(s)ds + h(t), \quad t \in [0,b], \\ x(0) &= z \in X. \end{aligned}\right. \tag{2.4}$$

Motivated by Grimmer [11], we adopt the following concepts of mild, classical and strong solution for the non-homogeneous system (2.4).

Definition 2.2 *A function* $u \in C([0,b], X)$ *is called a mild solution of (2.4) on* $[0,b]$*, if* $u(0) = z$ *and*

$$u(t) = R(t)z + \int_0^t R(t-s)h(s)ds, \ t \in [0,b]. \tag{2.5}$$

Definition 2.3 *A function* $u \in C([0,b], X)$ *is called a classical solution of (2.4) on* $[0,b]$*, if* $u(0) = z$*,* $u(\cdot) \in C^1([0,b], X) \bigcap C([0,b] : [D(A)])$ *and (2.4) is satisfied.*

Definition 2.4 *A function* $u \in C([0,b], X)$ *is called a strong solution of (2.4), if* $u(0) = z$*;* $u(\cdot)$ *is differentiable a.e. on* $[0,b]$*;* $u' \in L^1(0,b; X)$*;* $u(t) \in D(A)$ *a.e.* $t \in [0,b]$*, and (2.4) is satisfied a.e. on* $[0,b]$*.*

Proceeding as in the proof [11, Corollary 2] we can prove the following Lemma.

Lemma 2.1 *The function* $u(t) = \int_0^t R(t-s)h(s)ds$ *is a strong solutions of abstract Cauchy problem (2.4) with* $z = 0$*, if and only if,* $u(\cdot)$ *is differentiable a.e. on* $[0,b]$ *and* $u' \in L^1(0,b;X)$*.*

For additional details on the existence and qualitative properties of resolvent of operators for integro-differential systems, we refer the reader to [6, 7, 11, 25] and the references therein.

In this work, we employ an axiomatic definition of the phase space $\mathcal{B}$ which is similar to that introduced in [23]. Specifically, $\mathcal{B}$ will be a linear space of functions mapping $(-\infty, 0]$ into X endowed with a seminorm $\| \cdot \|_{\mathcal{B}}$ and verifying the following axioms.

(**A**) If $x : (-\infty, \sigma + a) \to X$, $a > 0, \sigma \in \mathbb{R}$, is continuous on $[\sigma, \sigma + a)$ and $x_\sigma \in \mathcal{B}$, then for every $t \in [\sigma, \sigma + a)$ the following conditions hold:

(i) x_t is in $\mathcal{B}$.

(ii) $\| x(t) \| \leq H \| x_t \|_{\mathcal{B}}$.

(iii) $\| x_t \|_{\mathcal{B}} \leq K(t-\sigma) \sup\{\| x(s) \| : \sigma \leq s \leq t\} + M(t-\sigma) \| x_\sigma \|_{\mathcal{B}}$,

where $H > 0$ is a constant; $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(\cdot)$.

(**A1**) For the function $x(\cdot)$ in (**A**), the function $t \to x_t$ is continuous from $[\sigma, \sigma + a)$ into $\mathcal{B}$.

(**B**) The space $\mathcal{B}$ is complete.

To study the regularity of solutions, we consider the next axiom introduced in [22].

(C3) Let $x : (-\infty, \sigma + b] \to X$ be a continuous function such that $x_\sigma = 0$ and the right derivative $x'(\sigma^+)$ exists. If the function $\psi : (-\infty, 0] \to X$ defined by $\psi(\theta) = 0$ for $\theta < 0$ and $\psi(0) = x'(\sigma^+)$ belongs to $\mathcal{B}$, then $\| \frac{1}{h} x_{\sigma+h} - \psi \|_{\mathcal{B}} \to 0$ as $h \downarrow 0$.

Example 2.1 The phase space $\mathbf{C_r} \times \mathbf{L^p}(\rho, \mathbf{X})$

Let $r \geq 0$, $1 \leq p < \infty$ and let $\rho : (-\infty, -r] \to \mathbb{R}$ be a nonnegative measurable function which satisfies the conditions (g-5), (g-6) in the terminology of [23]. Briefly, this means that ρ is locally integrable and there exists a non-negative, locally bounded function γ on $(-\infty, 0]$ such that $\rho(\xi + \theta) \leq \gamma(\xi)\rho(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(\rho, X)$ consists of all classes of functions $\varphi : (-\infty, 0] \to X$ such that φ is continuous on $[-r, 0]$, Lebesgue-measurable, and $\rho \parallel \varphi \parallel^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(\rho, X)$ is defined by

$$\parallel \varphi \parallel_{\mathcal{B}} := \sup\{\parallel \varphi(\theta) \parallel : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} \rho(\theta) \parallel \varphi(\theta) \parallel^p d\theta \right)^{1/p}.$$

The space $\mathcal{B} = C_r \times L^p(\rho; X)$ satisfies axioms **(A), (A-1), (B).** Moreover, when $r = 0$ and $p = 2$, we can take $H = 1$, $M(t) = \gamma(-t)^{1/2}$ and $K(t) = 1 + \left(\int_{-t}^{0} \rho(\theta)\, d\theta \right)^{1/2}$, for $t \geq 0$, see [23, Theorem 1.3.8] for details.

Let $(Z, \| \cdot \|_Z)$ and $(W, \| \cdot \|_W)$ be Banach spaces. In this paper, the notation $\mathcal{L}(Z, W)$ stands for the Banach space of bounded linear operators from Z into W endowed with the uniform operator topology and we abbreviate this notation to $\mathcal{L}(Z)$ when $Z = W$. The notation, $B_r(x, Z)$ stands for the closed ball with center at x and radius $r > 0$ in Z. Additionally, for a bounded function $\gamma : [0, a] \to Z$ and $t \in [0, a]$ we use the notation

$$\parallel \gamma \parallel_{Z,t} = \sup\{\parallel \gamma(s) \parallel_Z : s \in [0, t]\}, \tag{2.6}$$

and we simplify this notation to γ_t when no confusion about the space Z arises.

3 Existence and regularity of mild solutions

In this section we study the existence and regularity of mild solutions of the abstract integro-differential system (1.1)-(1.2). Motivated by (2.5), we consider the following concept of mild solution.

Definition 3.1 *A function $u : (-\infty, b] \to X$, $0 < b \leq a$, is called a mild solution of (1.1)-(1.2) on $[0, b]$, if $u_0 = \varphi$; $u|_{[0,b]} \in C([0, b] : X)$; the functions $\tau \to AR(t - \tau)f(\tau, u_\tau)$ and $\tau \to \int_0^\tau B(\tau - \xi)R(t - \tau)f(\xi, u_\xi)$ are integrable on $[0, t)$ for every $t \in (0, b]$ and*

$$\begin{aligned} u(t) &= R(t)(\varphi(0) + f(0, \varphi)) - f(t, u_t) - \int_0^t AR(t - s)f(s, u_s)ds \\ &\quad - \int_0^t \int_0^s B(s - \xi)R(t - s)f(\xi, u_\xi)d\xi ds + \int_0^t R(t - s)g(s, u_s)ds, \;\; t \in [0, b]. \end{aligned}$$

To establish our results we introduce the following assumption. In the sequel, $[D(A)]$ is the domain of A endowed with the graph norm.

($\mathbf{H_1}$) There exists a Banach space $(Y, \| \cdot \|_Y)$ continuously included in X such that the following conditions are verified.

(a) For every $t \in (0,a]$, $R(t) \in \mathcal{L}(X,Y) \cap \mathcal{L}(Y,[D(A)])$ and $B(t) \in \mathcal{L}(Y,X)$. In addition, $AR(\cdot)x \in C((0,a],X)$ and $B(\cdot)x \in C(I,X)$ for every $x \in Y$.

(b) There are functions $\mathcal{H}_1(\cdot), \mathcal{H}_2(\cdot)$, $\mu(\cdot)$ in $L^1(I;\mathbb{R}^+)$ such that

$$\begin{aligned} \| AR(t) \|_{\mathcal{L}(Y,X)} &\leq \mathcal{H}_1(t), & t \in (0,a], \\ \| B(s)R(t) \|_{\mathcal{L}(Y,X)} &\leq \mu(s)\mathcal{H}_2(t), & 0 \leq s < t \leq a. \end{aligned}$$

($\mathbf{H_2}$) The function $g : I \times \mathcal{B} \to X$ verifies the following conditions.

(i) The function $g(t,\cdot) : \mathcal{B} \to X$ is continuous for every $t \in I$.

(ii) For every $x \in X$, the function $g(\cdot,x) : I \to X$ is strongly measurable.

(iii) There are a function $m_g \in C(I;[0,\infty))$ and a continuous non-decreasing function $\Omega_g : [0,\infty) \to (0,\infty)$ such that $\| g(t,\psi) \| \leq m_g(t)\Omega_g(\| \psi \|_{\mathcal{B}})$ for every $(t,\psi) \in I \times \mathcal{B}$.

