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Criteria for Disfocality and Disconjugacy for Third Order Differential Equations*

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Abstract

In this paper, lower bounds for the spacing $(b-a)$ of the zeros of the solutions and the zeros of the derivative of the solutions of third order differential equations of the form

$$y''' + q(t)y' + p(t)y = 0 \qquad (*)$$

are derived under the some assumptions on p and q. The concept of disfocality is introduced for third order differential equations (*). This helps to improve the Liapunov-type inequality, when y(t) is a solution of (*) with (i) $y(a) = 0 = y'(b)$ or $y'(a) = 0 = y(b)$ with $y(t) \neq 0,\ t \in (a,b)$ or (ii) $y(a) = 0 = y'(a),\ y(b) = 0 = y'(b)$ with $y(t) \neq 0,\ t \in (a,b)$.

If y(t) is a solution of (*) with $y(t_i) = 0,\ 1 \leq i \leq n,\ n \geq 4,\ (t_1 < t_2 < ... < t_n)$ and $y(t) \neq 0,\ t \in \bigcup_{i=1}^{i=n-1}(t_i, t_{i+1})$, then lower bound for spacing $(t_n - t_1)$ is obtained. A new criteria for disconjugacy is obtained for (*) in $[a,b]$. This papers improves many known bounds in the literature.

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1 Introduction

In [15], Russian mathematician A. M. Liapunov proved the following remarkable inequality:

If y(t) is a nontrivial solution of

$$y'' + p(t)y = 0, \tag{1.1}$$

with $y(a) = 0 = y(b)(a < b)$ and $y(t) \neq 0$ for $t \in (a, b)$, then

$$\frac{4}{b-a} < \int_a^b |p(t)|dt, \tag{1.2}$$

where $p \in L^1_{loc}$. This inequality provides a lower bound of the distance between consecutive zeros of $y(t)$. If $p(t) = p > 0$, then (1.2) yields

$$(b-a) > 2/\sqrt{p}.$$

In [12], the inequality (1.2) is strengthened to

$$\frac{4}{b-a} < \int_a^b p_+(t)dt, \tag{1.3}$$

where $p_+(t) = max\{p(t), 0\}$. The inequality (1.3) is the best possible in the sense that if the constant 4 in (1.3) is replaced by any larger constant, then there exists an example of (1.1) for which (1.3) no longer holds (see [12, p.345], [13]). However stronger results were obtained in [2], [13]. In [13] it is shown that

$$\int_a^c p_+(t)dt > \frac{1}{c-a} \quad \text{and} \quad \int_c^b p_+(t)dt > \frac{1}{b-c},$$

where $c \in (a, b)$ such that $y'(c) = 0$. Hence

$$\int_a^b p_+(t)dt > \frac{1}{c-a} + \frac{1}{b-c} = \frac{(b-a)}{(c-a)(b-c)} > \frac{4}{b-a}.$$

In [2], the authors obtained (see Cor. 4.1)

$$\frac{4}{b-a} < \left| \int_a^b p(t)dt \right|$$

from which (1.2) can be obtained. The inequality finds applications in the study of boundary value problems. It may be used to provide a lower bound on the first positive proper value of the Sturm-Liouville problems

$$\begin{aligned} y''(t) + \lambda q(t)y &= 0 \\ y(c) = 0 &= y(d) \quad (c < d) \end{aligned}$$

and

$$\begin{aligned} y''(t) + (\lambda + q(t))y &= 0 \\ y(c) = 0 &= y(d) \quad (c < d) \end{aligned}$$

by letting $p(t)$ to denote $\lambda q(t)$ and $\lambda + q(t)$ respectively in (1.2). The disconjugacy of (1.1) also depends on (1.2). Indeed, equation (1.1) is said to be disconjugate if

$$\int_a^b |p(t)|dt \leq 4/(b-a).$$

(Equation (1.1) is said to be disconjugate on $[a, b]$ if no non trivial solution of (1.1) has more than one zero). Thus (1.2) may be regarded as a necessary condition for conjugacy of (1.1). The inequality (1.2) finds lot of applications in areas like eigen value problems, stability,etc. A number of proofs are known and generalizations and improvement have also been given (see [12], [13], [23], [24]). Inequality (1.3) generalized to the condition

$$\int_a^b (t-a)(b-t)p_+(t)dt > (b-a) \tag{1.4}$$

by Hartman and Wintner [11]. An alternate proof of the inequality (1.4), due to Nehari [17], is given in [12, Theorem 5.1 Ch XI]. For the equation

$$y''(t) + q(t)y' + p(t)y = 0, \tag{1.5}$$

where $p, q \in C([0, \infty), R)$, Hartman and Wintner [11] established the inequality

$$\int_a^b (t-a)(b-t)p_+(t)dt + max\left\{ \int_a^b (t-a)|q(t)|, \int_a^b (b-t)|q(t)|dt \right\} > (b-a) \tag{1.6}$$

