Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **28**, 1-7; **http://www.math.u-szeged.hu/ejqtde/**

Some existence results for boundary value problems of fractional semilinear evolution equations

Bashir Ahmad

Department of Mathematics, Faculty of Science, King Abdulaziz University
P.O. Box 80203, Jeddah 21589, Saudi Arabia
E-mail: bashir_qau@yahoo.com

Abstract

In this paper, we study the existence of solutions for a two-point boundary value problem of fractional semilinear evolution equations in a Banach space. Our results are based on the contraction mapping principle and Krasnoselskii's fixed point theorem.

Keywords: Evolution equations of fractional order, boundary conditions, existence of solutions, fixed point theorem.
AMS Subject Classifications (2000): 34A34, 34B15.

1 Introduction

In some real world problems, fractional-order models are found to be more adequate than integer-order models. Fractional derivatives provide an excellent tool for the description of memory and hereditary properties of various materials and processes. The mathematical modelling of systems and processes in the fields of physics, chemistry, aerodynamics, electro dynamics of complex medium, polymer rheology, etc. involves derivatives of fractional order. In consequence, the subject of fractional differential equations is gaining much importance and attention. For examples and details, see [1-13] and the references therein.
In this paper, we consider a two-point boundary value problem involving fractional semilinear evolution equations and prove some existence results in a Banach space. Precisely, we study the following boundary value problem:

$$\begin{cases} {}^cD^q x(t) = A(t)x(t) + f(t, x(t)), \quad 0 < t < 1, \quad 1 < q \leq 2, \\ \alpha x(0) + \beta x'(0) = \gamma_1, \qquad \alpha x(1) + \beta x'(1) = \gamma_2, \end{cases} \tag{1.1}$$

where ${}^cD$ is the Caputo fractional derivative, $A(t)$ is a bounded linear operator on X for each $t \in [0,1]$ (the function $t \to A(t)$ is continuous in the uniform operator topology), $f : [0,1] \times X \to X$ and $\alpha > 0$, $\beta \geq 0$, $\gamma_{1,2}$ are real numbers. Here, $(X, \|.\|)$ is a Banach space and $\mathcal{C} = C([0,1], X)$ denotes the Banach space of all continuous functions from $[0,1] \to X$ endowed with a topology of uniform convergence with the norm denoted by $\|.\|_{\mathcal{C}}$.

2 Preliminaries

Let us recall some basic definitions [8, 11, 13] on fractional calculus.

Definition 2.1. For a function $g : [0, \infty) \to \mathbb{R}$, the Caputo derivative of fractional order q is defined as

$$ {}^cD^q g(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} g^{(n)}(s) ds, \quad n-1 < q < n, n = [q]+1, $$

where $[q]$ denotes the integer part of the real number q.

Definition 2.2. The Riemann-Liouville fractional integral of order q is defined as

$$ I^q g(t) = \frac{1}{\Gamma(q)} \int_0^t \frac{g(s)}{(t-s)^{1-q}} ds, \quad q > 0, $$

provided the integral exists.

Definition 2.3. The Riemann-Liouville fractional derivative of order q for a function $g(t)$ is defined by

$$ D^q g(t) = \frac{1}{\Gamma(n-q)} \Big(\frac{d}{dt}\Big)^n \int_0^t \frac{g(s)}{(t-s)^{q-n+1}} ds, \quad n = [q]+1, $$

provided the right hand side is pointwise defined on $(0, \infty)$.

Now, we state a known result due to Krasnoselskii [14] which is needed to prove the existence of at least one solution of (1.1).

Theorem 2.1. Let M be a closed convex and nonempty subset of a Banach space X. Let A, B be the operators such that: (i) $Ax + By \in M$ whenever $x, y \in M$; (ii) A is compact and continuous; (iii) B is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.