($\mathbf{H_3}$) The function $f : I \times \mathcal{B} \to X$ is completely continuous, $f(I \times \mathcal{B}) \subset Y$ and there are positive constants c_1, c_2 such that $\| f(t,\psi) \|_Y \leq c_1 \| \psi \|_{\mathcal{B}} + c_2$ for every $(t,\psi) \in I \times \mathcal{B}$.

($\mathbf{H_4}$) The function $f : \mathbb{R} \times \mathcal{B} \to X$ is continuous, $f(I \times \mathcal{B}) \subset Y$ and there exists a continuous function $L_f : [0,\infty) \to [0,\infty)$, such that

$$\| f(t,\psi_1) - f(t,\psi_2) \|_Y \leq L_f(r) \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \quad (t,\psi_j) \in I \times B_r(\varphi,\mathcal{B}).$$

($\mathbf{H_5}$) The function $g : \mathbb{R} \times \mathcal{B} \to X$ is continuous and there exists a continuous function $L_g : [0,\infty) \to [0,\infty)$ such that

$$\| g(t,\psi_1) - g(t,\psi_2) \| \leq L_g(r) \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \quad (t,\psi_j) \in I \times B_r(\varphi,\mathcal{B}).$$

($\mathbf{H_6}$) The functions $f,g : \mathbb{R} \times \mathcal{B} \to$ are continuous, $f(I \times \mathcal{B}) \subset Y$ and there are continuous functions $L_f, L_g : I \to [0,\infty)$ such that $L_f(0) \| i_c \|_{\mathcal{L}(Y,X)} K(0) < 1$ (i_c is the inclusion from Y into X) and

$$\begin{aligned} \| f(t,\psi_1) - f(s,\psi_2) \|_Y &\leq L_f(r)(| t-s | + \| \psi_1 - \psi_2 \|_{\mathcal{B}}), \\ \| g(t,\psi_1) - g(s,\psi_2) \| &\leq L_g(r)(| t-s | + \| \psi_1 - \psi_2 \|_{\mathcal{B}}), \end{aligned}$$

for every $t \in I$ and all $\psi_j \in B_r(\varphi,\mathcal{B})$.

($\mathbf{H}_\varphi$) Let $0 < b \leq a$ and $S(b) = \{x : (-\infty,b] \to X : x_0 = 0, x|_{[0,b]} \in C([0,b] : X)\}$ endowed with the uniform convergence topology on $[0,b]$. For every bounded set $Q \subset S(b)$, the set of functions $\{t \to f(t,x_t + y_t) : x \in Q\}$ is equicontinuous on $[0,b]$.

Remark 3.1 We note that the condition $(\mathbf{H_1})$ is linked to the integrability of the functions $s \to AR(t-s)$, $\tau \to \int_0^\tau B(\tau-\xi)R(t-\tau)d\xi$, since except trivial cases, these functions are not integrable. Consider, for instance, the case in which A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X and $B(t) = 0$ for every $t \geq 0$. In this case $R(t) = T(t)$ for $t \in I$, and by assuming that $AR(\cdot) \in L^1(0,t;\mathcal{L}(X))$, we have that $T(t)x - x = A\int_0^t T(s)ds = \int_0^t AT(s)ds$, which implies that the semigroup is continuous at zero in the uniform operator topology and, as consequence, that A is a bounded linear operator on X.

On the other hand, if condition $\mathbf{H_1}$ is verified and $x \in C(I,Y)$, then from the Bochner criterion for integrable functions and the estimates

$$\begin{aligned} \| AR(t-s)x(s) \| &\leq \mathcal{H}_1(t-s) \| x(s) \|_Y, \\ \| B(s-\xi)R(t-s)x(\xi) \| &\leq \mu(s-\xi)\mathcal{H}_2(t-s) \| x(\xi) \|_Y, \end{aligned}$$

we infer that the functions $s \to AR(t-s)x(s)$ and $s \to \int_0^s B(s-\xi)R(t-\xi)x(\xi)d\xi$ are integrable on $[0,t]$. For similar remarks concerning condition $(\mathbf{H_1})$, see [19, 20]

Remark 3.2 In the rest of this paper, $y : (-\infty, a] \to X$ is the function defined by $y(t) = \varphi(t)$ for $t \leq 0$ and $y(t) = R(t)\varphi(0)$ for $t \geq 0$. Additionally, $M = \sup_{s\in[0,b]} \| R(s) \|_{\mathcal{L}(X)}$, $K_b = \sup_{s\in[0,b]} K(s)$, $M_b = \sup_{s\in[0,b]} M(s)$, $L_{f,b} = \sup_{s\in[0,b]} L_f(s)$ and $L_{g,b} = \sup_{s\in[0,b]} L_g(s)$.

Now, we can establish our first existence result.

Theorem 3.1 *Let conditions* $(\mathbf{H_1})$-$(\mathbf{H_3})$ *and* $(\mathbf{H}_\varphi)$ *be hold. Assume* $c_1K(0) \| i_c \|_{\mathcal{L}(Y,X)} < 1$, $R(\cdot) \in C((0,a];\mathcal{L}(X;Y)) \bigcap C((0,a];\mathcal{L}(Y))$, $B(\cdot) \in L^1([0,a];\mathcal{L}(Y,X))$ *and that the following condition is verified.*

- **(a)** *For all* $t \in (0,a]$ *and* $r > 0$, *there exit a compact set* W_r^1 *in* Y *and a compact set* W_r^2 *in* X *such that* $R(t)f(s,\psi) \in W_r^1(t)$ *and* $R(t)g(s,\psi) \in W_r^2(t)$ *for every* $(s,\psi) \in [0,t] \times B_r(0,\mathcal{B})$.

Then there exists a mild solution of (1.1)-(1.2) on $[0,b]$ *for some* $0 < b \leq a$.

Proof: Let $0 < b \leq a$ be such that

$$\frac{K_a M}{(1-\mu)} \int_0^b m_g(s) < \int_C^\infty \Omega_g(s)ds, \tag{3.1}$$

$$\mu = c_1 K_b \left(\| i_c \|_{\mathcal{L}(Y,X)} + \int_0^b \mathcal{H}_1(s)ds + \int_0^b \mu(\xi)d\xi \int_0^b \mathcal{H}_2(s)ds \right) < 1,$$

where $C = \dfrac{1}{1-\mu}((M_a + K_a M H) \parallel \varphi \parallel_{\mathcal{B}} + K_a M \parallel f(0,\varphi) \parallel + \dfrac{c_2 \mu}{c_1})$. On the space of functions $S(b) = \{x : (-\infty, b] \to X : x_0 = 0, x|_{[0,b]} \in C([0,b] : X)\}$ endowed with the uniform convergence topology, we define the operator $\Gamma : S(b) \to S(b)$ by $(\Gamma x)_0 = 0$ and

$$
\begin{aligned}
\Gamma x(t) &= R(t) f(0,\varphi) - f(t, x_t + y_t) - \int_0^t AR(t-s) f(s, x_s + y_s) ds \\
&\quad - \int_0^t \int_0^s B(s-\xi) R(t-s) f(\xi, x_\xi + y_\xi) d\xi ds \\
&\quad + \int_0^t R(t-s) g(s, x_s + y_s) ds,
\end{aligned}
$$

for $t \in [0,b]$. We note that from Remark 3.1 and condition $\mathbf{H_3}$ it follows that $\Gamma S(b) \subset S(b)$.

Next, we prove that Γ verifies the conditions of the Leray Schauder Alternative Theorem (see [9, Theorem 6.5.4]). Let $(x^n)_{n\in\mathbb{N}}$ be a sequence in $S(b)$ and $x \in S(b)$ such that $x^n \to x$ in $S(b)$. From the phase space axioms, we have that the set $U = \{x_s^n + y_s, x_s + y_s : s \in [0,b], n \in \mathbb{N}\}$ is relatively compact in $\mathcal{B}$ and that $x_s^n \to x_s$ uniformly for $s \in [0,b]$ as $n \to \infty$, which permit to infer that $f(s, x_s^n + y_s) \to f(s, x_s + y_s)$ uniformly on $[0,b]$ as $n \to \infty$. Now, a standard application of the Lebesgue dominated convergence Theorem permits to conclude that Γ is continuous.