which reduces to (1.4) if $q(t) = 0$. In particular, (1.6) implies the "*de la vallee Poussin inequality*" (see [12]). In [10], Galbraith has shown that if a and b are successive zeros of (1.1) with $p(t) \geq 0$ is a linear function, then

$$(b-a)\int_a^b p(t)dt \leq \pi^2.$$

This inequality provides an upper bound for two successive zeros of an oscillatory solution of (1.1). Indeed, if $p(t) = p > 0$, then $(b - a) \leq \pi/(p)^{\frac{1}{2}}$. Fink [8], has obtained both upper and lower bounds of $(b - a)\int_a^b p(t)dt$, where $p(t) \geq 0$ is linear. Indeed, he has shown that

$$\frac{9}{8}\lambda_0^2 \leq (b - a)\int_a^b p(t)dt \leq \pi^2$$

and these are the best possible bounds, where λ_0 is the first positive zero of $J_{\frac{1}{3}}$ and J_n is the Bessele's function. The constant $\frac{9}{8}\lambda_0^2 = 9.478132...$ and $\pi^2 = 9.869604...$, so that it gives a delicate test for the spacing of the zeros for linear p. In [9], Fink has investigated the behaviour of the functional $(b - a)\int_a^b p(t)dt$, where p is in a certain class of sub or supper functions. Eliason [5], [6] has obtained upper and lower bound of the functional $(b-a)\int_a^b p(t)dt$, where $p(t)$ is concave or convex. In [16], St Mary and Eliason has considered the same problem for equation (1.5). In [1], Bailey and Waltman applied different techniques to obtained both uppper and lower bounds for the distance between two successive zeros of solution of (1.5). They also considered nonlinear equations. In a recent paper [2], Brown and Hinton used Opial's inequality to obtain lower bounds for the spacing of the zeros of a solution of (1.1) and lower bounds of the spacing $\beta - \alpha$, where $y(t)$ is a solution of (1.1) satisfying $y(\alpha) = 0 = y'(\beta)$ and $y'(\alpha) = 0 = y(\beta)(\alpha < \beta)$.

The inequality (1.2) is generalized to second order nonlinear differential equatiton by Eliason [5], to delay differential equations of second order by [6], [7] and Dahiya and Singh [3] and to higher order differential equation by Pachpatte [18]. However, very limited work has been done in this direction for differential equations for third and higher order. In [20],the authors considered the differential equations of the form

$$y''' + q(t)y' + p(t)y = 0, \tag{1.7}$$

where p and q are real-valued continuous functions on $[0, \infty)$ such that q is once differentiable and each $p(t)$ and $q'(t)$ is locally integrable. Let $y(t)$ be a nontrivial solution of (1.7) with $y(a) = 0 = y(b)$,$y(t) \neq 0$, $t \in (a, b)$. If there exists a $d \in (a, b)$ such that $y''(d) = 0$, then (see [20, Theorem 2])

$$(b - a)\left[\int_a^b |q(t)|dt + (b - a)|q(d)| + (b - a)\int_a^b |q'(t) - p(t)|dt\right] > 4. \tag{1.8}$$

Otherwise we consider $y(a) = 0 = y(b) = y(a')(a < b < a')$ with $y(t) \neq 0$ for $t \in (a, b)\bigcup(b, a')$. Then (see [20 ; Theorem 3])

$$(a' - a)\left[\int_a^{a'} |q(t)|dt + (a' - a)|q(d)| + (a' - a)\int_a^{a'} |q'(t) - p(t)|dt\right] > 4.$$

In this paper we have obtained the lower bounds of spacing $(b - a)$,where $y(t)$ is a solution of (1.7) satisfying $y(a) = 0 = y'(b)$ or $y'(a) = 0 = y(b)$. The concept of disfocality for the differential equation (1.7) has been introduced, which improves many more bounds in literature. Furthermore, the condition for disconjugacy of equation (1.7) is obtained. However, in this work we obtained a better bound than in (1.8) in some cases. The concept of disfocality for third order equations enables us to obtain this result.

2 Main Results

Liapunov Inequality, Disfocality and Disconjugacy

THEOREM 2.1 *Let $y(t)$ be a solution of (1.7)with $y(a) = 0 = y'(b), 0 \leq a < b$ and $y(t) \neq 0$, $t \in (a, b]$, where b is such that $|y(b)| = max\{|y(t)| : t \in [a, b]\}$. If $y''(d) = 0$ for some $d \in (a, b)$, then*

$$(b-a)\left[\int_a^b |q(t)|dt + (b-a)|q(d)| + (b-a)\left(\int_a^b |q'(t) - p(t)|dt\right)\right] > 1.$$

Proof. Let $M = max_{t\in[a,b]}|y(t)| = |y(b)|$. Then

$$M = |y(b)| = \left|\int_a^b y'(t)dt\right| \leq \int_a^b |y'(t)|dt. \tag{2.1}$$