As argued in reference [1], the solution of the boundary value problem (1.1) can be written as

$$ \begin{aligned} x(t) &= \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \Big(f(s, x(s)) + A(s)x(s) \Big) ds \\ &+ \int_0^1 \Big[\frac{(\beta - \alpha t)(1-s)^{q-1}}{\alpha \Gamma(q)} + \frac{\beta(\beta - \alpha t)(1-s)^{q-2}}{\alpha^2 \Gamma(q-1)} \Big] \Big(f(s, x(s)) + A(s)x(s) \Big) ds \\ &+ \frac{1}{\alpha^2} [(\alpha(1-t) + \beta)\gamma_1 + (\beta + \alpha t)\gamma_2]. \end{aligned} $$

3 Main results

Theorem 3.1. Let $f : [0,1] \times X \to X$ be a jointly continuous function mapping bounded subsets of $[0,1] \times X$ into relatively compact subsets of X, and

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|, \ \forall t \in [0,1], \ x,y \in X.$$

Then the boundary value problem (1.1) has a unique solution provided

$$(L + A_1) \leq \frac{1}{2}\Big[\frac{\beta+2\alpha}{\alpha\Gamma(q+1)} + \frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\Big]^{-1},$$

where $A_1 = \max_{t\in[0,1]} \|A(t)\|$.

Proof. Define $\digamma : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}
(\digamma x)(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(f(s,x(s)) + A(s)x(s)\Big)ds \\
&+ \int_0^1 \Big[\frac{(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big]\Big(f(s,x(s)) + A(s)x(s)\Big)ds \\
&+ \frac{1}{\alpha^2}[(\alpha(1-t)+\beta)\gamma_1 + (\beta+\alpha t)\gamma_2], \ t \in [0,1].
\end{aligned}$$

Setting $\sup_{t\in[0,1]} \|f(t,0)\| = M$ and choosing

$$r \geq 2\Big[M\Big(\frac{\beta+2\alpha}{\alpha\Gamma(q+1)} + \frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\Big) + \frac{\alpha+\beta}{\alpha^2}\Big(\gamma_1+\gamma_2\Big)\Big],$$

we show that $\digamma B_r \subset B_r$, where $B_r = \{x \in \mathcal{C} : \|x\| \leq r\}$. For $x \in B_r$, we have

$$\begin{aligned}
\|(\digamma x)(t)\| &\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(\|f(s,x(s))\| + \|A(s)\|\|x(s)\|\Big)ds \\
&+ \int_0^1 \Big|\beta - \alpha t\Big|\Big[\frac{(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big]\Big(\|f(s,x(s))\| + \|A(s)\|\|x(s)\|\Big)ds \\
&+ \frac{\alpha+\beta}{\alpha^2}\Big(|\gamma_1| + |\gamma_2|\Big) \\
&\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\| + \|A(s)\|\|x(s)\|\Big)ds \\
&+ \int_0^1 \Big|\beta - \alpha t\Big|\Big[\frac{(1-s)^{q-1}}{\alpha\Gamma(q)} + \frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big] \\
&\times \Big(\|f(s,x(s)) - f(s,0)\| + \|f(s,0)\| + \|A(s)\|\|x(s)\|\Big)ds + \frac{\alpha+\beta}{\alpha^2}\Big(|\gamma_1| + |\gamma_2|\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leq ((L+A_1)r+M)\Big[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
&\quad +\int_0^1 \Big|\beta-\alpha t\Big|\Big(\frac{(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big)ds\Big]+\frac{\alpha+\beta}{\alpha^2}\Big(|\gamma_1|+|\gamma_2|\Big) \\
&= ((L+A_1)r+M)\Big[\frac{t^q}{\Gamma(q+1)}+|\beta-\alpha t|\Big(\frac{1}{\alpha\Gamma(q+1)}+\frac{\beta}{\alpha^2\Gamma(q)}\Big)\Big] \\
&+ \frac{\alpha+\beta}{\alpha^2}\Big(|\gamma_1|+|\gamma_2|\Big) \\
&\leq (L+A_1)[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}]r+M\Big[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\Big] \\
&+ \frac{\alpha+\beta}{\alpha^2}\Big(|\gamma_1|+|\gamma_2|\Big)\leq r.
\end{aligned}
$$