Now, we establish *a priori* estimates for the solutions of the integral equation $z = \lambda \Gamma z$, $\lambda \in (0,1)$. Let x^λ be a solution of $z = \lambda \Gamma z$, $\lambda \in (0,1)$, and $\alpha^\lambda(\cdot)$ be the function defined by $\alpha^\lambda(s) = (M_b + K_b M H) \parallel \varphi \parallel_{\mathcal{B}} + K_b \sup_{\theta\in[0,s]} \parallel x^\lambda(\theta) \parallel$. By using that $\parallel x_s^\lambda \parallel_{\mathcal{B}} + \parallel y_s \parallel_{\mathcal{B}} \leq \alpha^\lambda(s)$, we find that

$$
\begin{aligned}
&\parallel x^\lambda(t) \parallel \\
&\leq M \parallel f(0,\varphi) \parallel + \parallel i_c \parallel_{\mathcal{L}(Y,X)} \left(c_1 \alpha^\lambda(t) + c_2\right) + \int_0^t \mathcal{H}_1(t-s) \left(c_1 \alpha^\lambda(s) + c_2\right) ds \\
&\quad + \int_0^t \int_0^s \mu(s-\xi) \mathcal{H}_2(t-s) \left(c_1 \alpha^\lambda(\xi) + c_2\right) d\xi ds + M \int_0^t m_g(s) \Omega_g(\alpha^\lambda(s)) ds \\
&\leq M \parallel f(0,\varphi) \parallel + c_2 \left(\parallel i_c \parallel_{\mathcal{L}(Y,X)} + \int_0^b \mathcal{H}_1(s) ds + \int_0^b \mathcal{H}_2(s) ds \int_0^b \mu(\xi) d\xi \right) \\
&\quad + \alpha^\lambda(t) c_1 \left(\parallel i_c \parallel_{\mathcal{L}(Y,X)} + \int_0^b \mathcal{H}_1(s) ds + \int_0^b \mathcal{H}_2(s) ds \int_0^b \mu(\xi) d\xi \right) \\
&\quad + M \int_0^t m_g(s) \Omega_g(\alpha^\lambda(s)) ds,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\alpha^{\lambda}(t) \\
&\leq \;\; (M_b + K_bMH) \parallel \varphi \parallel_{\mathcal{B}} + K_bM \parallel f(0,\varphi) \parallel \\
&\quad +(\alpha^{\lambda}(t)c_1K_b + c_2K_b)\left(\parallel i_c \parallel_{\mathcal{L}(Y,X)} + \int_0^b \mathcal{H}_1(s)ds + \int_0^b \mathcal{H}_2(s)ds \int_0^b \mu(\xi)d\xi \right) \\
&\quad +K_bM \int_0^t m_g(s)\Omega_g(\alpha^{\lambda}(s))ds.
\end{aligned}
$$

Consequently,

$$
\alpha^{\lambda}(t) \;\; \leq \;\; C + \frac{K_aM}{(1-\mu)} \int_0^t m_g(s)\Omega_g(\alpha^{\lambda}(s))ds, \quad t \in [0,b].
$$

Denoting by $\beta_{\lambda}(t)$ the right hand side of (3.2), we obtain $\beta_{\lambda}'(t) \leq \frac{K_aM}{(1-\mu)} m_g(t)\Omega_g(\beta_{\lambda}(t))$ and

$$
\int_{\beta_{\lambda}(t)=C}^{\beta_{\lambda}(t)} \frac{1}{\Omega_g(s)}ds \;\; \leq \;\; \frac{K_aM}{(1-\mu)} \int_0^b m_g(s) < \int_C^{\infty} \frac{1}{\Omega_g(s)}ds.
$$

This inequality and (3.1) proves that the set of functions $\{\beta^{\lambda} : \lambda \in (0,1)\}$ is uniformly bounded on $[0,b]$ and, as consequence, that $\{x^{\lambda} : \lambda \in (0,1)\}$ is bounded in $S(b)$.

To prove that Γ is completely continuous, we introduce the decomposition $\Gamma = \sum_{i=1}^{3} \Gamma_i$, where

$$
\begin{aligned}
\Gamma_1 x(t) &= R(t)f(0,\varphi) - f(t, x_t + y_t) - \int_0^t AR(t-s)f(s, x_s + y_s)ds, \qquad t \in I, \\
\Gamma_2 x(t) &= \int_0^t \int_0^s B(s-\xi)R(t-s)f(\xi, x_{\xi} + y_{\xi})d\xi ds, \qquad t \in I, \\
\Gamma_3 x(t) &= \int_0^t R(t-s)g(s, x_s + y_s)ds, \qquad t \in I,
\end{aligned}
$$

and $(\Gamma_i x)_0 = 0$ for $i = 1,2,3$. From [21, Lemma 3.1] we infer that Γ_1 and Γ_3 are completely continuous. Next, we prove that Γ_2 is also completely continuous. In the sequel, for $q > 0$ we use the notations $B_q = B_q(0, S(b))$ and $q^* = (M_a + K_aMH) \parallel \varphi \parallel_{\mathcal{B}} + K_a q$.

Step 1. For each $t \in [0,b]$, the set $(\Gamma_2 B_q)(t) = \{\Gamma_2 x(t) : x \in B_q\}$ is relatively compact in X.

The case $t = 0$ is trivial. Let $0 < \varepsilon < t < b$. From the assumptions, we can fix numbers $0 = t_0 < t_1 < \cdots < t_n = t - \epsilon$ such that $\parallel R(t-s) - R(t-s') \parallel_{\mathcal{L}(Y)} \leq \varepsilon$ for every $s, s' \in [t_i, t_{i+1}]$, $i = 0,1,2,\cdots n-1$. Let $x \in B_q$. From the mean value theorem for the

Bochner integral (see [26, Lemma 2.1.3]) we get

$$\begin{aligned}
\Gamma_2 x(t) &= \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\int_0^s B(s-\xi)R(t-t_i)f(\xi,x_\xi+y_\xi)d\xi ds \\
&\quad +\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\int_0^s B(s-\xi)(R(t-s)-R(t-t_i))f(\xi,x_\xi+y_\xi)d\xi ds \\
&\quad +\int_{t_n}^{t}\int_0^s B(s-\xi)R(t-s)f(\xi,x_\xi+y_\xi)d\xi ds \\
&\in \sum_{i=1}^{n}(t_i-t_{i-1})\overline{co(\{s\,co(\{B(\theta)x : x\in W^1_{q^*}(t-t_i),\theta\in[0,s]\}) :\ s\in[0,t]\})} \\
&\quad (c_1q^*+c_2)\left(\varepsilon a\int_0^a \parallel B(s)\parallel_{\mathcal{L}(Y,X)} ds+\int_{t-\varepsilon}^{t} H_2(s)ds\int_0^a \mu(s)ds\right)B_1(0,X) \\
&\subset \sum_{i=1}^{n}(t_i-t_{i-1})\overline{co(\{sB(\theta)x : x\in W^1_{q^*}(t-t_i), 0\le\theta,s\le t\})}+\mathcal{C}_\varepsilon=\mathcal{K}_\varepsilon+\mathcal{C}_\varepsilon,
\end{aligned}$$

which implies that $(\Gamma_2 B_q)(t)$ is relatively compact in X since $\mathcal{K}_\varepsilon$ is a compact and $diam(\mathcal{C}_\varepsilon)\to 0$ as $\varepsilon\to 0$.

Step 2. The set $\Gamma_2 B_q=\{\Gamma_2 x : x\in B_q\}$ is equicontinuous on $[0,b]$.

Let $0<\varepsilon<t<b$ and $0<\delta<\varepsilon$ such that $\parallel R(s)-R(s')\parallel_{\mathcal{L}(Y)}\le\varepsilon$ for every $s,s'\in[\varepsilon,b]$ with $\mid s-s'\mid\le\delta$. Under these conditions, for $x\in B_q$ and $0<h<\delta$ with $t+h\in[0,b]$, we get

$$\begin{aligned}
&\parallel \Gamma_2 x(t+h)-\Gamma_2 x(t)\parallel \\
&\quad\le \int_0^{t-\varepsilon}\int_0^s \parallel B(s-\xi)[R(t+h-s)-R(t-s)]f(\xi,x_\xi+y_\xi)\parallel d\xi ds \\
&\qquad +\int_{t-\varepsilon}^{t}\int_0^s \parallel B(s-\xi)[R(t+h-s)-R(t-s)]f(\xi,x_\xi+y_\xi)\parallel d\xi ds \\
&\qquad +\int_t^{t+h}\int_0^s \parallel B(s-\xi)R(t+h-s)f(\xi,x_\xi+y_\xi)\parallel d\xi ds \\
&\quad\le \varepsilon(c_1q^*+c_2)a\int_0^a \parallel B(s)\parallel_{\mathcal{L}(Y,X)} ds \\
&\qquad +(c_1q^*+c_2)\int_{t-\varepsilon}^{t}[\mathcal{H}_2(t+h-s)+\mathcal{H}_2(t-s)]ds\int_0^a \mu(s)ds \\
&\qquad +(c_1q^*+c_2))\int_t^{t+h}\mathcal{H}_2(t+h-s)ds\int_0^a \mu(s)ds,
\end{aligned}$$

which shows that $\Gamma_2 B_q$ is right equicontinuity at $t\in(0,a)$. A similar procedure allows us to prove the right equicontinuity at zero and the left equicontinuity at $t\in(0,b]$. Thus, $\Gamma_2 B_q$ is equicontinuous. This completes the proof that Γ_2 is completely continuous.

From the above remarks and [9, Theorem 6.5.4] we conclude that Γ has a fixed point $x \in S(b)$. Clearly, the function $u = y + x$ is a mild solution of (1.1)-(1.2) on $[0, b]$. $\blacksquare$

In the next result, we employ the notations introduced in Remark 3.2.

Theorem 3.2 *Let conditions* $(\mathbf{H_1}), (\mathbf{H_4})$ *and* $(\mathbf{H_5})$ *be hold. If* $L_f(0)K(0) \parallel i_c \parallel_{\mathcal{L}(Y,X)} < 1$, *then there exists a unique mild solution of (1.1)-(1.2) on* $[0, b]$ *for some* $0 < b \leq a$.