Squaring both the sides of (2.1), applying Cauchy-Schwarz inequality and integrating by parts, we obtain

$$\begin{aligned} M^2 &\leq (b-a)\int_a^b (y'(t))^2dt \\ &= -(b-a)\int_a^b y''(t)y(t)dt \\ &\leq (b-a)\int_a^b |y''(t)||y(t)|dt. \end{aligned}$$

Integrating (1.7) from d to t ($a \leq d < t$ or $t < d \leq b$) we get

$$y''(t) = -q(t)y(t) + q(d)y(d) + \int_d^t (q'(s) - p(s))y(s)ds,$$

that is,

$$\begin{aligned} |y''(t)| &\leq M[|q(t)| + |q(d)| + \int_d^t |q'(s) - p(s)|ds] \\ &= M[|q(t)| + |q(d)| + \int_a^b |q'(s) - p(s)|ds]. \end{aligned}$$

Hence

$$\begin{aligned} M^2 &\leq (b-a)\int_a^b M\left[|q(t)|dt + |q(d)| + \int_a^b |q'(s)-p(s)|ds\right]|y(t)|dt \\ &< M^2(b-a)\left[\int_a^b |q(t)|dt + (b-a)|q(d)| + (b-a)\int_a^b |q'(s)-p(s)|ds\right], \end{aligned}$$

from which the required inequality follows. Hence the proof of the theorem is complete.

THEOREM 2.2 *Let $y(t)$ be a solution of (1.7) with $y'(a) = 0 = y(b), 0 \leq a < b$ and $y(t) \neq 0$, $t \in [a, b)$, where a is such that $|y(a)| = max\{|y(t)| : t \in [a, b]\}$. If $y''(d) = 0$ for some $d \in (a, b)$, then*

$$(b-a)\left[\int_a^b |q(t)|dt + (b-a)|q(d)| + (b-a)\left(\int_a^b |q'(t)-p(t)|dt\right)\right] > 1$$

.

The proof is similar to that of Theorem 2.1 and hence is omitted.

DEFINITION 2.3 *Equation (1.7) is said to be right(left) disfocal in $[a, b]$ $(a < b)$ if the solutions of (1.7) with $y'(a) = 0$, $y(a) \neq 0$ $(y'(b) = 0$, $y(b) \neq 0)$ do not have two zeros (counting multiplicities) in $(a, b]$ $([a, b))$. Equation (1.7) is disconjugate in $[a, b]$ if no nontrivial solution of (1.7) has more than two zeros(counting multiplicities). By a solution of (1.7), we understand a non-trivial solution of (1.7).*

THEOREM 2.4 *If equation (1.7) is disconjugate in $[a, b]$, then it is right disfocal in $[c, b]$ or left disfocal in $[a, c]$ for every $c \in (a, b)$. If equation (1.7) is left disfocal in $[a, c]$ and right disfocal in $[c, b]$ for every $c \in (a, b)$, then it is disconjugate on $[a, b]$.*

Proof. Let equation(1.7) be disconjugate in $[a, b]$. Let y(t) be a solution of (1.7) with $y'(c) = 0$ and $y(c) \neq 0$ where $c \in (a, b)$. Then y(t) has atmost two zeros in $[c, b]$ or two zeros in $[a, c]$ (counting multiplicities). Hence (1.7) is left disfocal in $[a, c]$ or right disfocal in $[c, b]$.

Suppose that (1.7) is left disfocal in $[a, c]$ and right disfocal in $[c, b]$ for every c in (a, b). We claim that (1.7) is disconjugate in $[a, b]$. If not, then (1.7) admits a solution y(t) which has at least three zeros (counting multiplicities) in $[a, b]$. Let these three zeros be simple and $a \leq t_1 < t_2 < t_3 \leq b$ with $y(t_i) = 0, 1 \leq i \leq 3$. Then there exist $c_1 \in (t_1, t_2)$ and $c_2 \in (t_2, t_3)$ such that $y'(c_1) = 0 = y'(c_2)$. Hence (1.7) is not right disfocal in $[c_1, b]$ and not left disfocal in $[a, c_2]$. Thus we obtain a contradiction. Suppose $y(t)$ has a double zero at t_1 and a simple zero at t_2 or a simple zero at $t_1$and a double zero at t_2, where $a \leq t_1 < t_2 \leq b$. Let $c \in (t_1, t_2)$

such that $y'(c) = 0$. In the former case (1.7) is not left disfocal in $[a, c]$ and in the latter case (1.7) is not right disfocal in $[c, b]$. Thus we obtain a contradiction again. Hence the proof of the theorem is complete.

THEOREM 2.5 *If*

$$\int_a^b |q(t)|dt + \frac{1}{2}||q||(b-a) + \frac{1}{2}(b-a)\int_a^b |p(t) - q'(t)|dt \le 1/(b-a) \ ,$$

then equation (1.7) is right disfocal in $[a, b]$*, where* $||q|| = max\{|q(t)| : a \le t \le b\}$.