Now, for $x, y \in \mathcal{C}$ and for each $t \in [0,1]$, we obtain

$$
\begin{aligned}
&\|(\mathcal{F}x)(t)-(\mathcal{F}y)(t)\| \\
&\leq \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}|\Big(\|f(s,x(s))-f(s,y(s))\|+\|A(s)(x(s)-y(s)\|\Big)ds \\
&\quad +\int_0^1 \Big|\beta-\alpha t\Big|\Big[\frac{(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big] \\
&\times \Big(\|f(s,x(s))-f(s,y(s))\|+\|A(s)(x(s)-y(s)\|\Big)ds \\
&\leq (L+A_1)\|x-y\|_{\mathcal{C}}\Big[\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}ds \\
&\quad +\int_0^1 |\beta-\alpha t|\Big(\frac{(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big)ds\Big] \\
&\leq (L+A_1)\|x-y\|_{\mathcal{C}}\Big[\frac{t^q}{\Gamma(q+1)}+|\beta-\alpha t|\Big(\frac{1}{\alpha\Gamma(q+1)}+\frac{\beta}{\alpha^2\Gamma(q)}\Big)\Big] \\
&\leq (L+A_1)\Big[\frac{1}{\alpha\Gamma(q+1)}(2\alpha+\beta)+\frac{\beta^2+\alpha\beta)}{\alpha^2\Gamma(q)}\Big]\|x-y\|_{\mathcal{C}} \\
&\leq \Lambda_{\alpha,\beta,q,L,A_1}\|x-y\|_{\mathcal{C}},
\end{aligned}
$$

where $\Lambda_{\alpha,\beta,q,L,A_1}=(L+A_1)\Big[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\Big]$, which depends only on the parameters involved in the problem. As $\Lambda_{\alpha,\beta,q,L,A_1}<1$, therefore $\mathcal{F}$ is a contraction. Thus, the conclusion of the theorem follows by the contraction mapping principle.

Theorem 3.2. Assume that $f:[0,1]\times X\to X$ is a jointly continuous function and maps bounded subsets of $[0,1]\times X$ into relatively compact subsets of X. Furthermore, assume that

($\mathbf{H_1}$) $\|f(t,x)-f(t,y)\|\leq L\|x-y\|,\ \forall t\in[0,1],\ x,y\in X;$

($\mathbf{H_2}$) $\|f(t,x)\|\leq \mu(t),\quad \forall (t,x)\in[0,1]\times X,$ and $\mu\in L^1([0,1],R^+)$.

If $(L+A_1)(\frac{\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)})<1$, then the boundary value problem (1.1) has at least one solution on $[0,1]$.

Proof. Let us fix

$$r \geq \frac{\|\mu\|_{L^1}[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}]+\frac{\alpha+\beta}{\alpha^2}(|\gamma_1|+|\gamma_2|)}{1-A_1[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}]},$$

and consider $B_r=\{x\in\mathcal{C}:\|x\|\leq r\}$. We define the operators Φ and Ψ on B_r as

$$(\Phi x)(t)=\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\Big(f(s,x(s))+A(s)x(s)\Big)ds,$$

$$\begin{aligned}(\Psi x)(t) &= \int_0^1 \Big[\frac{(\beta-\alpha t)(1-s)^{q-1}}{\alpha\Gamma(q)}+\frac{\beta(\beta-\alpha t)(1-s)^{q-2}}{\alpha^2\Gamma(q-1)}\Big]\Big(f(s,x(s))+A(s)x(s)\Big)ds\\ &+ \frac{1}{\alpha^2}[(\alpha(1-t)+\beta)\gamma_1+(\beta+\alpha t)\gamma_2].\end{aligned}$$

For $x,y\in B_r$, we find that

$$\|\Phi x+\Psi y\|\leq(\|\mu\|_{L^1}+A_1 r)\Big[\frac{2\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\Big]+\frac{\alpha+\beta}{\alpha^2}\Big(|\gamma_1|+|\gamma_2|\Big)\leq r.$$

Thus, $\Phi x+\Psi y\in B_r$. It follows from the assumption (H_1) that Ψ is a contraction mapping for

$$(L+A_1)\Big(\frac{\alpha+\beta}{\alpha\Gamma(q+1)}+\frac{\beta^2+\alpha\beta}{\alpha^2\Gamma(q)}\Big)<1.$$

The continuity of f implies that the operator Φ is continuous. Also, Φ is uniformly bounded on B_r as

$$\|\Phi x\|\leq\frac{(\|\mu\|_{L^1}+A_1 r)}{\Gamma(q+1)}.$$

Now we prove the compactness of the operator Φ. Setting $\Omega=[0,1]\times B_r$, we define $\sup_{(t,x)\in\Omega}\|f(t,x)\|=\overline{f}$, and consequently we have