Proof: Let r, b_1, C_f, C_g be positive constants such that $\parallel f(t, \psi) \parallel_Y \leq C_f$, $\parallel g(t, \psi) \parallel \leq C_g$ for all $(t, \psi) \in [0, b_1] \times B_r(\varphi, \mathcal{B})$. Now, we choose $0 < b < b_1$ and $0 < \rho < b$ such that $\mu = L_{f,b} \parallel i_c \parallel_{\mathcal{L}(Y,X)} K_b < 1$, $\rho K_b + \sup\limits_{\theta \in [0,b]} \parallel y_\theta - \varphi \parallel_{\mathcal{B}} < r$ and

$$
\begin{aligned}
\parallel (R(\theta) - I)f(\theta, y_\theta) \parallel_b + M \parallel f(0, \varphi) - f(\theta, y_\theta) \parallel_b &\leq \frac{(1-\mu)\rho}{3}, \\
C_f \int_0^b \mathcal{H}_1(s)ds + C_f \int_0^b \mathcal{H}_2(\xi)d\xi \int_0^b \mu(s)ds + bMC_g &\leq \frac{(1-\mu)\rho}{3}, \\
K_b \left[L_{f,b} \int_0^b \mathcal{H}_1(s)ds + L_{f,b} \int_0^b \mathcal{H}_2(\xi)d\xi \int_0^b \mu(s)ds + bML_{g,b} \right] &< \frac{1-\mu}{2},
\end{aligned}
$$

where the notation introduced in (2.6) have been used.

Let Γ be the map introduced in the proof of Theorem 3.1. Next, we prove that Γ is a contraction from $B_\rho(0, S(b))$ into $B_\rho(0, S(b))$. For $x \in S(b)$, we have that

$$
\parallel x_t + y_t - \varphi \parallel_{\mathcal{B}} \leq \parallel x_t \parallel_{\mathcal{B}} + \parallel y_t - \varphi \parallel_{\mathcal{B}} \leq K_b \rho + \sup_{t \in [0,b]} \parallel y_t - \varphi \parallel_{\mathcal{B}} < r,
$$

and hence,

$$
\begin{aligned}
&\parallel \Gamma x(t) \parallel \\
&\leq \parallel R(t)(f(0, \varphi) - f(t, y_t)) \parallel + \parallel R(t)f(t, y_t) - f(t, y_t) \parallel \\
&\quad + \parallel f(t, y_t) - f(t, x_t + y_t) \parallel + \int_0^t \parallel AR(t-s)f(s, x_s + y_s) \parallel ds \\
&\quad + \int_0^t \int_0^s \parallel B(s-\xi)R(t-s)f(\xi, x_\xi + y_\xi) \parallel d\xi ds + MC_g b \\
&\leq M \parallel f(0, \varphi) - f(\theta, y_\theta) \parallel_b + \parallel (R(\theta) - I)f(\theta, y_\theta) \parallel_b + \parallel i_c \parallel_{\mathcal{L}(Y,X)} L_{f,b} \parallel x_t \parallel_{\mathcal{B}} \\
&\quad + C_f \int_0^b \mathcal{H}_1(s)ds + C_f \int_0^b \mathcal{H}_2(\tau)d\tau \int_0^b \mu(\xi)d\xi + MC_g b \\
&\leq (1-\mu)\rho + \parallel i_c \parallel_{\mathcal{L}(Y,X)} L_{f,b} K_b \parallel x \parallel_b
\end{aligned}
$$

which implies that $\Gamma x \in B_\rho(0, S(b))$. On the other hand, for $u, v \in S(b)$ and $t \in [0, b]$ we get

$$\begin{aligned}
&\| \Gamma u(t) - \Gamma v(t) \| \\
&\leq \ \| i_c \|_{\mathcal{L}(Y,X)} L_{f,b} K_b \| u - v \|_b + L_{f,b} K_b \int_0^b \mathcal{H}_1(t-s) ds \| u - v \|_b \\
&\quad + \left(L_{f,b} K_b \int_0^t \int_0^s \mu(s-\xi) \mathcal{H}_2(t-s) d\xi ds + M L_{g,b} K_b a \right) \| u - v \|_b \\
&\leq \ \left(\| i_c \|_{\mathcal{L}(Y,X)} L_{f,b} K_b + \frac{1-\mu}{2} \right) \| u - v \|_b,
\end{aligned}$$

which shows that Γ is a contraction on $B_\rho(0, S(b))$. This complete the proof. ■

By using [26, Theorem 4.3.2], we also obtain the existence of mild solution.

Theorem 3.3 *Let assumptions* $(\mathbf{H_1}), (\mathbf{H_4})$ *be hold. Suppose,* $L_f(0) \| i_c \|_{\mathcal{L}(Y,X)} K(0) < 1$, $R(\cdot) \in C((0, a], \mathcal{L}(X))$ *and that the function* $g(\cdot)$ *verifies the condition* $(\mathbf{a})$ *of Theorem 3.1. Then there exists a mild solution of (1.1)-(1.2) on* $[0, b]$ *for some* $0 < b \leq a$.

Proof: Let r, b, ρ and Γ as in the proof of Theorem 3.2 and consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where $(\Gamma_i x)_0 = 0$ and

$$\begin{aligned}
\Gamma_1 x(t) &= R(t) f(0, \varphi) - f(t, x_t + y_t) - \int_0^t AR(t-s) f(s, x_s + y_s) ds \\
&\quad + \int_0^t \int_0^s B(s-\xi) R(t-s) f(\xi, x_\xi + y_\xi) d\xi ds, \\
\Gamma_2 x(t) &= \int_0^t R(t-s) g(s, x_s + y_s) ds,
\end{aligned}$$

for $t \in [0, b]$. From the proof of Theorem 3.2, we infer that $\Gamma(B_\rho(0, S(b))) \subset B_\rho(0, S(b))$ and that Γ_1 is a contraction on $B_\rho(0, S(b))$. Moreover, from [21, Lemma 3.1] it follows that Γ_2 is completely continuous on $B_\rho(0, S(b))$ which implies that Γ is a condensing operator. Now, the assertion is a consequence of [26, Theorem 4.3.2]. The proof is ended. ■

In the rest of this section, we discuss the regularity of the mild solutions of (1.1)-(1.2). Motivated by the definitions 2.3 and 2.4, we introduce the following concepts of solution.

Definition 3.2 *A function* $u : (-\infty, b] \to X$ *is called a classical solution of (1.1)-(1.2) on* $[0, b]$ *if* $u_0 = \varphi$; $u \in C^1([0, b] : X)$; $u(t) \in D(A)$ *for every* $t \in [0, b]$; *the function* $s \to B(t-s)u(s)$ *is integrable on* $[0, t)$ *for every* $t \in [0, b)$ *and (1.1)-(1.2) is verified on* $[0, b]$.

Definition 3.3 *A function* $u : (-\infty, b] \to X$ *is called a strong solution of (1.1)-(1.2) on* $[0, b]$ *if* $u_0 = \varphi$; *the functions* $u(\cdot)$ *and* $t \to f(t, u_t)$ *are differentiable a.e. on* $[0, b]$; $u'(\cdot)$ *and* $\frac{d}{dt} f(t, u_t)$ *are functions in* $L^1(0, b; X)$; $u(t) \in D(A)$ *a.e.* $t \in [0, b]$ *and (1.1)-(1.2) are verified a.e. on* $[0, b]$.

We need to introduce some additional notations. Let $(Z, \|\cdot\|_Z)$ and $(W, \|\cdot\|_W)$ be Banach spaces. For a function $\gamma : I \to Z$ and $h \in \mathbb{R}$, we represent by $\partial_h \gamma$ the function defined by $\partial_h \gamma(t) = \frac{\gamma(t+h)-\gamma(t)}{h}$. Moreover, if $\zeta : I \times W \to Z$ is differentiable, we employ the decomposition

$$\begin{aligned} \zeta(s,w) - \zeta(t,z) &= D_1\zeta(t,z)(s-t) + D_2\zeta(t,z)(w-z) \\ &\quad + \|(s-t, w-z)\| \; R(\zeta, t, z, s-t, w-z), \end{aligned} \tag{3.2}$$

where $\|R(\zeta(t,z), \tilde{s}, \tilde{w})\|_Z \to 0$ as $\|(\tilde{s}, \tilde{z})\| = |\tilde{s}| + \|\tilde{w}\|_W \to 0$.

For $x \in X$, we will use the notation χ_x for the function $\chi_x : (-\infty, 0] \to X$ given for $\chi_x(\theta) = 0$ for $\theta < 0$ and $\chi_x(0) = x$. We also define the operator functions $S(t) : \mathcal{B} \to \mathcal{B}$, $t \geq 0$, by $[S(t)\varphi](\theta) = R(t+\theta)\varphi(0)$ for $-t \leq \theta \leq 0$ and $[S(t)\varphi](\theta) = \varphi(t+\theta)$ for $-\infty < \theta < -t$.