Proof. Suppose that equation (1.7) is not right disfocal in [a,b]. Then (1.7) has a solution $y(t)$ with $y'(a) = 0, y(a) \ne 0$ and $y(t)$ has two zeros (counting multiplicities) in (a,b]. If $a < t_1 \le b$ with $y(t_1) = 0 = y'(t_1)$ and $y(t) \ne 0, \quad t \in [a, t_1)$, then there exists a $d \in (a, t_1)$ such that $y''(d) = 0$. Integrating (1.7) from d to t, where $a < t \le t_1$, we have

$$y''(t) + \int_d^t q(s)y'(s)ds + \int_d^t p(s)y(s)ds = 0.$$

Further integration from a to t ($a < t \le t_1$) yields

$$y'(t) + \int_a^t \left(\int_d^u q(s)y'(s)ds\right)du + \int_a^t \left(\int_d^u p(s)y(s)ds\right)du = 0.$$

that is,

$$y'(t) + \int_a^t q(u)y(u)du - q(d)y(d)(t-a) + \int_a^t \left(\int_d^u (p(s) - q'(s))y(s)ds\right)dt = 0.$$

Integrating from a to t_1, we obtain

$$\begin{aligned} y(a) = \int_a^{t_1} \left(\int_a^t q(u)y(u)du\right)dt \quad &- \quad \int_a^{t_1} q(d)y(d)(t-a)dt \\ &+ \quad \int_a^{t_1} \left(\int_a^t \left(\int_d^u (p(s) - q'(s))y(s)ds\right)du\right)dt. \end{aligned}$$

Hence

$$\begin{aligned} |y(a)| \ < \ |y(a)|\Bigg[&(t_1 - a)\int_a^{t_1} |q(u)|du + \frac{1}{2}(t_1 - a)^2||q|| \\ &+ \int_a^{t_1} \left(\int_a^t |\int_d^u |p(s) - q'(s)|ds|du\right)dt\Bigg]. \end{aligned}$$

Since $|y(a)| \ne 0$, then

$$1 < (b-a)\int_a^b |q(u)|du + \frac{1}{2}(b-a)^2||q|| + \int_a^b \left(\int_a^t |\int_d^u |p(s) - q'(s)|ds|du\right)dt \qquad (2.2)$$

Since ,

$$|\int_d^u |p(s)-q'(s)|ds| \ \leq \ \int_a^b |p(s)-q'(s)|ds. \ for \ \ d,u \in [a,b]$$

then (2.2) yields

$$\int_a^b |q(t)|dt + \frac{1}{2}(b-a)||q|| + \frac{1}{2}(b-a)\int_a^b |p(s)-q'(s)|ds > 1/(b-a), \tag{2.3}$$

a contradiction to the given hypothesis. If there exists a $T \in (a, t_1)$ such that $y'(T) = 0$ and $y(T) \neq 0$, then we work over the interval $[T, b]$ to obtain

$$\int_T^b |q(t)|dt + \frac{1}{2}||q||(b-T) + \frac{1}{2}(b-T)\int_T^b |p(s)-q'(s)|ds > 1/(b-T).$$

However, this inequality yields (2.3). If $a < t_1 < t_2 \leq b$ with $y(t_1) = 0 = y(t_2)$ and $y(t) \neq 0$ for $t \in [a, t_1) \bigcup (t_1, t_2)$, then there exists a $c \in (t_1, t_2)$ such that $y'(c) = 0$. Hence there exists a $d \in (a, c)$ such that $y''(d) = 0$. Let $|y(a)| \geq |y(c)|$. Integrating (1.7) from d to t, where $a < t < t_2$, we have

$$y^{''}(t) + \int_d^t q(s)y'(s)ds + \int_d^t p(s)y(s)ds = 0 \ , \quad t \in [a, t_2] \ .$$

Further integration from a to t $(a < t \leq t_2)$ yields

$$y'(t) + \int_a^t \left(\int_d^u q(s)y'(s)ds\right)du + \int_a^t \left(\int_d^u p(s)y(s)ds\right)du = 0,$$

that is,

$$y'(t) + \int_a^t q(u)y(u)du - q(d)y(d)(t-a) + \int_a^t \left(\int_d^u (p(s)-q'(s))y(s)ds\right)du = 0.$$

Integrating from a to t_2, we obtain

$$\begin{aligned} y(a) = & \int_a^{t_2}\left(\int_a^t q(u)y(u)du\right)dt - \int_a^{t_2} q(d)y(d)(t-a)dt \\ & + \int_a^{t_2}\left(\int_a^t \left(\int_d^u (p(s)-q'(s))y(s)ds\right)du\right)dt = 0. \end{aligned}$$

Hence

$$\begin{aligned} |y(a)| < |y(a)| \Big[& (t_2-a)\int_a^{t_2} |q(u)|du + \frac{1}{2}(t_2-a)^2||q|| \\ & + \int_a^{t_2}\left(\int_a^t |\int_d^u |p(s)-q'(s)|ds|du\right)dt. \end{aligned}$$