$$\begin{aligned}&\|(\Phi x)(t_1)-(\Phi x)(t_2)\|\\ &=\Big\|\frac{1}{\Gamma(q)}\int_0^{t_1}[(t_2-s)^{q-1}-(t_1-s)^{q-1}](f(s,x(s))+A(s)x(s))ds\\ &+\int_{t_1}^{t_2}(t_2-s)^{q-1}f(s,x(s))ds\Big\|\\ &\leq\frac{(\overline{f}+A_1 r)}{\Gamma(q+1)}|2(t_2-t_1)^q+t_1^q-t_2^q|,\end{aligned}$$

which is independent of x. So Φ is relatively compact on B_r. Hence, by the Arzela Ascoli Theorem, Φ is compact on B_r. Thus all the assumptions of Theorem 2.1 are satisfied and the conclusion of Theorem 2.1 implies that the boundary value problem (1.1) has at least one solution on $[0,1]$.

Example. Consider the following boundary value problem

$$\begin{cases} {}^cD^{\frac{3}{2}}x(t) = \frac{t}{20}x + \frac{1}{(t+5)^2}\frac{|x|}{1+|x|}, & t \in [0,1], \\ x(0) + x'(0) = 0, & x(1) + x'(1) = 0. \end{cases} \tag{3.1}$$

Here, $f(t,x(t)) = \frac{1}{(t+5)^2}\frac{|x|}{1+|x|}$, $A(t) = \frac{t}{20}$, $\alpha = 1$, $\beta = 1$, $\gamma_1 = 0 = \gamma_2$. Clearly $\|f(t,x) - f(t,y)\| \leq L\|x - y\|$ with $L = \frac{1}{25}$ and $A_1 = \frac{1}{20}$. Further,

$$2(L + A_1)\Big(\frac{\beta + 2\alpha}{\alpha\Gamma(q+1)} + \frac{\beta^2 + \alpha\beta}{\alpha^2\Gamma(q)}\Big) = \frac{36}{25\sqrt{\pi}} < 1.$$

Thus, all the assumptions of Theorem 3.1 are satisfied. So, the conclusion of Theorem 3.1 applies and the boundary value problem (3.1) has a unique solution on $[0,1]$.

Acknowledgement. The author thanks the anonymous referee for his/her valuable suggestions.

References

[1] B. Ahmad, J.J. Nieto, Existence results for nonlinear boundary value problems of fractional integrodifferential equations with integral boundary conditions. Boundary Value Problems, Article ID 708576 (2009), 11 pages doi:10.1155/2009/708576.

[2] B. Ahmad, S. Sivasundaram, Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations. Nonlinear Analysis: Hybrid Systems (2009), doi:10.1016/j.nahs.2009.01.008 (in press).

[3] B. Ahmad, J.J. Nieto, Existence results for higher order fractional differential inclusions with nonlocal boundary conditions. Nonlinear Studies, to appear.

[4] K. Balachandran, J.Y. Park, Nonlocal Cauchy problem for abstract fractional semilinear evolution equations. Nonlinear Anal. (2009), doi:10.1016/j.na.2009.03.005 (in press).

[5] Y.-K. Chang, J.J. Nieto, Some new existence results for fractional differential inclusions with boundary conditions. Math. Comput. Modelling, 49 (2009), 605-609.

[6] V. Daftardar-Gejji, S. Bhalekar, Boundary value problems for multi-term fractional differential equations. J. Math. Anal. Appl. 345 (2008), 754-765.

[7] V. Gafiychuk, B. Datsko, V. Meleshko, Mathematical modeling of time fractional reaction-diffusion systems. J. Comput. Appl. Math. 220(2008) 215-225.

[8] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[9] V. Lakshmikantham, S. Leela, J. Vasundhara Devi, Theory of Fractional Dynamic Systems, to be published by Cambridge Academic Publishers, U.K.

[10] M.P. Lazarević, Finite time stability analysis of PD$^\alpha$ fractional control of robotic time -delay systems. Mech. Res. Comm. 33 (2006), 269-279.

[11] I. Podlubny, Fractional Differential Equations. Academic Press, San Diego, 1999.

[12] S.Z. Rida, H.M. El-Sherbiny, A.A.M. Arafa, On the solution of the fractional nonlinear Schrödinger equation. Physics Letters A 372 (2008), 553-558.

[13] S.G. Samko, A.A. Kilbas, O.I. Marichev, Fractional Integrals and Derivatives, Theory and Applications. Gordon and Breach, Yverdon, 1993.

[14] D.R. Smart, Fixed Point Theorems. Cambridge University Press, 1980.

(Received March 19, 2009)