Remark 3.3 In the sequel, we always assume that the operators $A, R(t), B(s)$ commutes.

The proof of the next Lemma is a tedious routine based in the use of the phase space axioms. We choose to omit it.

Lemma 3.1 *Let conditions* $(\mathbf{H_1})$ *and* $(\mathbf{H_6})$ *be hold. Assume that the functions* $R(\cdot)\varphi(0)$, $S(\cdot)\varphi$ *and* $R(\cdot)f(0,\varphi)$ *are Lipschitz on* $[0,a]$, $\| i_c \|_{\mathcal{L}(Y,X)} K_b L_f(0) < 1$ *and the set*

$$\{AR(s)f(t,\psi), B(s)R(t)f(\xi,\psi) : 0 \leq t, s, \xi \leq b, \psi \in B_r(0,\mathcal{B})\}$$

is bounded in X *for every* $r > 0$. *If* $u(\cdot)$ *is a mild solution of (1.1)-(1.2) on* $[0,b]$, $0 \leq b \leq a$, *then the functions* $u(\cdot)$ *and* $t \to u_t$ *are Lipschitz on* $[0,b]$.

Theorem 3.4 *Assume the conditions of Lemma 3.1 be valid; X reflexive;* $\varphi(0) \in D(A)$ *and* $Y = [D(A)]$. *If* $u(\cdot)$ *is a mild solution of (1.1)-(1.2) on* $[0,b]$, *then* $u(\cdot)$ *is a strong solution on* $[0,b]$.

Proof: From Lemma 3.1, condition $(\mathbf{H_6})$ and the fact that the functions $B(\theta)$, $\theta \in I$, are uniformly bounded in $\mathcal{L}([D(A)], X))$, we infer that the functions $v_1(t) = -Af(t, u_t)$, $v_2(t) = g(t, u_t)$ and $v_3(t) = -\int_0^t B(t-s)f(s,u_s)ds$ are Lipschitz on $[0,b]$, which in turn implies that $z(t) = \int_0^t R(t-s)(v_1(s)+v_2(s)+v_3(s))ds$ is also Lipschitz on $[0,b]$. Since X is reflexive, the functions $z(\cdot)$ and $t \to f(t,u_t)$ are *a.e.* differentiable on $[0,b]$, $z' \in L^1(0,b;X)$ and $\frac{d}{dt} f(t,u_t) \in L^1(0,b;X)$. Under these conditions, we know from Lemma 2.1 that the mild solution $w(\cdot)$ of

$$\begin{cases} \dfrac{dx(t)}{dt} &= Ax(t) + \displaystyle\int_0^t B(t-s)x(s)ds + v_1(t) + v_2(t) + v_3(t), \quad t \in [0,b], \\ x(0) &= \varphi(0) + f(0,\varphi), \end{cases} \tag{3.3}$$

is a strong solution of (3.3) on $[0,b]$ and from the uniqueness of mild solutions of (3.3) it follows that $w(t) = u(t) + f(t,u_t)$ for every $t \in [0,b]$. These remarks permit to conclude that $u(\cdot)$ is a strong solution of (1.1)-(1.2) on $[0,b]$. The proof is complete. ■

Theorem 3.5 *Assume assumption* $(\mathbf{H_1})$ *be valid,* $Y = [D(A)]$ *and the following conditions.*

(a) *The function* $S(\cdot)\varphi$ *Lipschitz on* I, $\varphi(0) \in [D(A)]$ *and* $\frac{d^+}{dt}y_{t|_{t=0}}$ *exists.*

(b) $f \in C^1(I \times \mathcal{B} : Y)$, $g \in C^1(I \times \mathcal{B} : X)$ *and* $Df(0,\varphi) \equiv 0$.

(c) *The space* $\mathcal{B}$ *verifies axiom* **(C3)** *and* $\chi_{g(0,\varphi)} \in \mathcal{B}$ *or* $g(0,\varphi) \equiv 0$.

If $u(\cdot)$ *is a mild solution of (1.1)-(1.2) on* $[0,b]$, *then* $u(\cdot)$ *is a classical solution on* $[0,b_1]$ *for some* $0 < b_1 \le b$.

Proof: Using that $Df(0,\varphi) \equiv 0$, we can fix positive constants $0 < b_1 \le b$, L and r such that $\| u_s \|_{\mathcal{B}} \le r$ for all $s \in [0,b_1]$,

$$\mu = K_{b_1}[\Theta \| i_c \|_{\mathcal{L}(Y,X)} + \int_0^{b_1} \left(\mathcal{H}_1(s) + \int_0^{b_1} \mu(s)\mathcal{H}_2(\xi)ds \right) d\xi + Mb_1 \| D_2 g(s,u_s) \|_{b_1}] < 1,$$

$$\max\{\| f(t,\psi_1) - f(s,\psi_2) \|_Y, \| g(t,\psi_1) - g(s,\psi_2) \|\} \le L(| t-s | + \| \psi_1 - \psi_2 \|_{\mathcal{B}}),$$

for all $(t,\psi_i) \in [0,b_1] \times B_r(\varphi,\mathcal{B})$, where $\Theta = \max\{\| D_2 f(s,u_s) \|_{Y,b_1}, \| D_2 f(s,u_s) \|_{Y,b_1}\}$. We also select $\widetilde{L} > 0$ such that $\| u_s - u_{s'} \|_{\mathcal{B}} \le \widetilde{L} \mid s - s' \mid$ for all $s, s' \in [0,b]$ (see Lemma 3.1).

Let $z \in C([0,b_1] : X)$ be the solution of the integral equation

$$\begin{aligned} w(t) &= \Lambda(t) - D_2 f(t,u_t)w_t - \int_0^t AR(t-s)D_2 f(s,u_s)w_s ds \\ &\quad - \int_0^t \int_0^s B(s-\xi)R(t-s)D_2 f(\xi,u_\xi)w_\xi d\xi ds + \int_0^t R(t-s)D_2 g(s,u_s)w_s ds \end{aligned} \tag{3.4}$$

where $w_0 = \dfrac{d^+}{dt}y_{t|_{t=0}} + \chi_{g(0,\varphi)}$ and

$$\begin{aligned} \Lambda(t) &= R(t)A\varphi(0) + \int_0^t R(t-s)B(s)\varphi(0)ds - D_1 f(t,u_t) \\ &\quad + \int_0^t AR(t-s)D_1 f(s,u_s)ds + \int_0^t \int_0^s B(s-\xi)R(t-s)D_1 f(\xi,u_\xi)d\xi ds \\ &\quad + \int_0^t R(t-s)B(0)f(0,\varphi)ds + \int_0^t R(t-s)D_1 g(s,u_s)ds + R(t)g(0,\varphi). \end{aligned}$$

The existence and uniqueness of $z(\cdot)$ is a consequence of the contraction mapping principle, the condition $\mu < 1$ and the fact that $\frac{d^+}{dt}y_{t|_{t=0}}(0) = A\varphi(0)$. We omit details.

Next, we prove that $u'(\cdot) = z(\cdot)$ on $[0, b_1]$. Let $t \in [0, b_1)$ and $h > 0$ with $t+h \in [0, b_1]$. By using the notations introduced in (3.4) and remarking that $h\partial_h u_\xi = u_{\xi+h} - u_\xi$, we find that

$$\begin{aligned}
&\| \partial_h u(t) - z(t) \| \\
&\quad\leq \mathcal{H}_1(t,h) + \| i_c \|_{\mathcal{L}(Y,X)} \| D_2 f(t,u_t) \|_Y \| \frac{u_{t+h} - u_t}{h} - z_t \|_{\mathcal{B}} \\
&\quad\quad + \frac{\| (h, h\partial_h u_t) \|}{h} \| R(f,t,u_t,h,h\partial_h u_t) \|_Y \\
&\quad\quad + \int_0^t \mathcal{H}_1(t-s) \| D_2 f(s,u_s) \|_Y \| \frac{u_{s+h} - u_s}{h} - z_s \|_{\mathcal{B}} \, ds \\
&\quad\quad + \int_0^t \mathcal{H}_1(t-s) \frac{\| (h, h\partial_h u_s) \|}{h} \| R(f,s,u_s,h,h\partial_h u_s) \|_Y \, ds \\
&\quad\quad + \int_0^t \int_0^s \mu(s-\xi)\mathcal{H}_2(t-s) \| D_2 f(\xi,u_\xi) \|_Y \| \frac{u_{\xi+h} - u_\xi}{h} - z_\xi \|_{\mathcal{B}} \, d\xi ds \\
&\quad\quad + \int_0^t \int_0^s \mu(s-\xi)\mathcal{H}_2(t-s) \frac{\| (h, h\partial_h u_\xi) \|}{h} \| R(f,\xi,u_\xi,h,h\partial_h u_\xi) \|_Y \, d\xi ds \\
&\quad\quad + M \int_0^t \| D_2 g(s,u_s) \| \| \frac{u_{s+h} - u_s}{h} - z_s \|_{\mathcal{B}} \, ds \\
&\quad\quad + M \int_0^t \frac{\| (h, h\partial_h u_s) \|}{h} \| R(g,s,u_s,h,h\partial_h u_s) \| \, ds,
\end{aligned}$$