Since $|y(a)| \neq 0$, then

$$1 < (b-a)\int_a^b |q(u)|du + \frac{1}{2}(b-a)^2||q|| + \int_a^b \int_a^t |\int_d^u |p(s) - q'(s)|ds|du\, dt. \tag{2.4}$$

that is,

$$\int_a^b |q(u)|du + \frac{1}{2}(b-a)||q|| + \frac{1}{2}(b-a)\int_a^b |p(s) - q'(s)|ds > 1/(b-a).$$

Let $|y(a)| < |y(c)|$. Integrating (1.7) from d to t we obtain

$$y''(t) + \int_d^t q(s)y'(s)ds + \int_d^t p(s)y(s)ds = 0\ , \quad t \in (a, t_2]\ ,$$

that is,

$$y''(t) + q(t)y(t) - q(d)y(d) + \int_d^t (p(s) - q'(s))y(s)ds = 0.\ .$$

Then integrating from c to t we have

$$y'(t) + \int_c^t q(u)y(u)du - q(d)y(d)(t-c) + \int_c^t \left(\int_d^u (p(s) - q'(s))y(s)ds\right) du = 0\ , \quad t \in (a, t_2]\ . \tag{2.5}$$

If $t \in (c, t_2]$, then further integration of the above identity from c to t_2 yields

$$\begin{aligned} y(c) = &\int_c^{t_2} \left(\int_c^t q(u)y(u)du\right) dt - \int_c^{t_2} q(d)y(d)(t-c)dt \\ &+ \int_c^{t_2} \left(\int_c^t \left(\int_d^u (p(s) - q'(s))y(s)ds\right) du\right) dt. \end{aligned}$$

Hence

$$|y(c)| < |y(c)|\left[(t_2 - c)\int_c^{t_2} |q(s)|ds + \frac{1}{2}(t_2 - c)^2||q|| + \int_c^{t_2}\int_c^t |\int_d^u |p(s) - q'(s)|ds|du\, dt\right].$$

As $y(c) \neq 0$, then

$$1 < (b-a)\int_a^b |q(s)|ds + \frac{1}{2}(b-a)^2||q|| + \frac{1}{2}(b-a)^2\int_a^b |p(s) - q'(s)|ds\ ,$$

whether $u > d$ or $u < d$. If $t \in (a, c]$, then integrating (2.5) from a to c yields

$$\begin{aligned} y(c) - y(a) + &\int_a^c \left(\int_c^t q(u)y(u)du\right) dt - \int_a^c q(d)y(d)(t-c)dt \\ &+ \int_a^c \left(\int_c^t \left(\int_d^u (p(s) - q'(s))y(s)ds\right) du\right) dt = 0\ . \end{aligned}$$

Hence

$$|y(c)-y(a)| < |y(c)|\Big[(c-a)\int_c^b |q(u)|du + \frac{1}{2}||q||(c-a)^2 + \int_a^c |\int_c^t |\int_d^u |p(s)-q'(s)|ds|du|dt\Big].$$

that is,

$$\left|1-\frac{y(a)}{y(c)}\right| < (b-a)\int_a^b |q(t)|dt + \frac{1}{2}(b-a)^2||q|| + \frac{1}{2}(b-a)^2\int_a^b |p(t)-q'(t)|dt\ ,$$

whether $u > d$ or $u < d$. Since $y(a)y(c) < 0$, then

$$1 < 1-\frac{y(a)}{y(c)} < (b-a)\int_a^b |q(t)|dt + \frac{1}{2}(b-a)^2||q|| + \frac{1}{2}(b-a)^2\int_a^b |p(t)-q'(t)|dt.$$

Hence in either case (2.3) holds. If there exists a $T \in (a, t_1)$ such that $y'(T) = 0$ and $y(T) \neq 0$, then we work over the interval $[T, b]$ to obtain

$$\int_T^b |q(t)|dt + \frac{1}{2}(b-T)||q|| + \frac{1}{2}(b-T)\int_T^b |p(t)-q'(t)|dt > 1/(b-T)$$

which yields (2.3). As (2.3) contradicts the given hypothesis, then the theorem is proved.

THEOREM 2.6 *If*

$$\int_a^b |q(t)|dt + \frac{1}{2}||q||(b-a) + \frac{1}{2}(b-a)\int_a^b |p(t)-q'(t)|dt \ \leq\ 1/(b-a)\ ,$$

then equation (1.7) is left disfocal in $[a, b]$.

The proof is similar to that of Theorem (2.5) and hence is omitted.

THEOREM 2.7 *If equation (1.7) is not right disfocal in [c,b] and not left disfocal in [a,c], where* $c \in (a, b)$*, then*

$$\int_a^b |q(s)|ds + \frac{1}{2}||q||(b-a) + \frac{1}{2}(b-a)\int_a^b |p(s)-q'(s)|ds \ >\ 4/(b-a), \tag{2.6}$$

Proof. Since (1.7) is not right disfocal in [c,b] and not left disfocal in [a,c], where $c \in (a, b)$, then from Theorems 2.5 and 2.6 we obtain

$$\int_c^b |q(t)|dt + \frac{1}{2}(b-c)||q|| + \frac{1}{2}(b-c)\int_c^b |p(t)-q'(t)|dt \ >\ 1/(b-c), \tag{2.7}$$

and

$$\int_a^c |q(t)|dt + \frac{1}{2}(c-a)||q|| + \frac{1}{2}(c-a)\int_a^c |p(t)-q'(t)|dt \ > \ 1/(c-a). \tag{2.8}$$

Hence

$$\begin{aligned}\int_a^b |q(t)|dt + \frac{1}{2}(b-a)||q|| \ &+ \ \frac{1}{2}(c-a)\int_a^c |p(t)-q'(t)|dt \\ &+ \ \frac{1}{2}(b-c)\int_c^b |p(t)-q'(t)|dt \\ &> \ \frac{(b-a)}{(c-a)(b-c)}\end{aligned}$$

The function $f(c) = (c-a)(b-c)$ attains maximum at $c = (a+b)/2$ and $f((a+b)/2) = (b-a)^2/4$. Hence the required inequality follows. Thus the proof of the theorem is complete.

COROLLARY 2.8 *If $y(t)$ is a solution of (1.7) with $y(a) = 0 = y'(a)$, $\ y(b) = 0 = y'(b)$ and $y(t) \neq 0$, $\ t \in (a,b)$, then*

$$\int_a^b |q(t)|dt + \frac{1}{2}(b-a)||q|| + \frac{1}{2}(b-a)\int_a^b |p(t)-q'(t)|dt > 4/(b-a)\ .$$

Proof. There exists a $c \in (a,b)$ such that $y'(c) = 0$. Hence equation (1.7) is not right disfocal on [c,b] and not left disfocal on [a,c]. Then the result follows from Theorem 2.7.

REMARK 2.9 *Corollary 2.8 is an improvement of the inequality*

$$\int_a^b |q(t)|dt + (b-a)||q|| + (b-a)\int_a^b |p(t)-q'(t)|dt > 4/(b-a),$$

if y(t) is a solution of (1.7) with $y(a) = 0 = y'(a)$ and $y(b) = 0 = y'(b)(a < b)$ and $y(t) \neq 0 for\ t \in (a,b)$. However, if $y(t)$ is a solution of (1.7) with $y(a) = 0$, $\ y(b) = 0 = y'(b)$ and $y(t) \neq 0$, $\ t \in (a,b)$ or $y(a) = 0 = y'(a)$, $\ y(b) = 0$ and $y(t) \neq 0$, $\ t \in (a,b)$. Then (see [20; Theorem 1]) can be applied but Theorem 2.7 cannot be applied because (1.7) is left disfocal in [a,c] in the former case and right disfocal in latter case, where $c \in (a,b)$ with $y'(c) = 0$.

REMARK 2.10 *Suppose that $y(t)$ is a solution of (1.7) with $y(a) = 0 = y(b) = y(a')$ $\ (a < b < a')$ and $y(t) \neq 0$ for $t \in (a,b)\bigcup(b,a')$. Then there exist a $c_1 \in (a,b)$ and $c_2 \in (b,a')$ such that $y'(c_1) = 0 = y'(c_2)$. Theorem 2 (see [20],) can be applied to this situation but Theorem 2.7 cannot be applied because (1.7) is left disfocal on $[a,c_1]$ and right disfocal on $[c_2,b]$. However, the following result holds :*

COROLLARY 2.11 *If $y(t)$ is a solution of (1.7) with $y(t_i) = 0, \quad 1 \le i \le 4 \quad (t_1 < t_2 < t_3 < t_4)$ and $y(t) \ne 0, \quad t \in \bigcup_{i=1}^{3}(t_i, t_{i+1})$, then*

$$\int_{t_1}^{t_4} |q(t)|dt + \frac{1}{2}(t_4 - t_1)||q|| + \frac{1}{2}(t_4 - t_1)\int_{t_1}^{t_4} |p(t) - q'(t)|dt > 4/(t_4 - t_1),$$

if $t_2 < (t_1 + t_4)/2 < t_3$; otherwise,

$$\begin{aligned} \int_{t_1}^{t_4} |q(t)|dt + (t_4 - t_1)||q|| \quad &+ \quad (t_4 - t_1)\int_{t_1}^{t_4} |p(t) - q'(t)|dt \\ &> \quad 2\left[\frac{1}{(t_3 - t_1)} + \frac{1}{(t_4 - t_2)}\right]. \end{aligned}$$

If $t_1 = t_2$ is a double zero or $t_3 = t_4$ is a double zero, then

$$\int_{t_1}^{t_4} |q(t)|dt + \frac{1}{2}(t_4 - t_1)||q|| + \frac{1}{2}(t_4 - t_1)\int_{t_1}^{t_4} |p(t) - q'(t)|dt > \frac{1}{(t_4 - t_1)} + \frac{1}{(t_3 - t_1)}$$

or

$$\int_{t_1}^{t_4} |q(t)|dt + \frac{1}{2}(t_4 - t_1)||q|| + \frac{1}{2}(t_4 - t_1)\int_{t_1}^{t_4} |p(t) - q'(t)|dt > \frac{1}{(t_4 - t_1)} + \frac{1}{(t_4 - t_2)} \ .$$