where $\mathcal{H}_1(t,h) \to 0$ uniformly on $[0, b_1]$ as $h \to 0$. Since f, g are functions of class C^1, it follows that $\frac{\|(h,h\partial_h u_s)\|}{h} \| R(f,s,u_s,h,h\partial_h u_s) \|_Y \to 0$ and $\frac{\|(h,h\partial_h u_s)\|}{h} \| R(g,s,u_s,h,h\partial_h u_s) \| \to 0$ uniformly on $[0, b_1]$ as $h \to 0$, which allows re-write the last inequality in the form

$$\begin{aligned}
&\| \partial_h u(t) - z(t) \| \\
&\quad\leq \mathcal{H}_2(t,h) + \| i_c \|_{\mathcal{L}(Y,X)} \| D_2 f(t,u_t) \|_Y \| \frac{u_{t+h} - u_t}{h} - z_t \|_{\mathcal{B}} \\
&\quad\quad + \int_0^t \mathcal{H}_1(t-s) \| D_2 f(s,u_s) \|_Y \| \frac{u_{s+h} - u_s}{h} - z_s \|_{\mathcal{B}} \, ds \\
&\quad\quad + \int_0^t \int_0^s \mu(s-\xi)\mathcal{H}_2(t-s) \| D_2 f(\xi,u_\xi) \|_Y \| \frac{u_{\xi+h} - u_\xi}{h} - z_\xi \|_{\mathcal{B}} \, d\xi ds \\
&\quad\quad + M \int_0^t \| D_2 g(s,u_s) \| \| \frac{u_{s+h} - u_s}{h} - z_s \|_{\mathcal{B}} \, ds,
\end{aligned}$$

where $\mathcal{H}_2(t,h) \to 0$ uniformly on $[0, b_1]$ as $h \to 0$. Now, from axiom $(\mathbf{A})$(iii) and the fact that $\mu < 1$ we obtain

$$\| \partial_h u(\theta) - z(\theta) \|_t \;\leq\; \frac{1}{1-\mu} \sup_{s\in[0,t]} \mathcal{H}_2(s,h) + \frac{\mu M_{b_1}}{(1-\mu)K_{b_1}} \| \frac{u_h - \varphi}{h} - z_0 \|_{\mathcal{B}} \, .$$

Thus, to prove that $u'(\cdot) = z(\cdot)$ it is sufficient to show that $\| \frac{u_h - \varphi}{h} - z_0 \|_{\mathcal{B}} \to 0$ as $h \to 0$. Consider the decomposition $u = y + z^1 + z^2$, where $(z^i)_0 = 0$, $i = 1, 2$, and

$$\begin{aligned} z^1(\theta) &= R(\theta)f(0,\varphi) - f(\theta, u_\theta) - \int_0^\theta AR(\theta - s)f(s, u_s)ds \\ &\quad - \int_0^\theta R(\theta - s)\int_0^s B(s-\xi)f(\xi, u_\xi)d\xi ds, \\ z^2(\theta) &= \int_0^\theta R(\theta - s)g(s, u_s)ds, \end{aligned}$$

for $\theta \in [0, b_1]$. It is easy to see that $u_t = y_t + z_t^1 + z_t^2$ and

$$\| \frac{u_h - \varphi}{h} - z_0 \|_{\mathcal{B}} \leq \| \frac{y_h - \varphi}{h} - \frac{d^+}{dt} y_{t|_{t=0}} \|_{\mathcal{B}} + \| \frac{z_h^1}{h} \|_{\mathcal{B}} + \| \frac{z_h^2}{h} - \chi_{g(0,\varphi)} \|_{\mathcal{B}} .$$

From condition (**c**), we obtain that $\| \frac{z_h^2}{h} - \chi_{g(0,\varphi)} \|_{\mathcal{B}} \to 0$ as $h \to 0$. By using that the function $t \to u_t$ is Lipschitz on $[0, b_1]$, the fact that

$$R(t)x - x = \int_0^t AR(s)xds + \int_0^t \int_0^s B(s-\theta)R(\theta)xds, \qquad x \in D(A),$$

the Lagrange mean value Theorem for differentiable functions, for $0 < \theta < h < b_1$ we find that

$$\begin{aligned} \frac{\| z^1(\theta) \|}{h} &\leq \frac{1}{h} \| f(\theta, u_\theta) - f(0,\varphi) \|_h + \frac{1}{h}\int_0^\theta \| AR(s)[f(0,\varphi) - f(s,u_s)] \| ds \\ &\quad + \int_0^\theta \int_0^s \| R(\theta - s)B(s-\xi)[f(0,\varphi) - f(\xi, u_\xi)] \| d\xi ds \\ &\leq \sup_{(s,\zeta)\in[(0,\varphi),(\theta,u_\theta)]} \frac{\| Df(s,\zeta) \| \| (\theta, u_\theta - \varphi) \|}{h} \\ &\quad + \frac{\widetilde{L}\theta}{h}\left(\int_0^h \mathcal{H}_1(s)ds + \int_0^h \mathcal{H}_2(s)ds \int_0^h \mu(\xi)d\xi \right), \end{aligned}$$

where $[(0, \varphi), (\theta, u_\theta)]$ represent the segment of line between $(0, \varphi)$ and (θ, u_θ). This allows to conclude that $\| \frac{z_h^1}{h} \|_{\mathcal{B}} \to 0$ as $h \to 0$ since $Df(0, \varphi) \equiv 0$. Thus, $\partial_h u \to z$ uniformly on $[0, b_1]$ and $u' = z$ on $[0, b_1]$. Moreover, from the inequality

$$\| \partial_h u_t - z_t \|_{\mathcal{B}} \leq M_{b_1} \| \frac{u_h - \varphi}{h} - z_0 \|_{\mathcal{B}} + K_{b_1} \| \partial_h u(\theta) - u'(\theta) \|_t,$$

it follows that the function $t \to u_t$ is also of class C^1 on $[0, b_1]$.

Let $v \in C([0, b_1], X)$ be the mild solution of the abstract Cauchy problem

$$\left.\begin{aligned} w'(t) &= Aw(t) + \int_0^t B(t-s)w(s)ds - Af(t, u_t) - \int_0^t B(t-s)f(s, u_s)ds + g(t, u_t), \\ w(0) &= \varphi(0) + f(0, \varphi). \end{aligned}\right\} \tag{3.5}$$

From the above remarks, the functions $S(t) = -Af(t, u_t) - \int_0^t B(t-s)f(s, u_s)ds + g(t, u_t)$ and $t \to \int_0^t R(t-\theta)S(\theta)ds$ belong to $C^1([0, b_1] : X)$, which implies from [11, Corollary 2] that $v(\cdot)$ is a classical solution of (3.5). Finally, the uniqueness of mild solution of (3.5) implies that $v(t) = u(t) + f(t, u_t)$, which allows to conclude that $u(\cdot)$ is a classical solution of (1.1)-(1.2). ■

4 Applications

In this section we consider some applications of our abstract results. We begin considering the particular case which X is finite dimensional.

4.1 Ordinary neutral integro-differential equations

The literature for neutral systems for which $x(t) \in \mathbb{R}^n$ is very extensive and for this type of systems our results are easily applicable. In fact, in this case, the functions A, $B(s)$ are bounded linear operators, $R(\cdot) \in C([0, a], \mathcal{L}(X))$, $R(t)$ is compact for every $t \geq 0$ and the condition $\mathbf{H_1}$ is verified with $Y = X = \mathbb{R}^n$. The following results are reformulations of Theorems 3.1 and 3.2 respectively.

Proposition 4.1 *Let conditions* $(\mathbf{H_2}), (\mathbf{H}_\varphi)$ *be hold. Assume that* f, g *are continuous and that there are positive constants* c_1, c_2 *such that* $\| f(t, \psi) \| \leq c_1 \| \psi \|_{\mathcal{B}} + c_2$, *for every* $(t, \psi) \in I \times \mathcal{B}$. *If* $c_1 K(0) < 1$, *then there exists a mild solution of (1.1)-(1.2) on* $[0, b]$ *for some* $0 < b \leq a$.

Proposition 4.2 *Assume that* $f, g : I \times \mathcal{B} \to X$ *are continuous and that there are continuous functions* $L_f, L_g : I \to [0, \infty)$ *such that*

$$\begin{aligned} \| f(t, \psi_1) - f(t, \psi_2) \| &\leq L_f(r) \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \qquad (t, \psi_i) \in I \times B_r(0, \mathcal{B}), \\ \| g(t, \psi_1) - g(t, \psi_2) \| &\leq L_g(r) \| \psi_1 - \psi_2 \|_{\mathcal{B}}, \qquad (t, \psi_i) \in I \times B_r(0, \mathcal{B}). \end{aligned}$$

If $L_f(0)K(0) < 1$, *then there exists a unique mild solution of (1.1)-(1.2) on* $[0, b]$ *for some* $0 < b \leq a$.