Proof. There exists a $c \in [t_2, t_3]$ such that $y'(c) = 0$. Hence equation (1.7) is not left disfocal in $[t_1, c]$ and not right disfocal in $[c, t_4]$. From Theorems 2.5 and 2.6 it follows that

$$\int_{t_1}^{c} |q(t)|dt + \frac{1}{2}(c - t_1)||q|| + \frac{1}{2}(c - t_1)\int_{t_1}^{c} |p(t) - q'(t)|dt > 1/(c - t_1),$$

and

$$\int_{c}^{t_4} |q(t)|dt + \frac{1}{2}(t_4 - c)||q|| + \frac{1}{2}(t_4 - c)\int_{c}^{t_4} |p(t) - q'(t)|dt > 1/(t_4 - c).$$

Hence

$$\begin{aligned} \int_{t_1}^{t_4} |q(t)|dt \quad &+ \quad \frac{1}{2}(t_4 - t_1)||q|| + \frac{1}{2}(c - t_1)\int_{t_1}^{c} |p(t) - q'(t)|dt \\ &+ \quad \frac{1}{2}(t_4 - c)\int_{c}^{t_4} |p(t) - q'(t)|dt > (t_4 - t_1)/(c - t_1)(t_4 - c) \ . \end{aligned}$$

The function $f(c) = (c - t_1)(t_4 - c)$ attains maximum at $c = (t_1 + t_4)/2$ and $f((t_1 + t_4)/2) = (t_4 - t_1)^2/4$. Hence

$$\int_{t_1}^{t_4} |q(t)|dt + \frac{1}{2}(t_4 - t_1)||q|| + \frac{1}{2}(t_4 - t_1)\int_{t_1}^{t_4} |p(t) - q'(t)|dt > 4/(t_4 - t_1) \ .$$

Since $t_2 < c < t_3$ and $c = (t_1 + t_4)/2$, then $t_2 < (t_1 + t_4)/2 \ < t_3$. If we consider three consecutive zeros t_1, t_2 and t_3, then from (see[19 ; Theorem 2]) we obtain

$$\int_{t_1}^{t_3} |q(t)|dt + (t_3 - t_1)||q|| + (t_3 - t_1) \int_{t_1}^{t_3} |p(t) - q'(t)|dt > 4/(t_3 - t_1).$$

Hence

$$\int_{t_1}^{t_4} |q(t)|dt + (t_4 - t_1)||q|| + (t_4 - t_1) \int_{t_1}^{t_4} |p(t) - q'(t)|dt > 4/(t_3 - t_1).$$

Similarly, if we consider three consecutive zeros t_2, t_3 and t_4, then from (see [19 ; Theorem 2])it follows that

$$\int_{t_2}^{t_4} |q(t)|dt + (t_4 - t_2)||q|| + (t_4 - t_2) \int_{t_2}^{t_4} |p(t) - q'(t)|dt > 4/(t_4 - t_2).$$

Hence

$$\int_{t_1}^{t_4} |q(t)|dt + (t_4 - t_1)||q|| + (t_4 - t_1) \int_{t_1}^{t_4} |p(t) - q'(t)|dt > 4/(t_4 - t_2).$$

Thus

$$\begin{aligned} \int_{t_1}^{t_4} |q(t)|dt + (t_4 - t_1)||q|| + (t_4 - t_1) \int_{t_1}^{t_4} |p(t) - q'(t)|dt \\ > 2\left[\frac{1}{(t_3 - t_1)} + \frac{1}{(t_4 - t_2)}\right] . \end{aligned}$$

Let $t_1 = t_2$ be a double zero. There exists a $c \in (t_1, t_3)$ such that $y'(c) = 0$. Since equation (1.7) is not right disfocal on $[c, t_4]$, then from Theorem 2.5 it follows that

$$\int_{c}^{t_4} |q(t)|dt + \frac{1}{2}(t_4 - c)|q(d)| + \frac{1}{2}(t_4 - c) \int_{c}^{t_4} |p(t) - q'(t)|dt > 1/(t_4 - c).$$

Hence

$$\int_{c}^{t_4} |q(t)|dt + \frac{1}{2}(t_4 - c)||q|| + \frac{1}{2}(t_4 - c) \int_{c}^{t_4} |p(t) - q'(t)|dt > 1/(t_4 - t_1) \ .$$

As equation (1.7) is not left disfocal on $[t_1, c]$, then from Theorem 2.6 it follows that

$$\int_{t_1}^{c} |q(t)|dt + \frac{1}{2}(c - t_1)||q|| + \frac{1}{2}(c - t_1) \int_{t_1}^{c} |p(t) - q'(t)|dt > 1/(c - t_1).$$