As a practical application of Proposition 4.2, consider the system

$$\frac{d}{dt}\left[u(t) - \lambda \int_{-\infty}^{t} C(t-s)u(s)ds\right] = Au(t) + \lambda \int_{-\infty}^{t} B(t-s)u(s)ds - p(t) + q(t), \quad (4.1)$$

which arises in the study of the dynamics of income, employment, value of capital stock, and cumulative balance of payment, see [3] for details. In this system, λ is a real number, the state $u(t) \in \mathbb{R}^n$, $C(\cdot), B(\cdot)$ are $n \times n$ matrix continuous functions, A is a constant $n \times n$ matrix, $p(\cdot)$ represents the government intervention and $q(\cdot)$ the private initiative.

To treat system (4.1), we choose the space $\mathcal{B} = C_0 \times L^p(\rho, X)$ (see Example 2.1) with $X = \mathbb{R}^n$. Assume that the solution of (4.1) is known on $(-\infty, 0]$, $L_f := \mid \lambda \mid \left(\int_{-\infty}^{0} \frac{\|B(s)\|^2}{\rho(s)} ds\right)^{\frac{1}{2}} < \infty$ and $L_g := \left(\int_{-\infty}^{0} \frac{\|C(s)\|^2}{\rho(s)} ds\right)^{\frac{1}{2}} < \infty$. By defining the functions $f, g : [0, a] \times \mathcal{B} \to X$ by $f(\psi)(\xi) = \lambda \int_{-\infty}^{0} C(s)\psi(s, \xi)ds$ and $g(\psi)(\xi) = \int_{-\infty}^{0} B(s)\psi(s, \xi)ds$, we can transform system (4.1) into the abstract form (1.1)-(1.2). Moreover, f, g are bounded linear operators, $\| f(t, \cdot) \|_{\mathcal{L}(X)} \leq L_f$ and $\| g(t, \cdot) \|_{\mathcal{L}(X)} \leq L_g$ for every $t \in [0, a]$. The next result is a consequence of Proposition 4.2.

Corollary 4.1 *Let $\varphi \in \mathcal{B}$ and assume $L_f K(0) < 1$. Then there exists a mild local solution $u(\cdot)$ of (4.1) with $u_0 = \varphi$.*

4.2 Abstract neutral differential equations

Now, we consider briefly the abstract system

$$\left.\begin{aligned} \frac{d}{dt}(x(t) + f(t, x_t)) &= Ax(t) + g(t, x_t), \quad t \in I = [0, a], \\ x_0 &= \varphi \in \mathcal{B}, \end{aligned}\right\} \tag{4.2}$$

which have been studied, among others, in [16, 17, 19].

Assume that A is the infinitesimal generator of an analytic semigroup of bounded linear operators $(T(t))_{t \geq 0}$ on X, $0 \in \rho(A)$ and that $M > 0$ is such that $\|T(t)\| \leq M$ for every $t \in [0, a]$. Under theses conditions, is possible to define the fractional powers $(-A)^\alpha$ for $0 < \alpha \leq 1$, as a closed linear operators on their domains $D(-A)^\alpha$. Furthermore, $D(-A)^\alpha$ is dense in X and the expression $\|x\|_\alpha = \|(-A)^\alpha x\|$ defines a norm on $D(-A)^\alpha$. If X_α, $\alpha \in (0, 1)$, represents the space $D(-A)^\alpha$ endowed with the norm $\| \cdot \|_\alpha$, then $(X_\alpha, \| \cdot \|_\alpha)$ is a Banach space continuously included in X and there exists $C_\alpha > 0$ such that $\|(-A)^\alpha T(t)\| \leq C_\alpha t^{-\alpha}$ for every $t > 0$.

The following results are particular versions of Theorems 3.1, 3.2 and 3.3. We only note that in this case, $R(t) = T(t)$ for $t \geq 0$.

Theorem 4.1 *Assume the conditions $(\mathbf{H_2})$, $(\mathbf{H_3})$ and $(\mathbf{H}_\varphi)$ be verified with $Y = X_\alpha$, $\alpha \in (0, 1)$, the semigroup $(T(t))_{t \geq 0}$ is compact and $c_1 K(0) \| i_c \|_{\mathcal{L}(X_\alpha, X)} < 1$. Then there exists a mild solution of (4.2) on $[0, b]$ for some $0 < b \leq a$.*

Theorem 4.2 *Assume that the conditions $(\mathbf{H_4})$ and $(\mathbf{H_5})$ are verified with $Y = X_\alpha$, for some $\alpha \in (0, 1)$. If $L_f(0) K(0) \| i_c \|_{\mathcal{L}(X_\alpha, X)} < 1$, then there exists a unique mild solution of (4.2) on $[0, b]$ for some $0 < b \leq a$.*

Theorem 4.3 *Assume condition $(\mathbf{H_4})$ be satisfied with $Y = X_\alpha$, for some $\alpha \in (0, 1)$. Suppose that the semigroup $(T(t))_{t \geq 0}$ is compact and $L_f(0) \| i_c \|_{\mathcal{L}(X_\alpha, X)} K(0) < 1$. Then there exists a mild solution of (4.2) on $[0, b]$ for some $0 < b \leq a$.*

4.3 A neutral integro-differential system

To finish this section, we discuss the existence of solutions for the integro-differential system

$$
\begin{aligned}
\frac{\partial}{\partial t}\left[u(t,\xi)+\int_{-\infty}^{t}\int_{0}^{\pi} b(s-t,\eta,\xi)u(s,\eta)d\eta ds\right] &= \frac{\partial^2 u(t,\xi)}{\partial \xi^2}+\int_0^t a(s-t)\frac{\partial^2 u(s,\xi)}{\partial \xi^2}ds \\
&\quad +\int_{-\infty}^{t} a_0(s-t)u(s,\xi)ds, \qquad (4.3)\\
u(t,0) &= u(t,\pi)=0, \qquad (4.4)\\
u(\theta,\xi) &= \phi(\theta,\xi), \qquad \theta\leq 0, \qquad (4.5)
\end{aligned}
$$

for $t\in I=[0,a]$ and $\xi\in[0,\pi]$.

To treat this system in the abstract form (1.1)-(1.2), we choose the spaces $X = L^2([0,\pi])$, $\mathcal{B}=C_0\times L^p(\rho,X)$ (see Example 2.1) and the operator $A: D(A)\subset X\to X$ given by $Ax=x''$ on the domain $D(A)=\{x\in X: x''\in X,\, x(0)=x(\pi)=0\}$. It is well known that A is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on X. Moreover, A has discrete spectrum, the eigenvalues are $-n^2, n\in\mathbb{N}$, with corresponding normalized eigenvectors $z_n(\xi):=\left(\frac{2}{\pi}\right)^{1/2}\sin(n\xi)$ and the following properties hold:

(a) $\{z_n : n\in\mathbb{N}\}$ is an orthonormal basis of X.

(b) For $x\in X$, $T(t)x=\sum_{n=1}^{\infty} e^{-n^2t} < x, z_n > z_n$.

(c) For $\alpha\in(0,1)$, the fractional power $(-A)^{\alpha}: D((-A)^{\alpha})\subseteq X\to X$ of A is given by $(-A)^{\alpha}x=\sum_{n=1}^{\infty} n^{2\alpha} < x, z_n > z_n$, where $D((-A)^{\alpha})=\{x\in X: (-A)^{\alpha}x\in X\}$.

Lemma 4.1 *Assume that the family of operators $(a(t)A)_{t\geq 0}$ verifies the conditions* **H2** *and* **H3** *in Desch & Grimmer [8] and the assumptions in [12, Theorem 3.3]. Then, the system*

$$
\begin{aligned}
u'(t) &= Au(t)+\int_0^t a(t-s)Au(s)ds,\\
u(0) &= x,
\end{aligned}
$$

has an associated compact resolvent of operators $(R(t))_{t\geq 0}$ on X. Moreover, for $\eta\in(0,1)$ there exists $C_\eta>0$ such that $\|(-A)^{\eta}R(s)\|\leq \frac{C_\eta}{s^\eta}$, for each $s>0$.

Proof: The existence of a compact resolvent is a consequence of [8, Theorem 2]. The second assertion follows from [12, Theorem 3.3]. Related this last assertion, we only note that the operators $(-A)^{\eta}, R(t)$ commutes. The proof is complete. ∎

In the sequel, we assume that the conditions in Lemma 4.1 are fulfilled, $\varphi(\theta)(\xi) = \phi(\theta,\xi)$ belong to $\mathcal{B}$ and the following conditions.

(i) The functions $a_0 : \mathbb{R} \to \mathbb{R}$ are continuous and $L_g := \left(\int_{-\infty}^{0} \frac{(a_0(s))^2}{\rho(s)} ds \right)^{\frac{1}{2}} < \infty$.