Hence

$$\int_{t_1}^{c} |q(t)|dt + \frac{1}{2}(c - t_1)||q|| + \frac{1}{2}(c - t_1) \int_{t_1}^{c} |p(t) - q'(t)|dt > 1/(t_3 - t_1).$$

Thus

$$\int_{t_1}^{t_4} |q(t)|dt + \frac{1}{2}(t_4 - t_1)||q|| + \frac{1}{2}(t_4 - t_1)\int_{t_1}^{t_4} |p(t) - q'(t)|dt$$
$$> \left[\frac{1}{(t_4 - t_1)} + \frac{1}{(t_3 - t_1)}\right].$$

Similarly, if $t_3 = t_4$ is a double zero, then we have the other inequality.

REMARK 2.12 *Corollary 2.11 cannot be obtained from Theorems 1 and 2 in [19].*

REMARK 2.13 *If, in general, $y(t)$ is a solution of (1.7) with $y(t_i) = 0, \quad 1 \leq i \leq n$, $n \geq 4, \quad (t_1 < t_2 < .... < t_n)$ and $y(t) \neq 0, t \in \overset{n}{\underset{i=1}{\cup}}(t_i, t_{i+1})$, then*

$$\int_{t_1}^{t_n} |q(t)|dt + \frac{1}{2}(t_n - t_1)||q|| + \frac{1}{2}(t_n - t_1)\int_{t_1}^{t_n} |p(t) - q'(t)|dt > 4/(t_n - t_1),$$

if $t_{i-1} < (t_n + t_1)/2 < t_i, 3 \leq i \leq n-1$; otherwise,

$$\int_{t_1}^{t_n} |q(t)|dt + (t_n - t_1)||q|| + (t_n - t_1)\int_{t_1}^{t_n} |p(t) - q'(t)|dt$$
$$> 2\left[\frac{1}{(t_i - t_1)} + \frac{1}{(t_n - t_{i-1})}\right], \quad 3 \leq i \leq n-1.$$

This can be proved as in Corollary 2.11 by taking $c \in (t_{i-1}, t_i)$ such that $y'(c) = 0$.

THEOREM 2.14 *If, for every $c \in (a, b)$*

$$\int_a^c |q(t)|dt + \frac{1}{2}(c - a)||q|| + \frac{1}{2}(c - a)\int_a^c |p(t) - q'(t)|dt < 1/(c - a)$$

and

$$\int_c^b |q(t)|dt + \frac{1}{2}(b - c)||q|| + \frac{1}{2}(b - c)\int_c^b |p(t) - q'(t)|dt < 1/(b - c),$$

then (1.7) is disconjugate in $[a, b]$.

Proof. If possible, let $y(t)$ be a solution of (1.7) having three zeros (counting multiplicities) in $[a, b]$. Let $a \leq t_1 < t_2 < t_3 \leq b, \quad y(t_i) = 0, 1 \leq i \leq 3$ and $y(t) \neq 0$ for $t \in [a, b], \ t \neq t_i, \quad 1 \leq i \leq 3$. Then there exists a $c_1 \in (t_1, t_2)$ such that $y'(c_1) = 0$. Hence (1.7) is not right disfocal on $[c_1, t_3]$. Thus

$$\int_{c_1}^{t_3} |q(t)|dt + \frac{1}{2}(t_3 - c_1)||q|| + \frac{1}{2}(t_3 - c_1)\int_{c_1}^{t_3} |p(t) - q'(t)|dt > 1/(t_3 - c_1).$$

Consequently,

$$\int_{c_1}^{b} |q(t)|dt + \frac{1}{2}(b-c_1)||q|| + \frac{1}{2}(b-c_1)\int_{c_1}^{b} |p(t)-q'(t)|dt > 1/(b-c_1)$$

a contradiction. A similar contradiction is obtained if we take $c_2 \in (t_2, t_3)$ such that $y'(c_2) = 0$. Suppose $y(t_1) = 0 = y'(t_1)$ and $y(t_2) = 0$, where $a \le t_1 < t_2 \le b$. Then there exists a $c_3 \in (t_1, t_2)$ such that $y'(c_3) = 0$. Since (1.7) is not disfocal on $[t_1, c_3]$, then

$$\int_{t_1}^{c_3} |q(t)|dt + \frac{1}{2}(c_3-t_1)||q|| + \frac{1}{2}(c_3-t_1)\int_{t_1}^{c_3} |p(t)-q'(t)|dt > 1/(c_3-t_1).$$

Hence

$$\int_{a}^{c_3} |q(t)|dt + \frac{1}{2}(c_3-a)||q|| + \frac{1}{2}(c_3-a)\int_{a}^{c_3} |p(t)-q'(t)|dt > 1/(c_3-a),$$

a contradiction. A similar contradiction is obtained if $y(t_1) = 0, \quad y(t_2) = 0 = y'(t_2)$. This completes the proof of the Theorem.

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