(ii) The functions $b(\cdot)$, $\frac{\partial b(s,\eta,\xi)}{\partial \xi}$ are measurable, $b(s, \eta, \pi) = b(s, \eta, 0) = 0$ for all (s, η) and

$$L_f := \max\{ (\int_0^{\pi} \int_{-\infty}^{0} \int_0^{\pi} \rho^{-1}(\theta) \left(\frac{\partial^i}{\partial \xi^i} b(\theta, \eta, \xi) \right)^2 d\eta d\theta d\xi)^{1/2} : \ i = 0, 1 \} < \infty.$$

By defining the operators $f, g : \mathcal{B} \to X$ by

$$\begin{aligned} f(\psi)(\xi) &= \int_{-\infty}^{0} \int_0^{\pi} b(s, \eta, \xi) \psi(s, \eta) d\eta ds, \\ g(\psi)(\xi) &= \int_{-\infty}^{0} a_0(s) \psi(s, \xi) ds, \end{aligned}$$

we can transform (4.3)-(4.5) into the abstract form (1.1)-(1.2). Moreover, f, g are bounded linear operators, $\| f \|_{\mathcal{L}(\mathcal{B},X)} \leq L_f$, $\| g \|_{\mathcal{L}(\mathcal{B},X)} \leq L_g$ and a straightforward estimation using (**ii**) shows that $f(I \times \mathcal{B}) \subset D((-A)^{\frac{1}{2}})$ and $\| (-A)^{\frac{1}{2}} f(t, \cdot) \|_{\mathcal{L}(\mathcal{B},X)} \leq L_f$ for all $t \in I$.

Proposition 4.3 *If $L_f < 1$, then there exists a unique mild solution $u(\cdot)$ of (4.3)-(4.5) on $[0, b]$ for some $0 < b \leq a$. If in addition, $\varphi(0) \in D(A)$, $S(\cdot)\varphi$ is Lipschitz on $[0, b]$, the function $\frac{\partial^2}{\partial \xi^2} b(\theta, \eta, \xi)$ is continuous and*

$$\widetilde{L_f} := (\int_0^{\pi} \int_{-\infty}^{0} \int_0^{\pi} \rho^{-1}(\theta) (\frac{\partial^2}{\partial \xi^2} b(\theta, \eta, \xi))^2 d\eta d\theta d\xi)^{1/2} < \infty, \tag{4.6}$$

then $u(\cdot)$ is a strong solution.

Proof: From Lemma 4.1, $\| AR(t)x \| \leq C_{\frac{1}{2}} t^{-\frac{1}{2}} \| x \|$ and

$$\| a(t)AR(t)x \|_{\mathcal{L}(X_{\frac{1}{2}},X)} = \| a(s)(-A)^{\frac{1}{2}} R(t)(-A)^{\frac{1}{2}} x \| \leq C_{\frac{1}{2}} \mid a(\cdot) \mid_a t^{-\frac{1}{2}} \| x \|_{\frac{1}{2}},$$

for every $x \in D((-A)^{\frac{1}{2}})$. Thus, the condition $\mathbf{H_1}$ is verified. Now, the existence of a mild solution can be deduced from Theorem 3.2.

The regularity assertion follows from Theorem 3.4. We only note that under condition (4.6), the function f is $D(A)$-valued and that $\| Af \|_{\mathcal{L}(\mathcal{B},X)} \leq \max\{L_f, \widetilde{L_f}\}$. ∎

References

[1] Adimy, M.; Ezzinbi, K. A class of linear partial neutral functional-differential equations with nondense domain. *J. Diff. Eqns.* 147 (2) (1998), 285-332.

[2] Cannarsa, Piermarco.; Sforza, D, Global solutions of abstract semilinear parabolic equations with memory terms. *NoDEA Nonlinear Differential Equations. Appl.* 10 (2003), no. 4, 399–430.

[3] Chukwu, E. N. *Differential models and neutral systems for controlling the wealth of nations.* Series on Advances in Mathematics for Applied Sciences, 54. World Scientific Publishing Co., Inc., River Edge, NJ, 2001.

[4] Clément, Ph.; Nohel, J. A. Asymptotic behavior of solutions of nonlinear Volterra equations with completely positive kernels. *SIAM J. Math. Anal.* 12 (1981), no. 4, 514–535.

[5] Clément, Philippe; Prüss, Jan Global existence for a semilinear parabolic Volterra equation. *Math. Z.* 209 (1992), no. 1, 17–26.

[6] Da Prato, G.; Iannelli, M. Existence and regularity for a class of integro-differential equations of parabolic type. *J. Math. Anal. Appl.* 112 (1985), no. 1, 36–55.

[7] Da Prato, Giuseppe; Lunardi, Alessandra Solvability on the real line of a class of linear Volterra integro-differential equations of parabolic type. *Ann. Mat. Pura Appl.* (4) 150 (1988), 67-117.

[8] Desch, Wolfgang; Grimmer, Ronald; Schappacher, Wilhelm, Some considerations for linear integro-differential equations. *J. Math. Anal. Appl.* 104 (1984), no. 1, 219–234.

[9] Granas, A.; Dugundji, J., *Fixed Point Theory.* Springer-Verlag, New York, 2003.

[10] Grimmer, R. C. Resolvent operators for integral equations in a Banach space. *Trans. Amer. Math. Soc.* 273 (1982), no. 1, 333–349.

[11] Grimmer, R.; Prüss, J. On linear Volterra equations in Banach spaces. Hyperbolic partial differential equations, II. *Comput. Math. Appl.* 11 (1985), no. 1-3, 189–205.

[12] Grimmer, R. C. e Pritchard, A. J. Analytic resolvent operators for integral equations in Banach space. *J. Differential Equations* 50 (1983), no. 2, 234–259.

[13] M.E. Gurtin, A.C. Pipkin, A general theory of heat conduction with finite wave speed. *Arch. Rat. Mech. Anal.* 31 (1968), 113-126.

[14] Hale, Jack K.; Verduyn Lunel, Sjoerd M. *Introduction to functional-differential equations.* Applied Mathematical Sciences, 99. Springer-Verlag, New York, 1993.

[15] J. K. Hale Partial neutral functional-differential equations. *Rev. Roumaine Math Pures Appl* 39 (1994), no. 4, 339-344

[16] Hernández, E.; Henríquez, H. R., Existence results for partial neutral functional differential equations with unbounded delay. *J. Math. Anal. Appl.* 221 (1998), no. 2, 452–475.

[17] Hernández, E.; Henríquez, H. R., Existence of periodic solutions of partial neutral functional-differential equations with unbounded delay. *J. Math. Anal. Appl.* 221 (1998), no. 2, 499–522.

[18] Hernández M., Eduardo; Rabello, Marco; Henríquez, Hernán R. Existence of solutions for impulsive partial neutral functional differential equations. *J. Math. Anal. Appl.* 331 (2007), no. 2, 1135-1158.

[19] Hernández, Eduardo; Existence results for partial neutral integrodifferential equations with unbounded delay. *J. Math. Anal. Appl.* 292, (2004), no. 1, 194-210.

[20] Hernández, M. Eduardo A comment on the papers: "Controllability results for functional semilinear differential inclusions in Fréchet spaces" [Nonlinear Anal. 61 (2005), no. 3, 405–423; MR2123084] by M. Benchohra and A. Ouahab and "Controllability of impulsive neutral functional differential inclusions with infinite delay" [Nonlinear Anal. 60 (2005), no. 8, 1533-1552; MR2120031] by B. Liu. *Nonlinear Anal.* 66 (2007), no. 10, 2243-2245.

[21] Hernández, E.; McKibben, M.; Some comments on: " Existence of solutions of abstract nonlinear second-order neutral functional integrodifferential equations. " *Comput. Math. Appl.* 50 (2005), no. 5-6, 655-669.

[22] Henríquez, Hernán R. Regularity of solutions of abstract retarded functional differential equations with unbounded delay. *Nonlinear Anal.* 28 (1997), no. 3, 513-531.

[23] Hino, Y., Murakami, S. and Naito, T., *Functional-differential equations with infinite delay*. Lecture Notes in Mathematics, 1473. Springer-Verlag, Berlin, 1991.

[24] Lunardi, Alessandra On the linear heat equation with fading memory. *SIAM J. Math. Anal.* 21 (1990), no. 5, 1213–1224.

[25] Lunardi, Alessandra Laplace transform methods in integro-differential equations. Integro-differential evolution equations and applications (Trento, 1984). *J. Integral Equations* 10 (1985), no. 1-3, suppl., 185-211.

[26] Martin, R. H., *Nonlinear Operators and Differential Equations in Banach Spaces.* Robert E. Krieger Publ. Co., Florida, 1987.

[27] Miller, R. K. An integro-differential equation for rigid heat conductors with memory. *J. Math. Anal. Appl.* 66 (1978), no. 2, 313–332.

[28] Nunziato, J.W. On heat conduction in materials with memory. *Quart. Appl. Math.* 29 (1971), 187–204.

[29] Wu, J.; Xia, H. Rotating waves in neutral partial functional-differential equations. *J. Dynam. Differential Equations* 11 (1999), no. 2, 209–238.

[30] Wu, Jianhong, Xia Huaxing, Self-sustained oscillations in a ring array of coupled lossless transmission lines. *J. Differential Equations* 124 (1996), no. 1, 247–278.

[31] Wu, Jianhong, *Theory and applications of partial functional-differential equations.* Applied Mathematical Sciences, 119. Springer-Verlag, New York, 1996.

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